Journal of Theoretical Physics

Founded and Edited by M. Apostol **253** (2016)

ISSN 1453-4428

Elastic waves interacting with a plane surface

B. F. Apostol
Department of Engineering Seismology
Institute of Earth's Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: afelix@theory.nipne.ro

Abstract

The interaction of far-field spherical elastic waves (spherical shells) with the plane surface of a homogeneous isotropic elastic half-space is investigated in the transient regime. The transient regime is produced by wave sources localized both in space and time, like seismic sources. The problem investigated here may bear relevance upon the earthquake effects on Earth's surface, where the transient regime of wave propagation in finite bodies brings new technical aspects which received little attention. We consider spherical (shell) waves (primary waves) produced by a tensorial force located at an inner point in the half-space. On the surface, these waves create new wave sources which generate secondary waves. The secondary waves are computed here by means of a simplified model. It is shown that the secondary waves have a delay time in comparison with the primary waves and produce on the surface a main shock with a sharp wavefront and a long tail. Similar calculations are presented for an internal discontinuity surface parallel with the surface of the half-space; it is shown that the secondary waves generated on the surface of the half-space may be reduced appreciably in this case.

Introduction. This paper is motivated by the interaction of the elastic waves in the transient regime with a plane surface of an elastic half-space. This particular problem bears relevance on the effect of the seismic waves on the Earth's surface. It is well known that earthquakes are produced by forces localized both in space and time; the seismic source is located at an inner point beneath the Earth's surface and its action lasts a finite, relatively short, duration of time. The problem originates with the classical works of Rayleigh, Lamb and Love (and it is sometime known as Lamb's problem).[1]-[3] The transient regime due to the finite active life of the source implies relevant aspects, which have not received sufficient attention.

For our particular purpose we may approximate the local extension of the Earth by a homogeneous isotropic elastic half-space (a semi-infinite solid) bounded by a plane surface. It is widely accepted that a seismic source can be represented by a tensorial force with the components

$$F_i(\mathbf{R},t) = m_{ij}(t)\partial_j\delta(\mathbf{R}-\mathbf{R}_0) \ , \tag{1}$$

(see, *e.g.*, Ref. [4], 2nd edition, p.60, Exercise 3.6), where $m_{ij}(t)$ is the tensor of the seismic moment (divided by the density of the medium for convenience) and $\mathbf{R}_0$ is the point of location of the source; δ is the Dirac delta function and the symbol ∂_j stands for the derivative with respect to the coordinate x_j, $i,j = 1,2,3$, of the position vector $\mathbf{R} = (x_1, x_2, x_3)$. A short temporal impulse is represented by $m_{ij}(t) = Tm_{ij}\delta(t)$, where T is a measure of the duration of the impulse. While a general tensor m_{ij} of the seismic moment correspond to a faulting slip, an isotropic tensor m_{ij} is associated with an explosion.

The seismic moment offers the opportunity to give an estimation of the dimension of the focal region of the earthquake. Indeed, if, generically, we denote by M the magnitude of the seismic moment (and by $m = M/\rho$ the magnitude of the seismic moment divided by density ρ of the medium), we may assume that the rupture of the material in the volume V of the the focal region occurs for $M/V = \rho c^2$, where c is a mean velocity of the seismic waves; this equality indicates that an energy density released in an earhquake equals the density of the elastic energy. For $M = 10^{26} dyn \cdot cm$ (corresponding to an earthquake magnitude $M_w = 7$, from the Gutenberg-Richter definition[5, 6] $\lg M = 1.5M_w + 16.1$), $\rho = 5g/cm^3$ for the average Earth's density and $c = 5km/s$ for a mean value of the velocity of the elastic waves we get a volume $V = 8 \times 10^{13} cm^3$ of the focal region and a localization length $l = V^{1/3} \simeq 400m$. The Dirac delta function used in the representation of the tensorial force may be viewed as being localized over a distance of the order l (volume l^3). This spatial uncertainty leads to a time uncertainty of the order $T = l/c = 0.08s$ (for $c = 5km/s$; the rupture velocity may be much smaller than the wave velocity). The magnitude of the seismic moment is currently estimated from seismic records.[4, 7, 8]

Primary waves. The equation of the elastic waves in an isotropic body is

$$\ddot{\mathbf{u}} - c_t^2 \Delta \mathbf{u} - (c_l^2 - c_t^2) grad \cdot div \mathbf{u} = \mathbf{F} \ , \tag{2}$$

where $\mathbf{u}$ is the displacement vector, $c_{l,t}$ are the wave velocities and $\mathbf{F}$ is the force (per unit mass).[9] We consider this equation in an isotropic elastic half-space extending in the region $z < 0$ and bounded by the flat surface $z = 0$. The elementary source, which generates the force $\mathbf{F}$, is placed at $\mathbf{R}_0 = (0, 0, z_0)$, $z_0 < 0$; the force is given by equation (1) (with $m_{ij}(t) = Tm_{ij}\delta(t)$). The coordinates of the position vector $\mathbf{R}$ are denoted by (x_1, x_2, x_3); also, the notation $x = x_1$, $y = x_2$, $z = x_3$ is used. We use the Helmholtz decomposition $\mathbf{F} = grad\phi + curl\mathbf{H}$ ($div\mathbf{H} = 0$), whence

$$\Delta\phi = div\mathbf{F} \ , \ \Delta\mathbf{H} = -curl\mathbf{F} \ ; \tag{3}$$

similarly, the displacement $\mathbf{u}$ is decomposed as $\mathbf{u} = grad\Phi + curl\mathbf{A}$, with the notation $\mathbf{u}^l = grad\Phi$ and $\mathbf{u}^t = curl\mathbf{A}$, by using the Helmholtz potentials Φ and $\mathbf{A}$ ($div\mathbf{A} = 0$); equation (2) is transformed into two standard wave equations

$$\ddot{\Phi} - c_l^2 \Delta\Phi = \phi \ , \ \ddot{\mathbf{A}} - c_t^2 \Delta\mathbf{A} = \mathbf{H} \ ; \tag{4}$$

we can see that the l, t-waves are separated.

From equations (3), and making use of the force distribution given by equation (1), we get immediately

$$\begin{gathered}\phi = -\frac{1}{4\pi} Tm_{ij}\delta(t) \int d\mathbf{R}' \frac{1}{|\mathbf{R}-\mathbf{R}'|} \partial_i' \partial_j' \delta(\mathbf{R}' - \mathbf{R}_0) = \\ = -\frac{1}{4\pi} Tm_{ij}\delta(t) \partial_i \partial_j \frac{1}{|\mathbf{R}-\mathbf{R}_0|}\end{gathered} \tag{5}$$

and

$$\begin{gathered}H_i = \frac{1}{4\pi} T\varepsilon_{ijk} m_{kl}\delta(t) \int d\mathbf{R}' \frac{1}{|\mathbf{R}-\mathbf{R}'|} \partial_j' \partial_l' \delta(\mathbf{R}' - \mathbf{R}_0) = \\ = \frac{1}{4\pi} T\varepsilon_{ijk} m_{kl}\delta(t) \partial_j \partial_l \frac{1}{|\mathbf{R}-\mathbf{R}_0|} \ ,\end{gathered} \tag{6}$$

where ε_{ijk} is the totally antisymmetric tensor of rank three. Making use of these sources in equations (4), and using the Kirchhoff retarded solutions, we get the potentials

$$\begin{aligned}\Phi &= -\tfrac{T}{(4\pi c_l)^2} m_{ij} \int d\mathbf{R}' \tfrac{\delta(t-\left|\mathbf{R}-\mathbf{R}'\right|/c_l)}{\left|\mathbf{R}-\mathbf{R}'\right|} \partial_i' \partial_j' \tfrac{1}{\left|\mathbf{R}'-\mathbf{R}_0\right|} = \\ &= -\tfrac{T}{(4\pi c_l)^2} m_{ij} \partial_i \partial_j \int d\mathbf{R}' \tfrac{\delta(t-\left|\mathbf{R}-\mathbf{R}'\right|/c_l)}{\left|\mathbf{R}-\mathbf{R}'\right|} \tfrac{1}{\left|\mathbf{R}'-\mathbf{R}_0\right|}\end{aligned} \tag{7}$$

and

$$\begin{aligned}A_i &= \tfrac{T}{(4\pi c_t)^2} \varepsilon_{ijk} m_{kl} \int d\mathbf{R}' \tfrac{\delta(t-\left|\mathbf{R}-\mathbf{R}'\right|/c_t)}{\left|\mathbf{R}-\mathbf{R}'\right|} \partial_j' \partial_l' \tfrac{1}{\left|\mathbf{R}'-\mathbf{R}_0\right|} = \\ &= \tfrac{T}{(4\pi c_t)^2} \varepsilon_{ijk} m_{kl} \partial_j \partial_l \int d\mathbf{R}' \tfrac{\delta(t-\left|\mathbf{R}-\mathbf{R}'\right|/c_t)}{\left|\mathbf{R}-\mathbf{R}'\right|} \tfrac{1}{\left|\mathbf{R}'-\mathbf{R}_0\right|} \ .\end{aligned} \tag{8}$$

We extend the integral

$$\begin{aligned}I &= \int d\mathbf{R}' \tfrac{\delta(t-\left|\mathbf{R}-\mathbf{R}'\right|/c)}{\left|\mathbf{R}-\mathbf{R}'\right|} \tfrac{1}{\left|\mathbf{R}'-\mathbf{R}_0\right|} = \\ &= \int d\mathbf{R}'' \tfrac{\delta(t-R''/c)}{R''} \tfrac{1}{\left|\mathbf{R}-\mathbf{R}_0-\mathbf{R}''\right|}\end{aligned} \tag{9}$$

(where c stands for $c_{l,t}$) occurring in the above equations to the whole space, where it can be effected straightforwardly by using spherical coordinates; we get

$$I = 4\pi c \left[\theta\left(ct - \left|\mathbf{R}-\mathbf{R}_0\right|\right) + \frac{ct}{\left|\mathbf{R}-\mathbf{R}_0\right|} \theta\left(\left|\mathbf{R}-\mathbf{R}_0\right| - ct\right) \right] \ ; \tag{10}$$

inserting this result in equations (7) and (8) we get the Helmholtz potentials

$$\begin{aligned}\Phi &= -\tfrac{T}{4\pi c_l} m_{ij} \partial_i \partial_j \left[\theta\left(c_l t - \left|\mathbf{R}-\mathbf{R}_0\right|\right) + \tfrac{c_l t}{\left|\mathbf{R}-\mathbf{R}_0\right|} \theta\left(\left|\mathbf{R}-\mathbf{R}_0\right| - c_l t\right) \right] \ , \\ A_i &= \tfrac{T}{4\pi c_t} \varepsilon_{ijk} m_{kl} \partial_j \partial_l \left[\theta\left(c_t t - \left|\mathbf{R}-\mathbf{R}_0\right|\right) + \tfrac{c_t t}{\left|\mathbf{R}-\mathbf{R}_0\right|} \theta\left(\left|\mathbf{R}-\mathbf{R}_0\right| - c_t t\right) \right] \ .\end{aligned} \tag{11}$$

Making use of the notation

$$F_{l,t} = \frac{T}{4\pi c_{l,t}} \left[\theta\left(c_{l,t} t - \left|\mathbf{R}-\mathbf{R}_0\right|\right) + \frac{c_{l,t} t}{\left|\mathbf{R}-\mathbf{R}_0\right|} \theta\left(\left|\mathbf{R}-\mathbf{R}_0\right| - c_{l,t} t\right) \right] \ , \tag{12}$$

the potentials can be written as

$$\Phi = -m_{ij} \partial_i \partial_j F_l \ , \ \ A_i = \varepsilon_{ijk} m_{kl} \partial_j \partial_l F_t \ ; \tag{13}$$

it follows the displacement

$$\begin{aligned}u_i^l &= \partial_i \Phi = -m_{jk} \partial_i \partial_j \partial_k F_l \ , \\ u_i^t &= \varepsilon_{ijk} \partial_j A_k = m_{jk} \partial_i \partial_j \partial_k F_t - m_{ij} \partial_j \Delta F_t \ .\end{aligned} \tag{14}$$

We can see that these solutions consist of two parts: spherical waves propagating with velocities $c_{l,t}$, given by δ-functions and derivatives of δ-functions (arising from the derivatives of the θ-functions in equation (12)), and a quasi-static displacement which includes the functions $\theta(\left|\mathbf{R}-\mathbf{R}_0\right| - c_{l,t} t)$

and extends over the distance $\Delta R=(c_l-c_t)t$ (notation $\mathbf{R}$ is used for $\mathbf{R}-\mathbf{R}_0$). The quasi-static contributions, being proportional to third-order derivatives of t/R, are solutions of homogeneous wave equations. In the transient regime, the quasi-static contributions should be omitted, and we may limit ourselves to the δ-functions and derivatives of δ-functions arising from the derivatives of the θ-functions in equation (12). Outside the support of the δ-functions and their derivatives (*i.e.*, for $R=|\mathbf{R}-\mathbf{R}_0|\neq c_{l,t}t$) the displacement is zero. We note also that for $R\neq c_t t$ the function F_t in equation (12) is either $T/4\pi c_t$ or $Tt/4\pi R$; in both cases the term with the laplacian in the second equation (14) cancels out (including the limit $R=c_t t$), and $\mathbf{u}^t$ acquires the same expression as $-u_i^l$ with c_l replaced by c_t.

The solution is given by the potentials in equation (13), provided we leave aside the quasi-static displacement; expressions like $m_{jk}\partial_i\partial_j\partial_k F$ becomes

$$\begin{aligned}m_{jk}\partial_i\partial_j\partial_k F=&[\tfrac{m_{jj}x_i}{2R^3}(1-2ct/R)+\tfrac{m_{ij}x_j}{R^3}(1-3ct/r)-\tfrac{3m_{jk}x_ix_jx_k}{2R^5}(1-4ct/R)]\delta(R-ct)-\\&-[\tfrac{m_{jj}x_i}{2R^2}(1-ct/R)-\tfrac{m_{jk}x_ix_jx_k}{2R^4}(1-3ct/R)]\delta'(R-ct)\ ,\end{aligned}\tag{15}$$

where F is a generic notation for $F_{l,t}$ with $c_{l,t}$ replaced by c and the factor $1/4\pi c$ omitted; the coordinates of the position vector are given by $\mathbf{R}=(x_1,x_2,x_3)$. We may put $R=ct$ in this equation and get

$$\begin{aligned}u_i^l=&\tfrac{T}{8\pi c_lR^3}[m_{jj}x_i+4m_{ij}x_j-9m_{jk}x_ix_jx_k\tfrac{1}{R^2}]\delta(R-c_lt)+\\&+\tfrac{T}{4\pi c_l}m_{jk}x_ix_jx_k\tfrac{1}{R^4}\delta'(R-c_lt)\ ,\end{aligned}\tag{16}$$

where c_l and the factor $1/4\pi c_l$ are restored. Similarly, from equations (14) we get

$$\begin{aligned}u_i^t=&-\tfrac{T}{8\pi c_tR^3}[m_{jj}x_i+6m_{ij}x_j-9m_{jk}x_ix_jx_k\tfrac{1}{R^2}]\delta(R-c_tt)-\\&-\tfrac{T}{4\pi c_t}\left(m_{jk}x_ix_jx_k\tfrac{1}{R^4}-m_{ij}x_j\tfrac{1}{R^2}\right)\delta'(R-c_tt)\ .\end{aligned}\tag{17}$$

We can see that in the far-field region (wave region) the source generates two (double-shock) spherical waves (derivatives of the δ-function), propagating with velocities $c_{l,t}$, given by

$$u_i^f\simeq\tfrac{Tm_{ij}x_j}{4\pi c_tR^2}\delta'(R-c_tt)+\tfrac{Tm_{jk}x_ix_jx_k}{4\pi R^4}\left[\tfrac{1}{c_l}\delta'(R-c_lt)-\tfrac{1}{c_t}\delta'(R-c_tt)\right]\ ;\tag{18}$$

these are the leading contributions to the solution in the wave region. Equivalent formulae for spherical seismic waves produced by a localized vector force have been derived by Stokes long time ago[10] (see also Refs. [4],[7, 8] and [11]).

The waves propagating with velocity c_l are the primary P waves (compressional waves), while the waves propagating with velocity c_t are the primary S-waves (they include the shear contribution). The second term on the right in equation (18) is longitudinal ($\sim\mathbf{R}$), while the polarization of the first term depends on the moment tensor. It is worth noting that the far-field waves given by equations (18) have the shape of spherical shells (zero thickness). These waves are associated with the feeble tremor produced by the P, S-waves in earthquakes.[3, 4, 7]

Interaction with the surface. The wavefront of the spherical waves given by equation (18) intersects the surface $x_3=z=0$ along a circular line defined by $\overline{\mathbf{R}}=(x_1,x_2,-z_0)$, $\overline{R}=(r^2+z_0^2)^{1/2}$, where $r=(x_1^2+x_2^2)^{1/2}$ is the distance from the origin (placed on the surface, the epicentre) to the intersection points (we recall that $\mathbf{R}$ and $\overline{\mathbf{R}}$ are in fact $\mathbf{R}-\mathbf{R}_0$ and $\overline{\mathbf{R}}-\mathbf{R}_0$). The

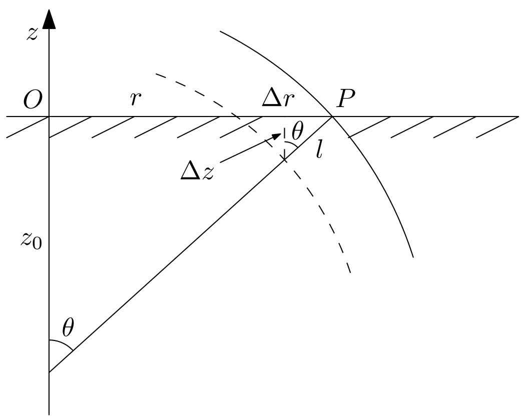

Figure 1: Spherical wave intersecting the surface $z=0$ at P.

radius $\overline{R}$ moves with velocity c, $\overline{R}=ct$, $t>\mid z_0\mid/c$, and the in-plane radius r moves according to the law $r=\sqrt{\overline{R}^2-z_0^2}=\sqrt{c^2t^2-z_0^2}$, where c stands for the velocities $c_{l,t}$; its velocity $v=dr/dt=c\overline{R}/r=c^2t/r$ is infinite for $r=0$ ($\overline{R}=ct=\mid z_0\mid$) and tends to c for large distances.

The finite duration T of the source makes the δ'-functions in equation (18) to be viewed as functions with a finite spread $l=\Delta R=cT\ll R$; consequently, the intersection line of the waves with the surface has a finite spread Δr, which can be calculated from

$$\overline{R}^2=r^2+z_0^2\ ,\ (\overline{R}+l)^2=(r+\Delta r)^2+z_0^2\ ; \tag{19}$$

hence,

$$\Delta r\simeq\frac{2\overline{R}l}{r+\sqrt{r^2+2\overline{R}l}}\ ; \tag{20}$$

we can see that for $r\rightarrow 0$ the width $\Delta r\simeq\sqrt{2\mid z_0\mid l}$ of the seismic spot on the surface is much larger than the width of the spot for large distances $\Delta r\simeq l$ ($2\mid z_0\mid\gg l$). We limit ourserves to an intermediate, limited range of variation of r arund a value of r of the order $\mid z_0\mid$ (not very close to the origin). A spherical wave intersecting the surface $z=0$ is shown in Fig.1.

The energy density of the spherical waves goes like $1/R^2$. As long as the spherical wave is fully included in the half-space its total energy E_0 is given by the energy density integrated over the spherical shell of radius R and thickness l. If the wave intersects the surface of the half-space, its energy E is given by the energy density integrated over the spherical sector which subtends the solid angle $2\pi(1+\cos\theta)$, where $\cos\theta=\mid z_0\mid/\overline{R}$ (see Fig.1). It follows $E=\frac{1}{2}E_0\,(1+\mid z_0\mid/ct)$ for $ct>\mid z_0\mid$. We can see that the energy of the wave decreases by the amount $E_s=\frac{1}{2}E_0\,(1-\mid z_0\mid/ct)$, $ct>\mid z_0\mid$. This amount of energy is transferred to the surface, which generates secondary waves (according to Huygens principle).

In the seismic spot with the width Δr and radius r generated on the surface by the far-field primary waves given by equation (20) we may expect a reaction of the (free) surface, such as to compensate the force exerted by the incoming spherical waves. This localized reaction force generates secondary waves, distinct from the incoming, primary spherical waves. The secondary waves can be viewed as waves scattered off the surface, from the small region of contact of the surface seismic spot (a circular line). If the reaction force is strictly limited to the zero-thickness surface (as, for instance, a surface force), it would not give rise to waves, since its source has a zero integration measure. We assume that this reaction appears in a surface layer with thickness Δz ($\Delta z\ll\mid z_0\mid$), where it is produced by volume forces. The thickness Δz of the superficial layer activated by the incoming primary wave may depend on $\overline{R}$ (and r), as the surface spread Δr does (equation (20)); for instance, from Fig.1 we have $\Delta z = l \mid z_0 \mid /\overline{R}$.

The volume elastic force per unit mass is given by $\partial_j \sigma_{ij}/\rho$, where $\sigma_{ij} = \rho \left[2c_t^2 u_{ij} + (c_l^2 - 2c_t^2) u_{kk}\delta_{ij}\right]$ is the stress tensor and u_{ij} is the strain tensor. The reaction force which compensates this elastic force is

$$f_i = -\partial_j \sigma_{ij}/\rho = -\partial_j \left[2c_t^2 u_{ij} + (c_l^2 - 2c_t^2) u_{kk}\delta_{ij}\right] \ . \tag{21}$$

We calculate the strain tensor from the displacement given by equation (18); in order to compute the secondary waves we use the decomposition in Helmholtz potentials. We denote by $\mathbf{u}_s$ the displacement vector in the secondary waves, and introduce the Helmholtz potentials ψ and $\mathbf{B}$ ($div\mathbf{B} = 0$) by $\mathbf{u}_s = grad\psi + curl\mathbf{B}$; then, we decompose the force $\mathbf{f}$ according to $\mathbf{f} = grad\chi + curl\mathbf{h}$ ($div\mathbf{h} = 0$), where $\Delta\chi = div\mathbf{f}$ and $\Delta\mathbf{h} = -curl\mathbf{f}$; by the equation of the elastic waves, the Helmholtz potentials satisfy the wave equations (4); by straightforward calculations we get $\chi = -c_l^2 u_{ii}$ and $\mathbf{h} = c_t^2 curl\mathbf{u}$, where $\mathbf{u}$ is $\mathbf{u}^f$ given by equation (18):

$$\begin{aligned} \chi &= -\tfrac{c_l T m_{jk} x_j x_k}{4\pi R^3}\delta''(R - c_l t) \ , \\ h_i &= \varepsilon_{ijk}\tfrac{c_t T m_{kl} x_j x_l}{4\pi R^3}\delta''(R - c_t t) \ ; \end{aligned} \tag{22}$$

we can see that the potentials χ and $\mathbf{h}$ "move" with velocities c_l and, respectively, c_t (v_l and, respectively, v_t in the plane $z = 0$).

We can calculate the displacement in the secondary waves $\mathbf{u}_s = grad\psi + curl\mathbf{B}$, by solving the wave equations (equations (4))

$$\ddot{\psi} - c_l^2 \Delta\psi = \chi \ , \ \ddot{\mathbf{B}} - c_t^2 \Delta\mathbf{B} = \mathbf{h} \ ; \tag{23}$$

with $\chi = -c_l^2 u_{ii}$ and $\mathbf{h} = c_t^2 curl\mathbf{u}$ restricted to the superficial layer of thickness Δz. Apart from appreciable technical complications, this procedure brings many superfluous features which obscure the relevant physical picture. This is why we prefer to use a simplified model which makes use of potentials of the form

$$\chi = \chi_0(r)\delta(z)\delta(r - v_l t) \ , \ \mathbf{h} = \mathbf{h}_0(r)\delta(z)\delta(r - v_t t) \tag{24}$$

($div\mathbf{h}_0 = 0$); equations (24) describe wave sources, distributed uniformly along circular lines on the surface, propagating on the surface with constant velocities $v_{l,t}$ and limited to a superficial layer with "zero" thickness and a circular line (with "zero" width); their magnitudes $\chi_0(r)$ and $\mathbf{h}_0(r)$ have an approximate $1/r$, $1/r^2$-dependence, which has a slow variation for $r \sim\mid z_0 \mid$ (and r not very close to the epicenter); for this range of the variable r we may consider χ_0 and $\mathbf{h}_0$ as being constant. The velocities $v_{l,t}$ in equation (24) correspond to the velocities $v_{l,t} = dr/dt = c_{l,t}\overline{R}/r = c_{l,t}^2 t/r$ calculated above, which are greater than $c_{l,t}$, depend on r and tends to $c_{l,t}$ for large values of the distance r. We make a further simplification and consider them as constant velocities slightly greater than $c_{l,t}$ (over an intermediate, limited range of variation of r). Also, in the subsequent calculations we consider the origin of the time at $r = 0$ (the epicentre) for each primary wave and the associated secondary source. The simplified model of secondary sources introduced here retains the main features of the exact problem, incorporated in the surface localization and propagation of the sources with velocities $v_{l,t}$ greater than wave velocities $c_{l,t}$; on the other hand, by using this model we lose the anisotropy induced by the tensor of the seismic moment and specific details regarding the dependence on the distance. Since the secondary seismic sources are sources moving on the surface we may call the secondary waves produced by these sources "surface seismic radiation".

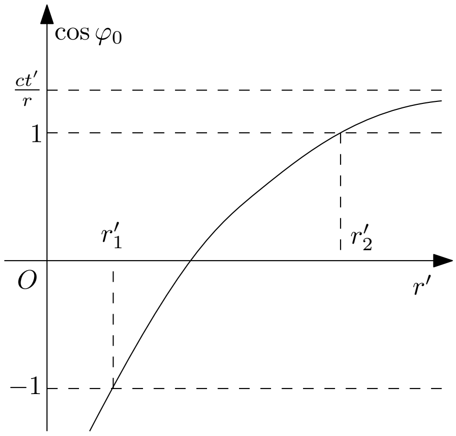

Figure 2: The function $\cos\varphi_0$ *vs* r' for $C > 0$ (equation (32)).

Secondary waves. Making use of the potentials given by equation (24), the solutions of equations (23) can be represented as

$$\begin{aligned} \psi &= \frac{1}{4\pi c_l^2}\int dt'\int d\mathbf{R}'\frac{\chi_0(r')\delta(z')\delta(r'-v_lt')}{\left|\mathbf{R}-\mathbf{R}'\right|}\delta\left(t-t'-\left|\mathbf{R}-\mathbf{R}'\right|/c_l\right) \ , \\ \mathbf{B} &= \frac{1}{4\pi c_t^2}\int dt'\int d\mathbf{R}'\frac{\mathbf{h}_0(r')\delta(z')\delta(r'-v_tt')}{\left|\mathbf{R}-\mathbf{R}'\right|}\delta\left(t-t'-\left|\mathbf{R}-\mathbf{R}'\right|/c_t\right) \ . \end{aligned} \tag{25}$$

First, we focus on the potential ψ, which can be written as

$$\begin{aligned} \psi = \frac{1}{4\pi vc^2}\int d\mathbf{r}'\frac{\chi_0(r')}{\left(r^2+r'^2-2rr'\cos\varphi+z^2\right)^{1/2}}\cdot \\ \cdot\delta\left[t-r'/v-\left(r^2+r'^2-2rr'\cos\varphi+z^2\right)^{1/2}/c\right] \ , \end{aligned} \tag{26}$$

where φ is the angle between the vectors $\mathbf{r}$ and $\mathbf{r}'$ and we use c and v for c_l and, respectively, v_l, for the sake of simplicity. In order to calculate the integral with respect to the angle φ in equation (26) we introduce the function

$$F(\cos\varphi) = t - r'/v - \left(r^2+r'^2-2rr'\cos\varphi+z^2\right)^{1/2}/c \tag{27}$$

and look for its zeroes, $F_0 = F(\cos\varphi_0) = 0$ ($r' < vt$); we note that, if there exists one root of this equation, there exists another one at least, in view of the symmetry $\cos\varphi = \cos(2\pi-\varphi)$. Then, we expand the function F in a Taylor series in the vicinity of its zero, according to

$$F = F_0 + (\cos\varphi - \cos\varphi_0)F_0' + ... = (\cos\varphi-\cos\varphi_0)F_0' + ... \ , \tag{28}$$

where F_0' is the derivative of the function F with respect to $\cos\varphi$ for $\cos\varphi = \cos\varphi_0$. It is easy to see that the integral reduces to

$$\psi = \frac{1}{2\pi cvr}\int_0^\infty dr'\frac{\chi_0(r')}{\sin\varphi_0} \ , \tag{29}$$

where φ_0 is the root of the equation $F(\cos\varphi_0) = 0$, lying between 0 and π.

The root $\cos\varphi_0$ is given by

$$F(\cos\varphi_0) = t - r'/v - \left(r^2+r'^2-2rr'\cos\varphi_0+z^2\right)^{1/2}/c = 0 \ , \tag{30}$$

or

$$\left(1-c^2/v^2\right)r'^2-2\left(r\cos\varphi_0-c^2t/v\right)r'-\left(c^2t^2-r^2-z^2\right)=0 \tag{31}$$

for $r^{'}<vt$. The important feature brought by the diference between the two velocities c and v can be accounted for conveniently by assuming that the two velocities are close to one another; we set $v=c(1+\varepsilon)$, $0<\varepsilon\ll 1$ (as for sufficiently large distances). In this circumstance we may neglect the quadratic term $\sim r'^2$ in equation (31) and replace t by the "retarded" time $\tau=t(1-\varepsilon)$ (*i.e.*, $\tau_{l,t}=t(1-\varepsilon_{l,t})$); we get

$$\cos\varphi_0\simeq\frac{2c\tau r^{'}-C}{2rr^{'}}\ ,\ C=c^2\tau^2-r^2-z^2 \tag{32}$$

for $r^{'}<vt=c\tau(1+2\varepsilon)$; the insertion of τ in place of t in C is a matter of technical convenience. It is easy to see that this equation has no solution for $C<0$ (because of the condition $r^{'}<vt$); for $C>0$ ($c^2\tau^2-r^2-z^2>0$) it has two solutions

$$r_1^{'}=\frac{C}{2(c\tau+r)}\ ,\ r_2^{'}=\frac{C}{2(c\tau-r)}\ , \tag{33}$$

corresponding to $\cos\varphi_0=-1$ ($\varphi_0=\pi$) and, respectively, $\cos\varphi_0=1$ ($\varphi_0=0$) (Fig.2). For $z=0$ the two roots $r_{1,2}^{'}$ reduce to $r_{1,2}^{'}=(c\tau\mp r)/2$; we can see that the sources of the secondary waves which arrive at r lie inside an anullus with radii $r_{1,2}^{'}$ and a constant width r, which expands on the surface with velocity $c/2$, after a time interval $\tau=r/c$. In the integral given by equation (29) we pass from the variable $r^{'}$ to the variable φ_0; for a limited range of integration r (from $r_1^{'}$ to $r_2^{'}$), we may take χ_0 out of the integral sign; we get

$$\psi\simeq\frac{C\chi_0}{4\pi c^2}\int_0^{\pi}d\varphi_0\frac{1}{(r\cos\varphi_0-c\tau)^2}\ , \tag{34}$$

or

$$\psi\simeq\frac{C\chi_0}{4\pi c^2r^2}\frac{\partial}{\partial x}\int_0^{\pi/2}d\varphi_0\left(\frac{1}{\cos\varphi_0-x}-\frac{1}{\cos\varphi_0+x}\right)\ ,\ x=c\tau/r>1. \tag{35}$$

The integrals in equation (35) can be effected immediately; we get the potential

$$\psi\simeq\frac{\chi_0}{4c_l^2}\frac{(c_l^2\tau_l^2-r^2-z^2)c_l\tau_l}{(c_l^2\tau_l^2-r^2)^{3/2}}\ , \tag{36}$$

where the velocity c_l is restored. Similarly, we get from equations (25) the vector potential

$$\mathbf{B}\simeq\frac{\mathbf{h}_0}{4c_t^2}\frac{(c_t^2\tau_t^2-r^2-z^2)c_t\tau_t}{(c_t^2\tau_t^2-r^2)^{3/2}}\ ; \tag{37}$$

these potentials are valid for $c_{l,t}^2\tau_{l,t}^2-r^2-z^2>0$. We can see that the wavefront $r^2+z^2=c_{l,t}^2\tau_{l,t}^2$ defines a spherical perturbation which moves with velocity $c_{l,t}$. The singular behaviour of these waves (for $z=0$) resembles the algebraic singularity of the waves in two dimensions produced by localized sources.[12, 13] The discontinuities exhibited by these functions are present irrespective of the particular dependence on r of the source potentials, as long as these potentials remain localized; they are related to a constant, finite velocity of propagation of the waves.

Making use of $\mathbf{u}_s=grad\psi+curl\mathbf{B}$ we can compute the displacement vector $\mathbf{u}_s$ in the secondary waves. We are interested mainly in the waves propagating on the surface ($z=0$). First, we note that the displacement is singular at $c_{l,t}\tau_{l,t}=r$; this indicates the existence of two main

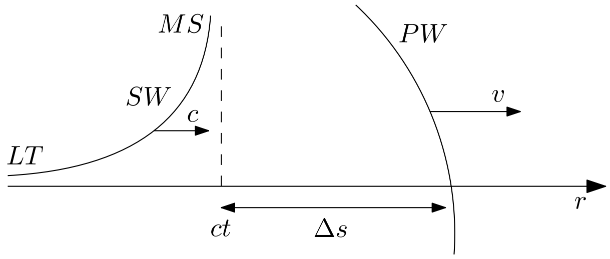

Figure 3: Primary wave (PW), moving with velocity v on the Earth's surface, secondary wave (SW), moving with velocity $c < v$, the main shock (MS) and the long tail (LT); the separation between the two wavefronts is $\Delta s = 2(v - c)t$ and the time delay is $\Delta t = (2r/c)(v/c - 1)$, where r is the distance on the surface from the epicentre.

shocks, occcurring after the arrival of the primary waves. Indeed, the primary waves arrive at the observation point $\mathbf{r}$ at the time $t_p = r/v_{l,t} = (r/c_{l,t})(1-\varepsilon_{l,t})$, while the main shocks occur at $t_m = \tau_{l,t}/(1-\varepsilon_{l,t}) \simeq (r/c_{l,t})(1+\varepsilon_{l,t})$; we can see that there exists a time delay $\Delta t \simeq t_m - t_p \simeq 2(r/c_{l,t})\varepsilon_{l,t}$ between the primary waves and the wavefronts of the secondary waves (the main shocks). The sharp singularity in equations (36) and (37) is related to our using constant velocities $v_{l,t}$; actually, an uncertainty of the form $\Delta v \simeq c\varepsilon$ exists in these velocities, which entails an uncertainty $\tau\varepsilon$ in the time τ, such that the smallest value of the denominator in equations (36) and (37) is of the order $c^2\tau^2\varepsilon$. In the vicinity of the two main shocks the leading contributions to the components of the surface displacement ($z = 0$, in polar cylindrical coordinates) are given by

$$\begin{gathered} u_{sr} \simeq \frac{\chi_0 \tau_l}{4c_l} \cdot \frac{r}{\left(c_l^2\tau_l^2 - r^2\right)^{3/2}} \ , \\ u_{s\varphi} \simeq -\frac{h_{0z}\tau_t}{4c_t} \cdot \frac{r}{\left(c_t^2\tau_t^2 - r^2\right)^{3/2}} \ , \\ u_{sz} \simeq \frac{h_{0\varphi}\tau_t}{4c_t} \cdot \frac{c_t^2\tau_t^2}{r\left(c_t^2\tau_t^2 - r^2\right)^{3/2}} \ ; \end{gathered} \tag{38}$$

we can see that there exists a horizontal component of the displacement perpendicular to the propagation direction ($u_{s\varphi}$) and both the r-component and the φ, z-components, which make right angles with the propagation direction, are of the same order of magnitude.[3] For long times ($c_{l,t}\tau_{l,t} \gg r$) the displacement (from equations (36) and (37)) goes like

$$u_{sr} \simeq \frac{\chi_0 r}{4c_l^4\tau_l^2} \ , \ u_{s\varphi} \simeq -\frac{h_{0z}r}{4c_t^4\tau_t^2} \ , \ u_{sz} \simeq \frac{h_{0\varphi}}{4c_t^2 r} \ , \tag{39}$$

which show that the displacement exhibits a long tail, especially the z-component; it subsides as a consequence of the time-dependence induced in the potential $\mathbf{h}_0$ by the integration variable r', a circumstance which is neglected in the calculations presented here. Primary and secondary waves, the main shock and the long tail are shown in Fig.3. These are the main characteristics of a seismogram.[3, 14, 15]

Internal discontinuity. Let us consider a homogeneous isotropic elastic half-space extending in the region $-\infty < z < z_1$ with a superposed homogeneous isotropic elastic layer extending from $z = z_1$ to $z = 0$, in welded contact with the half-space at the plane surface $z = z_1$; we assume $z_1 < 0$. The elastic properties of the half-space and the layer are distinct. This model can serve as a representation of an internal discontinuity in the elastic properties of the half-space investigated above. An elementary seismic source as given by equation (1) is located at depth z_0, either above ($z_0 > z_1$) or beneath the discontinuity ($z_0 < z_1$). In the subsequent calculations we assume $z_0 < z_1$. We denote the half-space by 1 and the superposed layer by 2. A primary spherical wave generated by the elementary z_0-source arrives at the z_1-interface, along a circular line of contact, where it generates secondary waves; the secondary waves propagate both in the half space 1 and in the layer 2, where they arrive at the surface $z = 0$; we estimate here these secondary waves generated by the z_1-interface.

By analogy with equations (24) we assume that the primary waves in the half-space 1 generate on the interface $z = z_1$ the force Helmholtz potentials

$$\chi = \chi_0(r)\delta(z - z_1)\delta(r - v_{l1}t) \ , \ \mathbf{h} = \mathbf{h}_0(r)\delta(z - z_1)\delta(r - v_{t1}t) \ ; \tag{40}$$

the velocities $v_{l,t1}$ are considered constant and the r-dependence in $\chi_0(r)$, $\mathbf{h}_0(r)$ is weak for a finite, intermediate range of distances r. It is easy to see, by analogy with the calculations described in the previous section, that the Helmholtz potentials ψ and $\mathbf{B}$ of the secondary displacement are given by

$$\psi \simeq \frac{\chi_0}{4c_{l2}^2}\frac{\left[c_{l2}^2\tau_l^2 - r^2 - (z - z_1)^2\right] c_{l2}\tau_l}{\left(c_{l2}^2\tau_l^2 - r^2\right)^{3/2}} \ . \tag{41}$$

(for z close to $z = 0$). Similarly, we get from the wave equation the vector potential

$$\mathbf{B} \simeq \frac{\mathbf{h}_0}{4c_{t2}^2}\frac{\left[c_{t2}^2\tau_t^2 - r^2 - (z - z_1)^2\right] c_{t2}\tau_t}{\left(c_{t2}^2\tau_t^2 - r^2\right)^{3/2}} \ ; \tag{42}$$

they differ from the potentials given above by equations (36) and (37) by the presence of $z_1 \neq 0$. The origin of the time is considered here the moment when the primary wave touches the z_1-interface. The above formulae are valid for $C_{l,t} = c_{l,t2}^2\tau_{l,t}^2 - r^2 - (z - z_1)^2 > 0$; for $C_{l,t} < 0$ the potentials are equal to zero.

We are interested mainly in the surface $z = 0$. The presence of $z_1 \neq 0$ in equations (41) and (42) gives rise to a qualitatively different behaviour of the secondary waves generated by the discontinuity. The difference arises fom the condition $C_{l,t} = c_{l,t2}^2\tau_{l,t}^2 - r^2 - z_1^2 > 0$, which prevents the singularity at $c_{l,t2}^2\tau_{l,t}^2 - r^2 = 0$ to be reached; consequently, the secondary waves in the presence of the discontinuity do not exhibit the singular main shock on the surface $z = 0$; the main shock is reduced appreciably in this case.

We note also the retarded time $\tau_{l,t} = t_{l,t}(1 - \varepsilon_{l,t})$ in the above formulae (where $\tau_{l,t}$ is measured from the moment the primary waves touches the z_1-interface). For $z_0 < z_1$ the primary waves do not arrive on the surface and the $\mathbf{u}_{s2}$-waves generated by the interface are the only secondary waves which arrive on the surface $z = 0$. For $z_1 < z_0 < 0$, primary waves arrive on the surface $z = 0$, (delayed) secondary waves are generated on the surface $z = 0$ and, afterwards, much reduced secondary waves generated by the interface arrive on the surface $z = 0$.

We emphasize also that the results given above are valid for small values of $\varepsilon_{l,t}$, *i.e.* for the elastic properties of the layer 2 differing slightly from the elastic properties of the half space 1. In addition, the secondary waves $\mathbf{u}_{s2}$ generate in their turn additional waves an the surface $z = 0$, which, however, are too small to present any further interest here (they may be called "tertiary" waves).

Concluding remarks. Primary elastic waves generated in a homogeneous isotropic elastic half-space by tensorial forces localized in space and time are derived here, as a model for the seismic waves produced by a seismic moment placed at an inner point in the half-space with a δ-type temporal dependence (temporal impulse). The primary waves have the shape of spherical shells. It is shown, mainly by using energy-balance arguments, that the primary waves interact with the surface of the half-space and transfer part of their energy to the surface; consequently, additional, secondary wave sources occurr on the surface, which generate secondary waves. Since the secondary sources move on the surface, the secondary waves they generate may be called "surface seismic radiation". A similar suggestion was implied long ago by Lamb.[2, 12] The secondary wave sources are localized on the surface along circular lines. It is worth noting that the secondary sources move on the surface with velocities greater than the elastic waves velocities. A simplified model is put forward here of secondary waves sources, which allows the estimation of the secondary waves produced by these sources. The model assumes a uniform distribution of sources along circular lines, moving with constant velocities greater than the velocities of the elastic waves; it does not account for the anisotropy of the sources, and gives only a qualitative dependence of the waves on the distance. The secondary waves generated by the surface sources are estimated within this model, with emphasis on the secondary waves propagating on the surface. It is shown that these secondary waves are responsible for the seismic main shock and the long tail exhibited usually by earthquakes in the seismic records. These two latter items have indeed been associated long ago to waves generated and propagating on the surface.[12], [14]-[16] The secondary waves generated by an internal discontinuity of the half-space are also estimated; it is shown that they exhibit a much reduced main shock.

Finally, a special situation deserves attention. If the source of the primary waves is located on the surface, the primary waves it generates are those given above in the corresponding section for $z_0 = 0$ (for an elementary source). The interaction of these primary waves with the surface is null, since the thickness $\Delta z = l \mid z_0 \mid /\overline{R}$ of the intersecting layer is zero for $z_0 = 0$ (see Fig.1). The support of the interaction force with the surface reduces to zero and, consequently, there will be no secondary waves in this case.

Acknowledgments. The author is indebted to his colleagues in the Department of Engineering Seismology, Institute of Earth's Physics, Magurele-Bucharest, for many enlightening discussions, and to the members of the Laboratory of Theoretical Physics at Magurele-Bucharest for many useful discussions and a throughout checking of this work. This work was partially supported by the Romanian Government Research Grant #PN16-35-01-07/11.03.2016.

References

[1] Lord Rayleigh, "On waves propagated along the plane surface of an elastic solid", Proc. London Math. Soc. **17** 4-11 (1885) (J. W. Strutt, Baron Rayleigh, *Scientific Papers*, vol. 2, 441-447, Cambridge University Press, London (1900)).

[2] H. Lamb, "On the propagation of tremors over the surface of an elastic solid", Phil. Trans. Roy. Soc. (London) **A203** 1-42 (1904).

[3] A. E. H. Love, *Some Problems of Geodynamics*, Cambridge University Press, London (1926).

[4] K. Aki and P. G. Richards, *Quantitative Seismology*, University Science Books, Sausalito, CA (2009).

[5] B. Gutenberg and C. F. Richter, "Earthquake magnitude, intensity, energy and acceleration", Bull. Seism. Soc. Am. **46** 105-145 (1956).

[6] H. Kanamori, "The energy released in great earthquakes", J. Geophys. Res. **82** 2981-2987 (1977).

[7] A. Ben-Menahem and J. D. Singh, *Seismic Waves and Sources*, Springer, NY (1981).

[8] A. Udias, *Principles of Seismology*, Cambridge University Press, NY (1999).

[9] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 7, *Theory of Elasticity*, Elsevier, Oxford (1986).

[10] G. G. Stokes, "On the dynamical theory of diffraction", Trans. Phil. Soc. Cambridge **9** 1-62 (1849) (reprinted in *Math. Phys. Papers*, vol. 2, Cambridge University Press, Cambridge (1883), pp. 243-328).

[11] M. Bath, *Mathematial Aspects of Seismology*, Elsevier, Amsterdam (1968).

[12] H. Lamb, "On wave-propagation in two dimensions", Proc. Math. Soc. London **35** 141-161 (1902).

[13] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, NY (1953).

[14] R. D. Oldham, "On the propagation of earthquake motion to long distances", Trans. Phil. Roy. Soc. London **A194** 135-174 (1900).

[15] C. G. Knott, *The Physics of Earthquake Phenomena*, Clarendon Press, Oxford (1908).

[16] H. Jeffreys, "On the cause of oscillatory movement in seismograms", Monthly Notices of the Royal Astron. Soc., Geophys. Suppl. **2** 407-415 (1931).