Journal of Theoretical Physics

Founded and Edited by M. Apostol **254** (2016)

ISSN 1453-4428

Singular solutions of the wave equation: a model description of the tsunami phenomenon

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Abstract

Compressional waves are considered in an ideal fluid, as produced by sources localized at an inner point in the fluid *(e.g.*, explosions) or just below the seabed (seafloor) of seas or oceans (earthquakes). It is shown that these primary waves, which have the shape of spherical shells, generate secondary-wave sources on the surface of the fluid, which, in turn, produce secondary waves in the fluid. We compute the secondary waves in a simplified model and show that they are singular waves, in the sense that, on the surface, these waves exhibit a sharp, singular wavefront and a long tail; their wavefront moves with the wave velocity which is smaller than the velocity of the secondary sources (primary waves) moving on the surface. The singular wavefront of the secondary waves on the surface of the fluid indicates a singular increase in height of the fluid at the position of the wavefront, which may be viewed as a model representation for the tsunami phenomenon.

Introduction. Let $\mathbf{u}(\mathbf{R},t)$ be a displacement field in a classical ideal fluid at rest and in equilibrium, where $\mathbf{R}$ denotes the position and t denotes the time; it produces a change $\delta n=-ndiv\mathbf{u}$ in the concentration of fluid particles n and a change $\delta p=(\partial p/\partial n)\delta n$ in the pressure p. Since the fluid is ideal (zero thermoconductivty), this change is adiabatic, *i.e.* it occurs at constant entropy s; we write $\delta p=-n(\partial p/\partial n)_s div\mathbf{u}$, or $\delta p=V(\partial p/\partial V)_s div\mathbf{u}=-(1/\beta)div\mathbf{u}$, where $V=1/n$ is the volume associated with the fluid particle and β is the adiabatic compressibility. It follows that a force density $-grad\delta p=(1/\beta)grad\cdot div\mathbf{u}$ appears in the fluid. We assume that the displacement $\mathbf{u}$ is a slowly varying function of position and time, such that $d\mathbf{u}/dt=\partial\mathbf{u}/\partial t+(\mathbf{v}grad)\mathbf{u}\simeq\partial\mathbf{u}/\partial t$, where $\mathbf{v}$ is the (transport) velocity; then, the equation of motion of the displacement $\mathbf{u}$ reads $\rho\ddot{\mathbf{u}}-(1/\beta)grad\cdot div\mathbf{u}=0$, or

$$\ddot{\mathbf{u}}-c^2grad\cdot div\mathbf{u}=0\ \ , \tag{1}$$

where ρ is the mass density of the fluid and $c=1/\sqrt{\rho\beta}$ is the sound velocity. This is the sound equation; it is a reduced form of the well-known Navier-Stokes equation for ideal fluids, and, at the same time, a reduced form of the Navier-Cauchy equation of the elastic motion of a homogeneous isotropic elastic body with only one Lame coefficients $\lambda=1/\beta=\rho c^2$ (the shear elastic modulus μ is zero).[1]

Let p_0 be a pressure that appears in the fluid; we assume that it lasts a short lapse of time T, such that it may be written as $Tp_0\delta(t)$, where δ is the Dirac function. Also, we assume that it is localized in a small volume v at some point $\mathbf{R}_0$ in the fluid, such that it generates a force density $-Tvp_0\delta(t)grad\delta(\mathbf{R}-\mathbf{R}_0)$. We may consider the fluid a sea or an ocean. The position $\mathbf{R}_0$

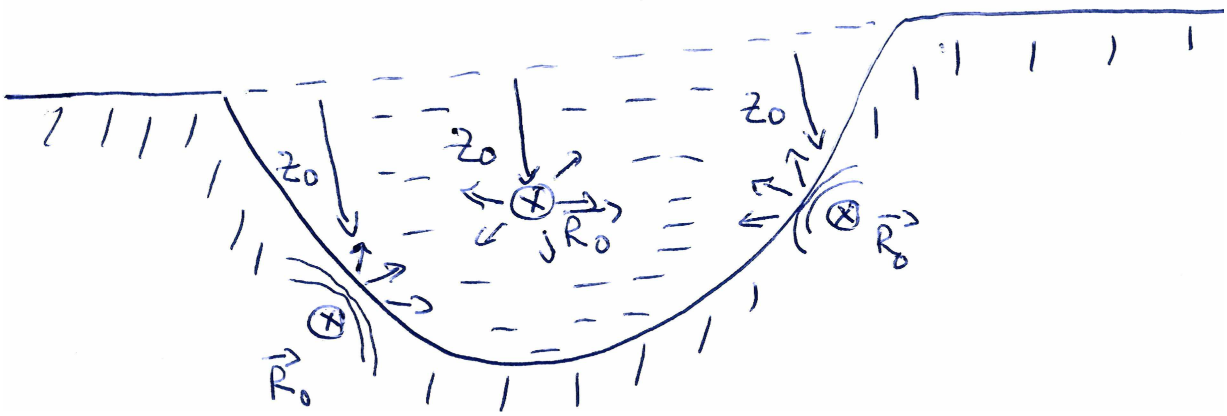

Figure 1: Various sources of compressional waves located at $\mathbf{R}_0$ in a fluid (depth z_0).

may be placed at some depth z_0 in the sea (ocean), the pressure being produced, for instance, by an explosion. Similarly, the position $\mathbf{R}_0$ may be viewed as the location of an earthquake focus placed just beneath the seabed (seafloor), as shown in Fig.1; the seismic waves with the polarization parallel with the seabed surface (tangential waves) do not affect the ideal fluid, while the waves normal to the seabed surface generate the pressure p_0. For convenience we denote vp_0 by $-M$, where M is similar with a seismic moment (a reduced scalar for the tensor of the seismic moment);[2, 3] also, we use the notation $m = M/\rho$. Then, the equation of the motion reads

$$\ddot{\mathbf{u}} - c^2 grad \cdot div\mathbf{u} = Tm\delta(t) grad\delta(\mathbf{R} - \mathbf{R}_0) \ . \tag{2}$$

We assume in addition $p_0 \ll \rho c^2$, an inequality which ensures a slow variation of the displacement $\mathbf{u}$.

Primary waves. We introduce the potential Φ through $\mathbf{u} = grad\Phi$, which satisfies the wave equation

$$\ddot{\Phi} - c^2 \Delta\Phi = Tm\delta(t)\delta(\mathbf{R} - \mathbf{R}_0) \ ; \tag{3}$$

the solution of this equation is the well-known spherical wave

$$\Phi = \frac{Tm}{4\pi c^2} \cdot \frac{\delta\left(t - |\mathbf{R} - \mathbf{R}_0| /c\right)}{|\mathbf{R} - \mathbf{R}_0|} \ , \tag{4}$$

which gives a far-field displacement ($|\mathbf{R} - \mathbf{R}_0| \gg v^{1/3}$)

$$\mathbf{u} = \frac{Tm}{4\pi c} \cdot \frac{\mathbf{R} - \mathbf{R}_0}{|\mathbf{R} - \mathbf{R}_0|^2} \cdot \delta' \left(|\mathbf{R} - \mathbf{R}_0| - ct\right) \ ; \tag{5}$$

we call this spherical-shell wave primary wave.

Interaction with the surface. We consider the plane surface of the fluid placed at $z = 0$ and take the origin on this surface (epicenter), such that $\mathbf{R}_0 = (0, 0, z_0)$, $z_0 < 0$, where z_0 is the depth where the p_0-perturbation occurs. For $ct >| z_0 |$ the primary wave intersects the surface $z = 0$ along a circular line with radius r, such that $r^2 + z_0^2 = |\mathbf{R} - \mathbf{R}_0|^2 = c^2t^2$ (Fig.2); we can see that the radius r "moves" on the surface with velocity $v = dr/dt = c^2t/r = cR/r > c$, which is greater than the wave velocity c; the notation $\mathbf{R}$ stands for $\mathbf{R} - \mathbf{R}_0$. Since we view the functions δ and δ' as functions localized over a small, undetermined distance $l \simeq v^{1/3}$, it follows that the surface spot of the primary wave is localized in an annulus with the width Δr given by

$$(r + \Delta r)^2 + z_0^2 = (ct + l)^2 \ , \ r^2 + z_0^2 = c^2t^2 \ , \tag{6}$$

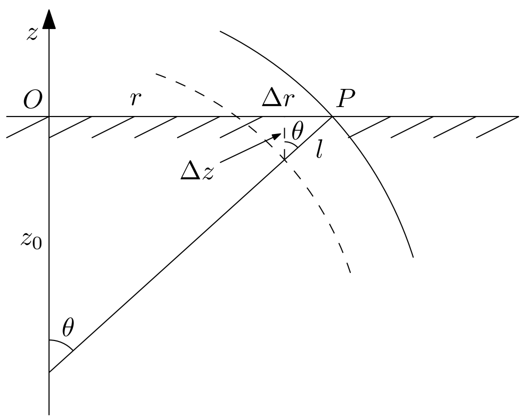

Figure 2: Primary wave (spherical shell) intersecting the surface $z = 0$ at P.

or

$$\Delta r = \frac{2ctl + l^2}{\sqrt{r^2 + 2ctl + l^2} + r} \quad ; \tag{7}$$

for $r \to 0$ this width is $\Delta r \simeq \sqrt{2 \mid z_0 \mid l} \gg l$ ($\mid z_0 \mid \gg l$), while for sufficiently large r the width Δr can be approximated by $\Delta r \simeq ctl/r = Rl/r > l$. For points on the fluid surface not very close to the origin we may use the approximation $\Delta r \simeq Rl/r$ (which has a weak dependence on r).

From equation (1) it is easy to get the energy density $\mathcal{E} = \frac{1}{2}[\dot{u}_i^2 + c^2(\partial_i u_i)^2]$ per unit mass (and the energy flux density $\mathbf{S} = -c^2\mathbf{u}(div\mathbf{u})$); from equation (5) we may use the representation $u \simeq Tm/cl^2R$ for the primary wave at distance R from the source; similarly, we may use the representation $\mathcal{E} \sim T^2m^2/l^6R^2$ for the energy density. As long as the primary wave is included entirely in the fluid (for $ct <\mid z_0 \mid$) its energy E_0 is equal to the energy density $\mathcal{E}$ multiplied by the volume $4\pi c^2t^2l$ of the spherical shell ($R = ct$); for $ct >\mid z_0 \mid$ the energy E of the primary wave reduces to the energy density $\mathcal{E}$ multiplied by the volume $2\pi c^2t^2(1 + \cos\theta)l$, $\cos\theta =\mid z_0 \mid /ct$, where θ is the angle of the spherical sector subtended by the wave in fluid; it follows that $E = \frac{1}{2}E_0(1 + \cos\theta) < E_0$; the missing energy is transferred to the surface spot, which becomes, according to the Huygens principle, an additional wave source; we call this source secondary source; it generates secondary waves in the fluid.

In the seismic spot with width Δr and radius r we may expect a reaction of the (free) surface, such as to compensate the force exerted by the incoming spherical wave. This localized reaction force generates secondary waves, distinct from the incoming, primary spherical wave. The secondary waves can be viewed as waves scattered off the surface, from the small region of contact of the primary wave with the surface. The primary wave given by equation (5) produces a pressure $p = -\rho c^2 div\mathbf{u}$; we expect the surface to react with a pressure $p_s = -p = \rho c^2 div\mathbf{u}$, which generates a force density $\mathbf{f}_s = -grad(p_s/\rho)$ (per unit mass), which derives from a potential $\chi = -p_s/\rho$; it follows that the secondary waves are given by $\mathbf{u}_s = grad\psi$, where the potential ψ satisfies the wave equation $\ddot{\psi} - c^2\Delta\psi = \chi$. The potential χ is given by $\chi = -p_s/\rho = -c^2 div\mathbf{u} = -c^2\Delta\Phi$, such that, using equation (3), we get immediately

$$\chi = -\ddot{\Phi} = -\frac{Tmc}{4\pi R}\delta''(R - ct) \tag{8}$$

in the far-field region (where R stands for $|\mathbf{R} - \mathbf{R}_0|$). This potential is localized on the annulus with radius r and width Δr on the surface ($R = ct = \sqrt{r^2 + z_0^2}$) and "moves" on the surface with the velocity $\mathbf{v}$ derived above. In addition, if the reaction force is strictly limited to a zero-thickness surface region, it would not give rise to waves, since the source has a zero integration measure in this case. We assume that the reaction appears in a surface layer with thickness Δz ($\Delta z \ll \mid z_0 \mid$); consequently, we multiply the potential χ given by equation (8) by $\Delta z \delta(z)$; from Fig.2 we can see that the thickness of the region of intersection of the primary wave with the surface is of the order $\Delta z = l \mid z_0 \mid /R$.

We can calculate the displacement in the secondary waves $\mathbf{u}_s = \mathrm{grad}\psi$, by solving the wave equation $\ddot{\psi} - c^2 \Delta \psi = \chi$ with χ given by equation (8). Apart from appreciable technical complications, this procedure brings superfluous features which obscure the relevant physical picture. This is why we prefer to use a simplified model for χ. First, we replace δ''in equation (8) by δ/l^2; this way, the potential χ becomes free of the undetermined parameter l. Then, we write

$$R - ct = R- \mid z_0 \mid -(ct- \mid z_0 \mid) \ , \tag{9}$$

which can be transformed into

$$\frac{r}{R+ \mid z_0 \mid} \left(r - \frac{R+ \mid z_0 \mid}{R} vt \right) \ , \tag{10}$$

where we measure the time form the moment the primary wave reaches the origin ($R = \mid z_0 \mid$). The function $(R+ \mid z_0 \mid)/R$ has a weak dependence on r, such that we can approximate it by a constant slightly greater than unity; for convenience we take it equal to unity (wich amounts to a redefinition of the velocity v). The potential χ becomes

$$\chi = \chi_0(r) \delta(r - vt) \delta(z) \ , \tag{11}$$

where

$$\chi_0(r) \simeq -\frac{m \mid z_0 \mid (R+ \mid z_0 \mid)}{4\pi R^2 r} \ . \tag{12}$$

For a limited range of variation of r around a value of the order $\mid z_0 \mid$(not very close to the origin) we may consider χ_0 constant. The potential given by equation (11) corresponds to wave sources distributed uniformly along circular lines on the surface, propagating on the surface with velocity v and limited to a superficial layer with thickness Δz. The velocity v in equation (11) corresponds to $v = dr/dt = cR/r$ calculated above, which is greater than c (noteworthy, the factor $(R+ \mid z_0 \mid)/R$ in front of v in equation (10) is also slightly greater than unity). We make a further simplification and consider v a constant velocity. The simplified model of secondary sources introduced here retains the main features of the exact problem, incorporated in the surface localization and propagation of the sources with velocity v greater than the wave velocity c; on the other hand, by using this model we lose specific details regarding the dependence on the distance. Since the secondary sources are sources moving on the surface, we may call the secondary waves produced by these sources "surface radiation".

Secondary waves. Making use of the potential given by equation (11), the solutions of the wave equation is

$$\psi = \frac{1}{4\pi c^2} \int d\mathbf{r}_1 \frac{\chi_0(r_1) \delta[r_1 - v(t - |\mathbf{R} - \mathbf{R}_1|/c)]}{|\mathbf{R} - \mathbf{R}_1|} \ , \tag{13}$$

where $\mathbf{R}_1 = (\mathbf{r}_1, 0)$; this equation can also be written as

$$\psi = \frac{1}{4\pi v c^2} \int d\mathbf{r}_1 \frac{\chi_0(r_1) \delta \left[t - r_1/v - (r^2 + r_1^2 - 2rr_1 \cos\varphi + z^2)^{1/2}/c \right]}{(r^2 + r_1^2 - 2rr_1 \cos\varphi + z^2)^{1/2}} \ , \tag{14}$$

where φ is the angle between the vectors $\mathbf{r}$ and $\mathbf{r}_1$. In order to calculate the integral with respect to the angle φ in equation (14) we introduce the function

$$F(\cos\varphi) = t - r_1/v - \left(r^2 + r_1^2 - 2rr_1 \cos\varphi + z^2 \right)^{1/2}/c \tag{15}$$

and look for its zeroes, $F_0 = F(\cos\varphi_0) = 0$ $(r_1 < vt)$; we note that, if there exists one root of this equation, there exists another one at least, in view of the symmetry $\cos\varphi = \cos(2\pi - \varphi)$. Then, we expand the function F in a Taylor series in the vicinity of its zero, according to

$$F = F_0 + (\cos\varphi - \cos\varphi_0) F_1 + ... = (\cos\varphi - \cos\varphi_0) F_1 + ..., \tag{16}$$

where F_1 is the derivative of the function F with respect to $\cos\varphi$ for $\cos\varphi = \cos\varphi_0$. It is easy to see that the integral reduces to

$$\psi = \frac{1}{2\pi cvr} \int_0^\infty dr_1 \frac{\chi_0(r_1)}{\sin\varphi_0} \ , \tag{17}$$

where φ_0 is the root of the equation $F(\cos\varphi_0) = 0$, lying between 0 and π.

The root $\cos\varphi_0$ is given by

$$F(\cos\varphi_0) = t - r_1/v - \left(r^2 + r_1^2 - 2rr_1\cos\varphi_0 + z^2\right)^{1/2}/c = 0 \tag{18}$$

or

$$\left(1 - c^2/v^2\right) r_1^2 - 2\left(r\cos\varphi_0 - c^2t/v\right) r_1 - \left(c^2t^2 - r^2 - z^2\right) = 0 \tag{19}$$

for $r_1 < vt$. The important feature brought about by the diference between the two velocities c and v can be accounted for conveniently by assuming that the two velocities are close to one another; we set $v = c(1+\varepsilon)$, $0 < \varepsilon \ll 1$ (as for sufficiently large distances). In this circumstance we may neglect the quadratic term $\sim r_1^2$ in equation (19) and replace t by the "retarded" time $\tau = t(1-\varepsilon)$; we get

$$\cos\varphi_0 \simeq \frac{2c\tau r_1 - C}{2rr_1} \ , \ C = c^2t^2 - r^2 - z^2 \tag{20}$$

for $r_1 < vt = ct(1+\varepsilon)$. It is easy to see that this equation has no solution for $C < 0$ (because of the condition $r_! < vt$); for $C > 0$ $(c^2t^2 - r^2 - z^2 > 0)$ it has two solutions

$$r_1^{(1)} = \frac{C}{2(c\tau + r)} \ , \ r_1^{(2)} = \frac{C}{2(c\tau - r)} \tag{21}$$

corresponding to $\cos\varphi_0 = -1$ $(\varphi_0 = \pi)$ and, respectively, $\cos\varphi_0 = 1$ $(\varphi_0 = 0)$. For $z = 0$ the two roots $r_1^{(1,2)}$ reduce, approximately, to $r_1^{(1,2)} \simeq (ct \mp r)/2$; we can see that the sources of the secondary waves which arrive at r lie inside an anullus with radii $r_1^{(1,2)}$ and a constant width r, which expands on the surface with velocity $c/2$, after a time interval $t \simeq r/c$.

In the integral given by equation (17) we pass from the variable r_1 to the variable φ_0; for a limited range of integration r (from $r_1^{(1)}$ to $r_1^{(2)}$), we may take χ_0 out of the integral sign; we get

$$\psi \simeq \frac{C\chi_0}{4\pi c^2} \int_0^\pi d\varphi_0 \frac{1}{(r\cos\varphi_0 - c\tau)^2} \ , \tag{22}$$

or

$$\psi \simeq \frac{C\chi_0}{4\pi c^2 r^2} \frac{\partial}{\partial x} \int_0^{\pi/2} d\varphi_0 \left(\frac{1}{\cos\varphi_0 - x} - \frac{1}{\cos\varphi_0 + x} \right) \ , \ x = c\tau/r > 1 \ . \tag{23}$$

The integrals in equation (23) can be effected immediately; we get the potential

$$\psi \simeq \frac{\chi_0}{4c^2} \frac{(c^2t^2 - r^2 - z^2)c\tau}{(c^2\tau^2 - r^2)^{3/2}} \ , \tag{24}$$

valid for $C = c^2t^2 - r^2 - z^2 > 0$ (and $c\tau > r$); otherwise, the potential ψ is zero.

We can see that the wavefront $r^2 + z^2 = c^2t^2$ defines a spherical perturbations which moves with velocity c. The singular behaviour of these waves for $z = 0$ resembles the algebraic singularity of the waves in two dimensions produced by localized sources.[4] The discontinuities exhibited by these functions are present even if we allow for a slight dependence on r of the source potential, as long as the potential remains localized; they are related to a constant, finite velocity of propagation of the waves.

Making use of $\mathbf{u}_s = \mathrm{grad}\psi$ we can compute the displacement vector $\mathbf{u}_s$ in the secondary waves. We are interested mainly in the waves propagating on the surface ($z = 0$). First, we note that the displacement is singular at $c\tau = r$; this indicates the existence of a shock occcurring after the arrival of the primary wave. Indeed, the primary wave arrives at the observation point $\mathbf{r}$ at the time $t_p = r/v = (r/c)(1-\varepsilon)$, while the shock occurs at $t_s = r/c$; we can see that there exists a time delay $\Delta t = t_s - t_p = 2(r/c)\varepsilon$ between the primary wave and the wavefront of the secondary wave. The sharp singularity in equation (24) is related to our using a constant velocitiy v; actually, an uncertainty of the form $\Delta v \simeq c\varepsilon$ exists in this velocity, which entails an uncertainty $\tau\varepsilon$ in the time τ, such that the smallest value of the denominator in equation (24) is of the order $c^2\tau^2\varepsilon$. In the vicinity of the wavefront the leading contributions to the components of the surface displacement ($z \rightarrow 0$, in polar cylindrical coordinates) are given by

$$u_{sr} \simeq \frac{\chi_0\tau}{4c} \cdot \frac{r}{(c^2\tau^2 - r^2)^{3/2}} \ , \ u_{sz} \simeq \frac{\chi_0\tau}{2c} \cdot \frac{|z|}{(c^2\tau^2 - r^2)^{3/2}} \rightarrow 0 \ . \tag{25}$$

We can see that the singular displacement tangential to the surface of the fluid (u_{sr}) generates an accumulation of fluid at the wavefront $r = c\tau$ along the vertical, with an infinite increase in height (while the level of the surface for $r < c\tau$ decreases); also, we can see that the pressure is decreased by the amount $\delta p = -\rho\ddot{\psi} \simeq -(3\rho\chi_0/4)r^3(c^2\tau^2 - r^2)^{-5/2}$ at $r = c\tau$, which brings about the great increase in height of the fluid at this point (under these conditions the negative velocity of the fluid particles given by equation (25) is irrelevant, as are the large spatial variations of u_{sr} near the wavefront, given by equation (25)). This may be viewed as a model picture for the tsunami phenomenon. The height h of the fluid with respect to the level of the equlibrium surface can be computed from the equilibrium equation in gravitational field $\delta p = -\rho g h$, where g is the gravitational constant and z in δp is replaced by h; the height h is a function r and t and is infinite at the wavefront $r = c\tau$.

Concluding remarks. Compressional waves are considered in an ideal fluid at equilibrium, generated by sources localized both in space and time, like explosions in seas and oceans or earthquakes occuring just beneath the seafloor (seabed). These primary waves are spherical shells which produce additional wave sources on the plane surface of the fluid (secondary sources). The secondary sources "move" on the surface with a velocity greater than the wave velocity and generate secondary waves in the fluid (radiation). The secondary waves are computed in a simplified model. It is shown that the secondary waves have a sharp spherical wavefront which is singular on the surface. This singularity may be viewed as a model representation of the tsunami phenomenon. The fluid accumulates on the vertical at the wavefront, while the fluid level decreases behind.

Acknowledgments. The author is indebted to the members of the Laboratory of Theoretical Physics at Magurele-Bucharest for many fruitful discussions. This work has been supported by the Scientific Research Agency of the Romanian Governments through Grants 04-ELI / 2016 (Program 5/5.1/ELI-RO), PN 16 42 01 01 / 2016 and PN (ELI) 16 42 01 05 / 2016.

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