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On the ringing of the bells

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Abstract

The ringing of the bells is analyzed within the frame of a simple model of vibrations of a thin spherical shell. Near- and far-field elastic waves propagating in air are derived, as produced by bell's vibrations. The audibility threshold distance and the limit of permanent daage to to human ear are estimated for a large bell. Also, a solution to the "lower octave" paradox is suggested.

An octave lower. The vibrations of a ringing bell is an instance of an intractable problem. The intractability arises from the fact that the bell is a shell, not a bulky body, and from its particular shape. Even if the bell's shape is approximated by an empty hemisphere, or a cone, or a hyperboloid, there still remain difficulties related to the shell structure, curvilinear coordinates and the boundary conditions. In particular, the bell's appex should be fixed and the bell's rim should vibrate freely. The shell elasticity is different from the bulk elasticity. The elastic energy should include invariants of the symmetry group of the body's shape constructed from the strain tensor; in most cases the non-linearities of the thin shell should be included. An example of elastic energy of an empty sphere is given in Ref. [1]. The curvilinear coordinates make the vibrations to take place in an inhomogeneous body, which greatly complicates the problem. It is believed that dividing the structure in finite elements and using classical elasticity, including free-surface conditions on the shell's surfaces, may, in principle, lead to a numerically solvable problem, though simplifying assumptions are needed. However, the elasticity inside the shell plays a minor role, and the elasticity moduli of a shell are different from those of a bulky body. A model is necessary, even for numerical calculations. An important help arises from the cutoff frequency related to the finite thickness of the shell.

A great simplification arises from limiting ourselves to the vibrations of the bell's rim. The excitation of the bell by a hammer, or a clapper, produces a sufficiently large deformation, which lasts a sufficiently long time in comparison with the elastic waves periods, to give rise to many stationary waves of the form $\cos\omega_n t\cos k_n x$ (and similar expressions with sin), where x is the coordinate along the rim and t denotes the time. The wave equation requires $\omega_n = vk_n$, where v is the velocity of the waves; the boundary conditions require $k_n = 2\pi n/L$, where L is the length of the rim and n is any positive integer. Since the excitation is sufficiently localized on the rim (δ-function), the amplitudes of these vibrations are equal, from initial conditions. The vibrations look like $\cos n\omega t\cos(2\pi nx/L)$, where $\omega = 2\pi v/L$.

Rayleigh[2] noticed that the pitch of the bells, *i.e.* the frequency heard by the human ear, is close to $\omega/2$, though the physical frequencies measured by resonators are, of course, $n\omega$; this is an octave below the fundamental frequency ω, and it does not belong to the series of eigenfrequencies. Other observations indicate other pitch frequencies.

Let us consider the sum

$$S = \cos\omega t + \cos 2\omega t + \cos 3\omega t + ... \cos N\omega t \quad , \tag{1}$$

where N is very large; S is the signal heard by the human ear. If we group the terms in pairs and transform the sums in products, we get

$$S = 2\cos\frac{\omega}{2}\left(\cos\frac{3}{2}\omega t + \cos\frac{7}{2}\omega t + \cos\frac{11}{2}\omega t + ...\right) + (\cos N\omega t) \quad ; \tag{2}$$

the last term in equation (2) is present if N is odd. The bracket in equation (2) is heard, practically, as a constant, the last term, if present, is not audible; therefore, the human ear hears $\omega/2$. If ω varies slighty with n, or there exist several subgroups of frequencies of the form given by equation (1), the pitch differs from $\omega/2$.

Waves produced by bells. Let us assume a spherical shell with radius a and thickness d ($d \ll a$). We may suppose that, at large distances, this shell approximates a bell. The longitudinal vibrations, which imply motion along the shell, do not affect the fluid the shell is immersed in (air). We adopt a simple model of transverse vibrations of the shell, consisting of a series of wavelengths $\lambda_n = v/\omega_n = a/n$, where v is the velocity of the elastic waves in shell, $\omega_n = (v/a)n$ are frequencies and n is any positive integer. $\omega = v/a$ is the fundamental frequency and the frequency series is limited by the cutoff frequency v/d, *i.e.* the integer n is limited by $N = a/d$. The energy stored by a transverse vibration with frequency ω_n may be approximated by $E_n = \mu V(u_0/\lambda_n)^2$, where μ is the elastic moduli, $V = a^2 d$ is the volume of the shell and u_0 is the vibration amplitude. This energy can also be written as $E_n = (\mu V/v^2)\omega_n^2 u_0^2$, or $E_n = \rho V\omega_n^2 u_0^2$, where ρ is the density of the shell. The total energy is given by

$$E = \sum_{n=1}^{N} E_n = \frac{\rho v^2 a^2}{d} u_0^2 \quad . \tag{3}$$

From equation (3) we get the amplitude $u_0 = (Ed/\rho)^{1/2}/va$. For numerical purposes we take $E = 10^8 erg$ (corresponding to $10kg$ with velocity $1m/s$), $d = 10cm$, $\rho = 10g/cm^3$, $a = 3m$ and $v = 3\times 10^5 cm/s$; we get $u_0 \simeq 10^{-4}cm$. Very likely, the cutoff frequency is much lower, such that, introducing the factor $(a/d)^3 \simeq 10^4$, we get the more realistic amplitude $u_0 \simeq 1cm$. We note that the result is the same when we restrict ourselves to the rim vibrations.

The force produced by the n-th vibration is obtained from the energy $\mu V(u_n/\lambda_n)^2$ as $F_n = (\mu V/\lambda_n^2)u_n$, where $u_n = u_0\cos\omega_n t$; this force can also be written as $F_n = \rho V\omega_n^2 u_0\cos\omega_n t$. The force F_n is concentrated on the shell, such that we may represent its density as

$$F_{nr} = \frac{\rho V}{a^3}\omega_n^2 u_0 d\delta(r-a)\cos\omega_n t = \frac{\rho d^2}{a}\omega_n^2 u_0\delta(r-a)\cos\omega_n t \quad ; \tag{4}$$

the suffix r indicates that this force density is the radial component. The equation of the (longitudinal) elastic displacement of the fluid is

$$\rho\ddot{u}_r - \rho c^2 grad_r(div\mathbf{u}) = \frac{\rho d^2}{a}\omega_n^2 u_0\delta(r-a)\cos\omega_n t \quad , \tag{5}$$

where c is the speed of the elastic waves in fluid; or,

$$\ddot{u}_r - c^2\frac{\partial}{\partial r}(div\mathbf{u}) = \frac{d^2}{a}\omega_n^2 u_0\delta(r-a)\cos\omega_n t \quad . \tag{6}$$

The displacement $\mathbf{u}$ derives from a potential Φ, through $\mathbf{u} = grad\Phi$; equation (6) becomes

$$\frac{\partial\ddot{\Phi}}{\partial r} - c^2\frac{\partial}{\partial r}\Delta\Phi = \frac{d^2}{a}\omega_n^2 u_0\delta(r-a)\cos\omega_n t \ , \tag{7}$$

or

$$\ddot{\Phi} - c^2\Delta\Phi = \frac{d^2}{a}\omega_n^2 u_0\theta(r-a)\cos\omega_n t \ . \tag{8}$$

For $r \gg a$ we may write this equation as

$$\ddot{\Phi} - c^2\Delta\Phi = \frac{d^2}{a}\omega_n^2 u_0\cos\omega_n t - d^2\omega_n^2 u_0\delta(r)\cos\omega_n t \tag{9}$$

($\theta(r) = 1$). The first term on the right in equation (9) gives a uniform contribution to Φ, which does not contribute to the displacement $\mathbf{u}$; the second term gives the solution

$$\Phi = -\frac{d^2\omega_n^2 u_0}{4\pi c^2}\int d\mathbf{r}'\frac{\delta(r')}{\mid \mathbf{r}-\mathbf{r}' \mid}\cos\omega_n(t- \mid \mathbf{r}-\mathbf{r}' \mid /c) \ . \tag{10}$$

Within our approximations we may replace $\delta(r')$ by $\delta(r') = 2\pi a^2\delta(\mathbf{r}')$; we get

$$\Phi \simeq -\frac{d^2a^2\omega_n^2 u_0}{2c^2 r}\cos\omega_n(t-r/c) \tag{11}$$

and

$$u_r \simeq \frac{d^2a^2\omega_n^2 u_0}{2c^2r^2}\cos\omega_n(t-r/c) - \frac{d^2a^2\omega_n^3 u_0}{2c^3 r}\sin\omega_n(t-r/c) \ , \tag{12}$$

where we can see the near-field and the far-field waves. The result given by equation is valid for $r \gg a$. For the fundamental mode $\omega_n \simeq v/a$ we can represent the far-fied waves as

$$u_r \simeq (v/c)^3(d^2u_0/2ar)\sin\omega(t-r/c) \ ; \tag{13}$$

a better numerical estimation is obtained by replacing a in equation (13) by a smaller value. For a bulk sphere d should be replaced in equation (13) by a.

For $v = 3\times10^5 cm/s$, $c = 3\times10^4 cm/s$, $a = 300cm$, $d = 10cm$ and $u_0 = 1cm$ we get the amplitude u_r^0 of the displacement u_r of the order $u_r^0 \simeq 10^2/r(cm)$ (fundamental frequency $\simeq 10^3 s^{-1}$). The elastic modulus μ for shells is lower than the elastic modulus for bulk materials, such that v in equation (13) should be smaller (say, $3\times10^4 cm/s$); also, the frequency should be smaller; more realistic values are $\omega \simeq 100s^{-1}$. The energy density in this wave is $\rho\dot{u}_r^{02} \simeq 10^7/r^2(erg/cm^3)$ (pressure) and the energy flux is $I = \rho\dot{u}_r^{02}c \simeq 10^{11}/r^2(erg/cm^2\cdot s)$, where $\rho \simeq 10^3 g/cm^3$ is the density of the air.

The sound intensity in decibels (dB) is given by $L = 10\lg(I/I_0)$, where I is the energy flux and I_0 is a reference energy flux of the order $I_0 \simeq 10erg/cm^2\cdot s$; making use of $I \simeq 10^{11}/r^2(erg/cm^s\cdot s)$ given above, we get $L \simeq 10\lg(10^{10}/r^2)(dB)$; the audibility limit is $L = 0$, which means that such a sound can be heard at a distance $r \simeq 1km$. Allowing for higher frequencies, this limit is probably of the order $10km$.

It is worth estimating the sound intensity in the near-field region; making use of the numerical data given above, we get the amplitude is $u_r^0 \simeq 10^4/r^2(cm)$, the energy density (pressure) is $\simeq 10^{11}/r^4(erg/cm^3)$ and the energy flux is $I \simeq 10^{15}/r^4(erg/cm^2\cdot s)$; the intensity is $L = 10\lg(10^{14}/r^4)$. For $r = 1cm$ we get the intensity $L = 140dB$, which is the threshold of permanent damage to the human ear.

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