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Statistical Seismology
(Studies in Seismology, I)

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Abstract

The geometric-growth model of seismic focus is presented and the geometric parameter r is introduced, related to the reciprocal number of effective dimensions of the focus. This parameter governs the relationship between the accumulation time and the energy accumulated in the seismic focus. The time, energy and magnitude earthquake probability distributions are derived, in particular the magnitude (Poisson) distribution $\sim e^{-\beta M}$, where $\beta = br$, $b = 3.5$ (3/2 in decimal logarithms) being the well-known Gutenberg-Richter constant (and M denoting the magnitude). The main concept employed in deriving these probabilities is the seismicity rate. The mean recurrence time, the empirical magnitude frequency, the excedence rate and the energy released by earthquakes are discussed. These laws are used to fitting 3640 earthquakes with magnitude $M \geq 3$ which occurred in Vrancea between $1981 - 2018$ (38 years). The results are compared with those corresponding to a similar analysis of 1999 earthquakes with magnitude $M \geq 3$, recorded in Vrancea between $1974 - 2004$ (31 years). The main result is the parameter r which increased from $r = 0.54$ in the former period to $r = 0.65$ in the more recent period; at the same time, an increased rate of seismicity is reported lately. These results suggest that the collective, generic seismic focus in the seismic region Vrancea (*i.e.* the spatial distribution of the seismic foci) has changed in the recent years, from a two-dimensional distribution towards a distribution with a more pronounced content of a one-dimensional distribution. These changes have important consequences in the accumulation time of the great earthquakes, which become longer. The results are very sensitive to small variations in the parameter r, such that a refined statistical anlysis (the only access way to to this parameter) is useful for estimating the seismic hazard.

By using the background distributions indicated above, the conditional probabilities for time, energy and magnitude distributions are derived and employed to analyze the short-term seismic activity, especially the accompanying (associated) distributions of foreshocks and aftershocks. It is shown that the temporal accompanying seismic activity is governed by the Omori law, with a (short-) time cutoff. The Omori distribution is derived by using the branching concept, as well as, equivalently, by means of the self-replication processes. While for long times a Poisson distribution may be included, power laws with a fractional exponent at short times (gamma distributions) are equivalent with Omori law (time exponent -1). Making use of the magnitude distribution the empirical Bath's law for aftershocks (foreshocks) is derived. On this occassion, the disparity between the temporal and magnitude (energy) scales is noted. The conditional probabilities have been used to analyze the next-earthquake distributions (inter-event time distributions) for 3640 earthquakes with magnitude $M \geq 3$ recorded in Vrancea between $1981 - 2018$. These probabilities exhibit a correlation extended over $20-25$ days, with a mean time $\simeq 6$ days and a deviation $\simeq 9.5$ days. The highest probability of occurence of an earthquake is the next day after an earthquake, at least for small and moderate earthquakes. Such probability tables can be used for short-term forecasting and estimating the seismic hazard and risk. The correlations disappear gradually with increasing magnitudes, such that such tables become useless in these (probably the most interesting) cases. Finally, a discussion is included, concerning the Statistical Seismology and its evolution ways.

1 Introduction

Seismic observations indicate that typical earthquakes release suddenly, in a short lapse of time, energy accumulated in a focus localized in Earth's crust. The dimension of the focus is very small in comparison with the distance to Earth's surface, such that the focus may be vieved as a point. Very likely, the energy accumulation in the focus is due to the movement of the tectonic plates, such that the focus may be viewed as a shearing fault.[1]

During an earthquake the seimographs placed on Earth's surface record a succession of longitudinal waves (P waves), transverse waves (S waves), followed by a surface main shock.[2] It was shown recently that the focal elastic force density can be represented as $f_i = M_{ij}\partial_j\delta(\mathbf{R}-\mathbf{R}_0)$, where M_{ij} is a seismic moment tensor associated to the focus placed at $\mathbf{R}_0$. The P and S waves and the main shock have been computed with this force density; the seismic waves are shell spherical waves, while the main shock has an abrupt front, followed by a long tail.[3] From the measurement of the P and S waves the seismic moment M_{ij} and the released energy $E = \overline{M}/2\sqrt{2}$, $\overline{M} = \left(M_{ij}^2\right)^{1/2}$, can de deduced, as well as all the parameters of the focal source, like the dimension of the focus, the orientation of the fault, the direction of the fault slip, the focal strain and the duration of the earthquake.[4]

2 Focal accumulation model

According to energy conservation, the energy accumulation in an earthquake focus obeys the continuity equation

$$\frac{\partial E}{\partial t} = -\mathbf{v} gradE \ , \tag{1}$$

where E is the energy at time t and $\mathbf{v}$ denotes the velocity of energy accumulation; the energy loss by dissipation (thermoconduction and friction by viscosity) may be taken into consideration by empirical parameters which appear immediately in this equation. For a localized focus, equation

[1]A. Wegener, "Die Herausbildung der Grossformen der Erdrinde (Kontinente und Ozeane) auf geophysikalischer Grundlage", Petermanns Geographische Mitteilungen **63** 185-195, 253-256, 305-309 (1912); A. Wegener, *Die Entstehung der Kontinente und Ozeane*, Vieweg & Sohn, Braunschweig (1929).

[2]C. G. Knott, *The Physics of Earthquake Phenomena*, Clarendon Press, Oxford (1908); A. E. H. Love, *Some Problems of Geodynamics*, Cambridge University Press, London (1926).

[3]B. F. Apostol, "Elastic waves inside and on the surface of an elastic half-space", Q. J. Mech. Appl. Math. **70** 289 (2017); *The Theory of Earthquakes*, Cambr. Int. Sci. Publ. (2017); *Introduction to the Theory of Earthquakes*, Cambr. Int. Sci. Publ. (2017).

[4]B. F. Apostol, "On an inverse problem in elastic wave propagation", Roum. J. Phys., to appear ("The inverse problem in Seismology. Seismic moment and energy of earthquakes. Seismic hyperbola", arxiv: 1808.03049v1 [physics. geo-ph], 9 August 2018).

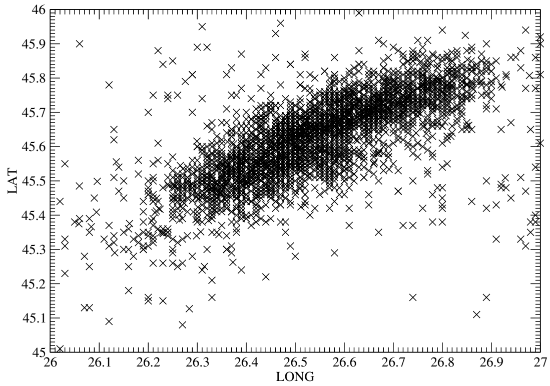

Figure 1: Geographical distribution in longitude and latitude of earthquakes in Vrancea between 1981-2018 with magnitude $M \geq 3$ (ROMPLUS Catalog).

(1) can be written as

$$\begin{gathered}\frac{dE}{dt} = v_x \frac{\Delta E}{\Delta x} + v_y \frac{\Delta E}{\Delta y} + v_z \frac{\Delta E}{\Delta z} = \\ = \left(\frac{v_x}{u_x} + \frac{v_y}{u_y} + \frac{v_z}{u_z} \right) \frac{\Delta E}{\Delta t} \ ,\end{gathered} \tag{2}$$

where $\mathbf{u}$ is the local velocity of the rocks in time lapse Δt and x, y, z denote the coordinates. In the absence of other agencies, we may take $\Delta E = E$ and $\Delta t = t$; in order to avoid the indeterminacy in the limit E, $t \to 0$ we introduce a cutoff energy E_0 and a cutoff time t_0. For long times the velocity $\mathbf{u}$ is small, such as to preserve the localization. In general, the velocity $\mathbf{u}$ is different from velocity $\mathbf{v}$. However, the energy accumulates by a direct elastic shear, so we may take $\mathbf{u} = \mathbf{v}$ (compression may appear in artificial foci, like explosions); we can see that these parameters disappear from equation (2), where we are left with a factor 3. However, the accumulation may proceed anisotropically, for instance along only one direction, or two directions as in a fault, so we get a factor 1 or 2 in equation (2). Consequently, we write equation (2) as

$$\frac{dE}{dt} = \frac{1}{r} \frac{E + E_0}{t + t_0} \ , \tag{3}$$

where the parameter r remains undetermined. In principle, this parameter may take any value, but, very likely, it lies in the range $1/3 < r < 1$. From equation (3) we get the energy E accumulated in time t as given by[5]

$$1 + t/t_0 = (1 + E/E_0)^r \ . \tag{4}$$

We note that the variables in this equation are the reduced (scaled) energy E/E_0 and the reduced time t/t_0; this feature and the power law are reminiscent of critical phenomena.[6]

Originally, a relationship $E/E_0 = (R/R_0)^3$ has been suggested, where two characteristic lengths R and R_0 are associated either with the central core of the critical focal zone where the seismic energy accumulates, or R may correspond to the characteristic length of the seismic region disrupted by the earthquake, R_0 being in this case a scale length.[7] For a uniform energy accumulation in the focus in the time interval t we get $E/E_0 = (R/R_0)^3 = (t/t_0)^3$, ($E \gg E_0$, $t \gg t_0$), hence $r = 1/3$.

[5] B. F. Apostol, PhD Thesis, Institute for Earth's Physics, University of Bucharest, Magurele, 2005; "A Model of Seismic Focus and Related Statistical Distributions of Earthquakes", Rom. Reps. Phys. **58** 583 (2006); "A Model of Seismic Focus and Related Statistical Distributions of Earthquakes", Phys. Lett. **A357** 462 (2006).

[6] P. Bak and C. Tang, "Earthquakes as a self-organized critical phenomenon", J. Geophys. Res. **94** 15635 (1989); P. Bak, *How Nature Works: The Science of Self-Organized Criticality*, Copernicus, NY (1996).

[7] C. G. Buffe and D. J. Varnes, "Predictive modeling of the seismic cycle of the Greater San Francisco Bay

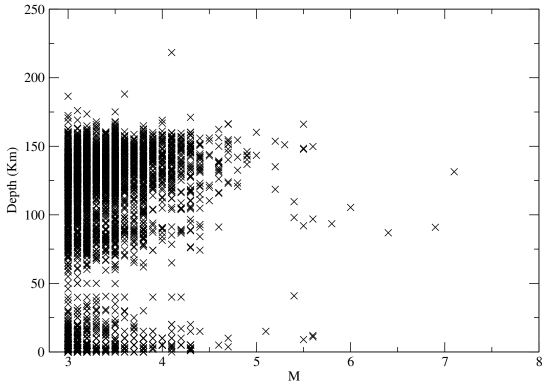

Figure 2: Depth distribution of earthquakes in Vrancea between 1981-2018 with magnitude $M \geq 3$ (ROMPLUS Catalog).

3 Probability distributions. Gutenberg-Richter law

In time $t = 0$ the energy released is $E = 0$, *i.e.* the energy E_0 accumulated in time t_0 is either the background energy, or it is released by quasi-static relaxation processes, or dissipated. It follows that in a time $T > t_0$ there may appear a maximum number $N_0 = T/t_0$ of "seismic events" with released energy $E = 0$. $1/t_0$ is called seismicity rate; t_0, defined as T/N_0, is a mean accumulation time. Similarly, the number N of earthquakes with an accumulation time t ($t_0 + t$, characteristic time t) and energy E is given by $N = T/(t_0 + t)$, whence

$$\frac{N}{N_0} = \frac{1}{1 + t/t_0} = (1 + E/E_0)^{-r} \quad . \tag{5}$$

Therefore, the density of time probability $P(t)$ is given by

$$P(t)dt = -d\left(\frac{1}{1 + t/t_0}\right) = \frac{1}{(1 + t/t_0)^2}\frac{dt}{t_0} \tag{6}$$

and the density of energy probability $P(E)$ is[8]

$$P(E)dE = \frac{r}{(1 + E/E_0)^{1+r}}\frac{dE}{E_0} \quad . \tag{7}$$

A power law

$$P(E) \sim E^{-\alpha} \tag{8}$$

for earthquake energy distributions has been suggested before, by an analysis of the earthquakes in Japan, where the exponent is $\alpha = 1.7 - 2.3$.[9] Since $\alpha = 1 + r$, we get the parameter r in the range $r = 0.7 - 1.3$.

region", J. Geophys. Res. **98** 9871 (1993); D. L. Wells, K. J. Coppersmith, "New empirical relationships among magnitude, rupture length, rupture width, rupture area, and surface displacement", Bull. Seism. Soc. Am. **84** 974 (1994); D. D. Bowman, G. Ouillon, C. G. Sammis, A. Sornette and D. Sornette, "An observational test of the critical earthquake concept", J. Geophys. Res. **103** 24359 (1998) and References therein.

[8] B. F. Apostol, "A Model of Seismic Focus and Related Statistical Distributions of Earthquakes", Rom. Reps. Phys. **58** 583 (2006); "A Model of Seismic Focus and Related Statistical Distributions of Earthquakes", Phys. Lett. **A357** 462 (2006).

[9] K. Wadati, "On the frequency distribution of earthquakes", J. Meteorol. Soc. Japan **10** 559 (1932).

The density of energy probability given by equation (7) can be written as

$$P(E) = \frac{r}{E_0} e^{-(1+r)\ln(1+E/E_0)} \simeq \frac{r}{E_0} e^{-(1+r)\ln(E/E_0)} \tag{9}$$

for $E \gg E_0$, which suggests the notation

$$\ln(E/E_0) = bM \ , \tag{10}$$

where b is a numerical coefficient, or

$$\ln E = a + bM \ , \tag{11}$$

where $a = \ln E_0$. This is the Gutenberg-Richter law, where M is the earthquake magnitude.[10] Since $E = \overline{M}/2\sqrt{2}$ we can also write

$$\ln \overline{M} = \left(a + \frac{3}{2}\ln 2\right) + bM \ ; \tag{12}$$

M is also called the moment magnitude. The values of the coefficients a and b are conventional.[11] The coefficient b has the value $b \simeq 3.5$, or $b = 3/2$ in decimal logarithms ($\ln 10 = 2.3$). It seems that this value comes from the estimation μSl of the seismic moment $\overline{M}$, where μ is the Lame elastic shear coefficient, S is the fault area and l denotes the fault slip; the volume Sl can be written as $A^{3/2}$, where A has the dimensions of an area; hence, the coefficient $b = 3/2$.[12] The value of the coefficient a is uncertain. For seismic moments measured in $dyn \cdot cm$ the value $a + \frac{3}{2}\lg 2 = 16.1$ is used in decimal logarithms, *i.e.* $a = 15.65$ ($\lg 2 = 0.3$); for moments measured in J this value corresponds to $a = 8.65$.[13] Since the measurement of $\lg E$ is expected to have ± 0.5 an error, it is likely that the error in magnitudes is ± 0.33.[14]

Making use of the definition (10) of the magnitude, we get from equation (9) the magnitude distribution[15]

$$P(M)dM = \beta e^{-\beta M} \ , \ \beta = br \ , \tag{13}$$

for $M \gg 1/b$ (a Poisson distribution). We can see that this distribution is free of the parameter a (E_0).

3.1 Magnitude frequency

The magnitude probability can be written as

$$P(M) = \frac{\Delta N}{N_0 \Delta M} = \frac{t_0 \Delta N}{T \Delta M} \ , \tag{14}$$

[10]B. Gutenberg and C. Richter, "Frequency of earthquakes in California", Bull. Seism. Soc. Am. **34** 185 (1944).

[11]T. Utsu and A. Seki, "A relation between the area of aftershock region and the energy of the mainshock", J. Seism. Soc. Japan **7** 233 (1954).

[12]T. Lay and T. Wallace, *Modern Global Seismology*, Academic Press, San Diego (1995); D. L. Wells, K. J. Coppersmith, "New empirical relationships among magnitude, rupture length, rupture width, rupture area, and surface displacement", Bull. Seism. Soc. Am. **84** 974 (1994).

[13]H. Kanamori, "The energy release in earthquakes", J. Geophys. Res. **82** 2981 (1977).

[14]It is worth noting that equation (10) and the relation $E \sim R^3$ lead to $R^2 \sim e^{2bM/3}$ and $\lg S \simeq 1.02M + const$, where S is interpreted as the area of the aftershock region and M is the magnitude of the main shock. This is an empirical law (T. Utsu and A. Seiki, cited above; T. Utsu, "Aftershocks and earthquake statistics (I): some parameters which characterize an aftershock sequence and their interaction", J. Faculty of Sciences, Hokkaido Univ., Ser. VII (geophysics) **3** 129 (1969)).

[15]B. F. Apostol, "A Model of Seismic Focus and Related Statistical Distributions of Earthquakes", Rom. Reps. Phys. **58** 583 (2006); "A Model of Seismic Focus and Related Statistical Distributions of Earthquakes", Phys. Lett. **A357** 462 (2006).

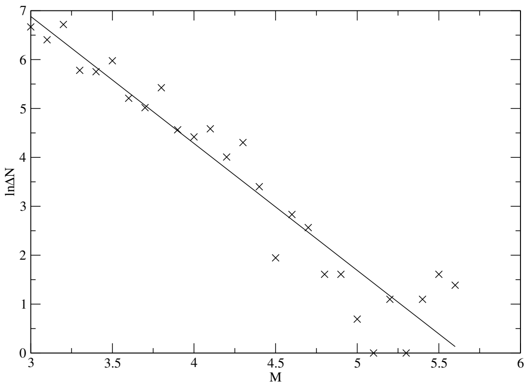

Figure 3: Logarithmic distribution $\ln \Delta N = \ln C - \beta M$ fitted to data given in Table 1.

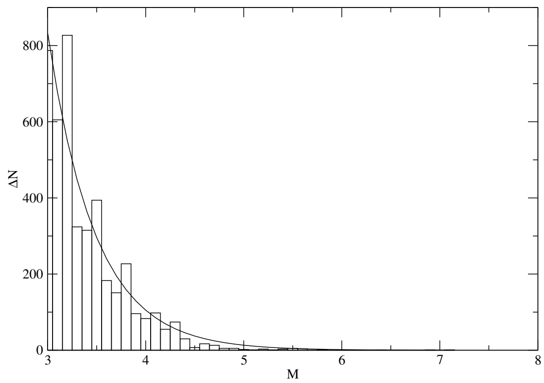

Figure 4: Exponential distribution $\Delta N = C \exp(-\beta M)$ fitted to data in Table 1.

where ΔN is the number of earthquakes with magnitude in the range $(M,\ M+\Delta M)$ and $N_0 = T/t_0$ is the total number of earthquakes with recurence time t_0 in time T ($1/t_0$ is the seismicity rate). From equation (13) we get

$$\lg\left(\Delta N/T\right) = A - BM \ , \tag{15}$$

where

$$A = \lg\left(\frac{\beta \Delta M}{t_0}\right) \ , \ B = \frac{\beta}{2.3} \ . \tag{16}$$

This magnitude-frequency ($\Delta N/T$) relationship has been checked for a large set of global earthquakes with $5.8 < M < 7.3$ ($\Delta M = 0.1$).[16] By fitting these experimental data the parameters $A = 4.6$ and $B = 0.6$ have been determined, leading to $\beta = 1.38$ and $1/t_0 = 10^{5.5}$ per year. We note that t_0 and β are fitting parameters. From $\beta = br$, $b = 3.5$, we get $r = 0.4$, which suggests an intermediate two/three-dimensional focal mechanism. It is worth noting that an isotropic three-dimensional accumulation mechanism leads to $r = 1/3$ and $\beta = 1.17$, while a two-dimensional accumulation mechanism leads to $r = 1/2$ and $\beta = 1.75$. Deviations from the linear law given by equation (15) exist for small magnitudes, which are consistent with the corrections brought by equation (7) to the Gutenberg-Richter law for low energies. Large deviations from this linear law occur for large magnitudes, which show that strong earthquakes, being rare, are not fully statistical, as expected.

[16] K. E. Bullen, *An Introduction to the Theory of Seismology*, Cambridge University Press, London (1963).

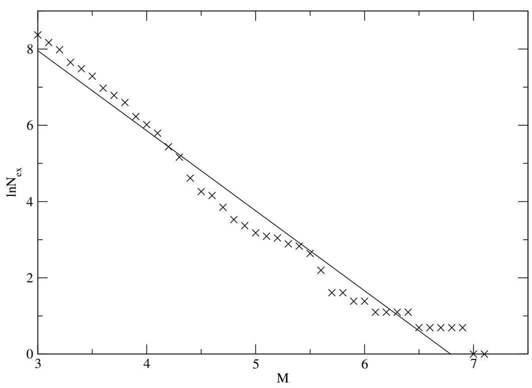

Figure 5: Excedence law $\ln N_{ex} = \ln N_0 - \beta M$ fitted to the cumulative distribution of data given in Table 1.

3.2 Mean recurrence time

From the magnitude frequency $\Delta N/T$ (equation (15)) we get the mean recurrence time

$$t_r = \frac{t_0}{\beta \Delta M} e^{\beta M} \tag{17}$$

for an earthquake with magnitude M (*i.e.* in the interval $(M,\ M + \Delta M)$). This time should be compared with the accumulation time $t_a = t_0 e^{\beta M}$, which can be derived from equations (4) and (10). These times are related by $t_a = (\beta \Delta M) t_r$, whence one can see that $t_a < t_r$, a relationship which shows that the energy may be lost by seismc events with lower energy, as expected. Since the parameters t_0, β, and ΔM are known from statistical analysis of data, equation (17) might suggest a possibility to predict the earthquake occurrence. This might be the case for small earthquakes, which are frequent (short t_r), though it is not very useful. In fact, the distribution $\beta e^{-\beta M}$ of the inverse of the recurrence times implies an error of the order $\left(\sqrt{\overline{M^2}} - \overline{M}\right)/\overline{M} = \sqrt{2} - 1$, *i.e.*, $\Delta t_r / t_r \simeq 0.41$, which is too large to be useful. For a maximal entropy with mean recurrence time t_r we get a Poisson distribution $(1/t_r) e^{-t/t_r}$ for the recurrence time, which has a standard deviation $\sqrt{\overline{(t - t_r)^2}} = t_r$.

In principle, there may exist a recurrence-time scale for some large earthquakes, which may lead, for instance, to a normal (Gauss) distribution; if the standard deviation is small, it may be used in forecasting such earthquakes. However, such a possibility is very unrealistic.

4 Earthquakes in Vrancea

The seismic area of the geographic region Vrancea is centered on 45.7° (northern) latitude and 26.6° (eastern) longitude. The earthquakes occur here in the depth range $60 - 180 km$. The region exhibits occasionally small crustal, or surface, earthquakes. Strong earthquakes occur sometimes in Vrancea, as, for instance, $M = 7.4$, March 4, 1977, depth $94 km$, or $M = 7.1$, August 30, 1986, depth $131 km$. Nine earthquakes with magnitude $M > 7$ have been recorded in Vrancea in the past two centuries. 3640 earthquakes with $M \geq 3$, which ocurred in Vrancea between 1981 and 2018 (38 years), are analyzed here ($\Delta M = 0.1$).[17] The distribution $\Delta N(M)$ of these data is given in Table 1 for $\Delta M = 0.1$.

[17] Romanian Earthquake Catalogue (ROMPLUS Catalog), National Institute for Earth Physics, Romania (2018).

M=3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
ΔN=787	605	827	324	315	394	183	151	227	96
M=4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9
ΔN=83	98	55	74	30	7	17	13	5	5
M=5.0	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
ΔN=2	1	3	1	3	5	4	0	1	0
M=6.0	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9...
ΔN=1	0	0	0	1	0	0	0	0	1
M=7.1	...								
ΔN=1	...								

Table 1: Magnitude distribution of earthquakes in Vrancea from 1981 to 2018 (magnitude $M \geq 3$, ROMPLUS Catalog).

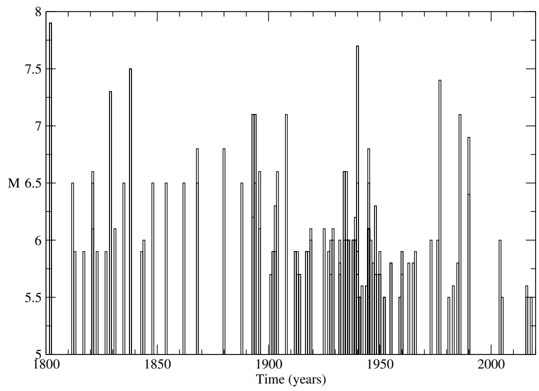

Figure 6: Earthquakes with magnitude $M > 5.5$ in the last two centuries in Vrancea (ROM{LUS Catalog).

The geographical distribution in longitude and latitude of earthquakes in Vrancea between $1981-2018$ with magnitude $M \geq 3$ is shown in Fig. 1, while the depth distribution of the same set of data is shown in Fig. 2.

The logarithmic distribution (15) is used in the form

$$\ln \Delta N = \ln C - \beta M \tag{18}$$

for fitting the data in Table 1, where $C = \beta N_0 \Delta M$, $\ln C = 14.67$, $\beta = 2.6$ (error $\simeq 10\%$, $\Delta M = 0.1$), as shown in Fig. 3. It follows a seismicity rate $-\ln t_0 \simeq 12.49$ (in years) and a coefficient $B = \beta/2.3 \simeq 1.13$. A similar fit, using $\Delta N = C \exp(-\beta M)$ is shown in Fig. 4, where $\ln C = 12.9$ and $\beta = 2.07$ (error $\simeq 30\%$), corresponding to a seismicity rate $-\ln t_0 \simeq 10.84$ and a coefficient $B = \beta/2.3 \simeq 0.9$. It is also convenient to introduce the so-called recurrence law, or the excedence rate, which gives the number N_{ex} of earthquakes with magnitude greater than M. The corresponding probability is readily obtained from (13) as $P_{ex} = e^{-\beta M}$, such that the excedence rate reads

$$\ln N_{ex} = \ln N_0 - \beta M \ . \tag{19}$$

This formula is fitted to the experimental data given in Table 1) with $\ln N_0 = 14.25$ and $\beta = 2.1$ (Fig. 5, error $\simeq 10\%$). These values correspond to a rate of seismicity $-\ln t_0 \simeq 10.62$ and $B = \beta/2.3 \simeq 0.9$.

The three fits described above have a different quality and a different accuracy. The direct exponential fit to the data has probably the best quality, because it takes into account all the data,

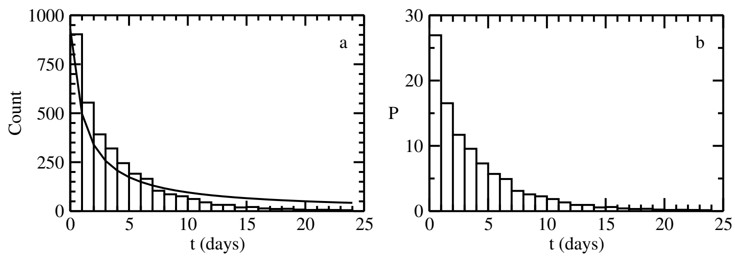

Figure 7: Next-earthquake distribution of seismic events in Vrancea 1981 − 2018 (3640 events, $M \geq 3$, panel *a*) and probabilities $P = P(t)$ (in %, panel *b*). The fitting curve in panel *a* is $1066.45/(1.15 + t)$ (coefficient of determination $\simeq 0.96$).

in contrast to the logarithmic fits were vanishing data are excluded. Unfortunately, its accuracy is poor, due to the abrupt variation of the exponential function and data scattering. Similarly, the best accuracy (10%) is for the logarithmic fit given by equation (18), as the function variation is the smallest. However, the quality of this fit is poor, as it loses all the vanishing data. Finally, the excedence law (19) produces a fit which is situated in-between, with a moderate accuracy and a moderate quality. It is reasonable to use an average value for the seismicity rate $-\overline{\ln t_o} = (12.49 + 10.84 + 10.62)/3 = 11.32$ and an average $\bar{\beta} = (2.6 + 2.07 + 2.1)/3 = 2.26$ (error $\simeq 18\%$). The value $\beta = 2.26$, which corresponds to a coefficient $B = \beta/2.3 \simeq 1$, indicates a value $r = \beta/b = 0.65$ for the parameter r of the focus model, where $b = 3.5$ is used. This may show that the geometry of the Vrancea focus is different from an isotropic point-like source, accumulating seismic energy with a uniform velocity; the focus seems to resemble more an intermediate one/two-dimensional geometry ($1/r = 1.5$) (with the hypothesis of uniform accumulating velocities).

The statistical analysis gives a generic image of a collective, global earthquake focus (a distribution of foci). Its results should be corroborated with the spatial distribution of the seismic foci in Vrancea, shown in Figs. 1, 2. Making use of the values obtained here for the average seismicity rate and average β-parameter, one may attempt to estimate the accumulation time, by using equation $t_a = t_0 e^{\beta M}$. The value $t_a \simeq 90$ years is obtained this way, for the accumulation time of earthquakes with magnitude $M = 7$ in Vrancea. It must be recalled that the error of such an estimation is $\sim 41\%$, *i.e.* ~ 37 years (leaving aside the errors in determining the statistical parameters t_0 and β). An error $\Delta M = -0.33$ reduces this accumulation time to $t_a \simeq 42$ years. If we use $-\ln t_0 = 10.62$ and $\beta = 2.1$ obtained from the fit to the excedence rate N_{ex}, then the accumulation time for $M = 7$ is $\simeq 59$ years. These periodicities can be compared with data in Fig. 6, where Vrancea earthquakes in Vrancea with magnitude $M > 5.5$ are shown for the last two hundreds years (ROMPLUS Catalog).

A similar statistical analysis has been done for 1999 earthquakes with $M \geq 3$, which occurred in Vrancea between 1974−2004 (31 years).[18] The parameters given by that analysis are $-\ln t_0 = 9.68$ and $\beta = 1.89$ ($r = 0.54$). An estimate of the accumulation time for $M = 7$ is $t_a \simeq 34.9$ years. In comparison with that previous period we note that the seismicity rate increased and the focus geometry changed from two dimensions towards one dimension. This latter change is responsible for an increase in the accumulation time. It seems that the seismic-focal region of Vrancea, *i.e.* the spatial distribution of the seismic foci, as shown in Figs. 1, 2, has changed in the last 30 years, in comparison with the previous period 1974 − 2004. The modification consists in the

[18] B. F. Apostol, "A Model of Seismic Focus and Related Statistical Distributions of Earthquakes", Rom. Reps. Phys. **58** 583 (2006); "A Model of Seismic Focus and Related Statistical Distributions of Earthquakes", Phys. Lett. **A357** 462 (2006).

apparition of more foci, responsible for small earthquakes, distributed mainly along the depth. This modification means an increase in the seismicity rate ($-\ln t_0$) and, at the same time, an evolution of the geometry of the collective focus towards a one-dimensional geometry. The latter change leads to an increase in the parameter r, which entails an appreciable increase in the long accumulation times.

Making use of the energy distribution given by equation (7), we can estimate the mean energy released by earthquakes with magnitudes $M < M_c$

$$\overline{E} \simeq \frac{r}{1-r} E_0 e^{b(1-r)M_c} = \frac{r}{1-r} e^{a+b(1-r)M_c} \ , \tag{20}$$

or

$$\overline{E} \simeq \frac{r}{1-r} 10^{8.65+1.5(1-r)M_c} \tag{21}$$

(energy in J). Leaving aside a few great earthquakes, the upper cutoff magnitude may be taken $M_c = 5.6$, since, beyond this value there exist only a few great earthquakes in Vrancea between $1981-2018$. We get $\overline{E} \simeq 10^{12} J/yr$ ($r = 0.65$). An additional factor $\simeq \sqrt{10}$ may occur if we include the great earthquakes; the rate of the seismic moment is $\overline{E}/2\sqrt{2}$. We can see from equation (20) that the energy released by all the earthquakes with magnitude smaller than M_c is much smaler than the energy of the earthquake with magnitude M_c (by a factor $e^{-\beta M_c}$).

5 Conditional distributions

What is the probability for an earthquake to occur at time τ after the occurrence of an earthquake at time t_m? We can view this process as a fictitious seismic event which would occur at time $t = t_m + \tau$, where t_m is fixed and $0 < \tau < \infty$. The probability distribution $\sim 1/(1 + t/t_0)^2$ (equation (6)) becomes

$$P(\tau) = \frac{1 + t_m/t_0}{(1 + t_m/t_0 + \tau/t_0)^2} \frac{1}{t_0} \tag{22}$$

(properly normalized). Obviously, it is also the probability for an earthquake to occur at time τ before an earthquake which will occur surely at time t_m. If we measure time τ with negative value in the past with respect to time t_m, τ should be replaced by $\mid \tau \mid$ in equation (22). The distribution is symmetric with respect to reflection $\tau \longrightarrow -\tau$. This is a conditional distribution. The earthquake occurring at time t_m is called conventionally main shock. The earthquakes occurring at times $\tau > 0$ are called succeeding earthquakes, while those occurring at times $\tau < 0$ are called preceding earthquakes. All form a seismic activity associated with the main shock.

Similarly, the conditional probability for an earthquake to occur with an energy ε when an earthquake with energy E_m occurred (or will surely occur) is

$$P(\varepsilon) = \frac{r(1 + E_m/E_0)^r}{(1 + E_m/E_0 + \varepsilon/E_0)^{1+r}} \frac{1}{E_0} \tag{23}$$

(equation (7)). We may relate the energies ε and E_m to the times τ and t_m, respectively, through equation (4), but the sum $E_m + \varepsilon$ is not related to $t_m + \tau$ through this relation.

An associated earthquake with magnitude μ has a probability $\sim e^{-\beta\mu}$, like any earthhquake (equation (13)). The magnitude μ may be viewed either as $\mu = M_m - M$, where $M < M_m$, or $\mu = M - M_m$ for $M > M_m$; we may use $\mu = \mid M_m - M \mid$ in both cases, or, if $M > M_m$, we may inter-change $M \longleftrightarrow M_m$, *i.e.* the main shock is the earthquake with magnitude M and the

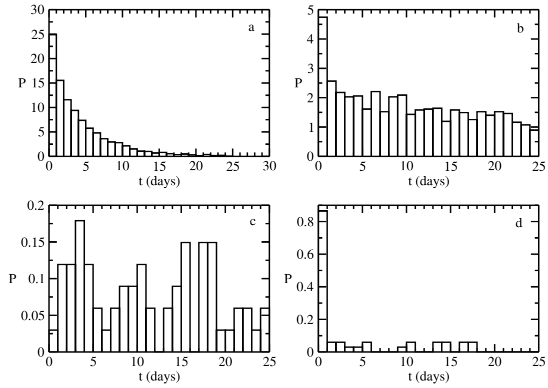

Figure 8: Next-earthquake distributions $P = P(M,t)$ (in %) for Vrancea $1981-2018$ for $M_0 \geq 3$ and $3 \leq M < 4$ (panel a), $4 \leq M < 5$ (panel b), $5 \leq M < 6$ (panel c) and $6 \leq M$ (panel d).

earthquake with magnitude M_m is an earthquake with a magnitude $M < M_m$. In any case, μ becomes the difference m between the higher magnitude and the lower magnitude; its probability distribution is $\beta e^{-\beta m}$ (which may be written also $\beta e^{-\beta|m|}$, either for $m > 0$, or $m < 0$).[19] Since $\sqrt{\overline{m^2}} = \sqrt{2}/\beta$, we may conclude that the highest earthquake associated with a main shock is by $\sqrt{2}/\beta$ lower in magnitude than the main shock. Making use of $\beta = 1.17$ given above (for $r = 0.33$) we get $\sqrt{2}/\beta = 1.2$, an estimate known as Bath's law.[20] For Vrancea, making use of $\beta = 2.26$ obtained before, we get $\sqrt{2}/\beta \simeq 0.62$.

It is worth noting the different scales of τ, ε and m. We may view these variables as small changes δt, δE and δM, such that $\delta t/t_0 = r(E/E_0)^{r-1}\delta E/E_0$, $\delta E/E_0 = be^{(b-1)M}\delta M$ and $\delta t/t_0 = \beta e^{\beta M}\delta M$; we can see that, for large energies, $\delta M \ll \delta t/t_0 \ll \delta E/E_0$ (for $r < 1$). It follows that for short times τ the energies differ much from the energy of the main shock, while higher magnitudes occur at longer times, as expected (hence the "highest aftershock" with magnitude lower by $\sqrt{2}/\beta$ than the main shock).

6 Short-term seismic activity

Let us assume $t_m \gg t_0$ in equation (22) and $\tau \ll t_m$; if we limit ourselves to $0 < \tau < t_m/2$, the time distribution becomes

$$P(\tau) = \frac{1}{\ln 2}\frac{1}{t_m/2 + \tau} \ . \tag{24}$$

This is the distribution of short-term seismic events associated with a main shock occurring at t_m. Since $\tau < t_m/2$, it includes events which are lower in energy than the main shock. The parameter t_m in equation (24) remains an empirical parameter.

In the time interval $0 < \tau < t_m/2$, measured from t_m both in the future and in the past, there may exist seismic events lower in energy than the main shock which are causaly related to the main shock. These are the aftershocks (suceeding events) and the foreshocks (preceding events). Their distribution is given by equation (24); they are expected to dominate the short-term activity in the limit $\tau \ll t_m/2$. In the same time interval $0 < \tau < t_m/2$ there may exist regular earthquakes, some with higher energy than the main shock. We can use a cutoff time t_c instead of $t_m/2$, in

[19] D. Vere-Jones, "A note on the statistical interpretation of Bath's law", Bull. Seismol. Soc. Amer. **59** 1535 (1969).

[20] M. Bath, "Lateral inhomogeneities of the upper mantle", Tectonophysics **2** 483 (1965).

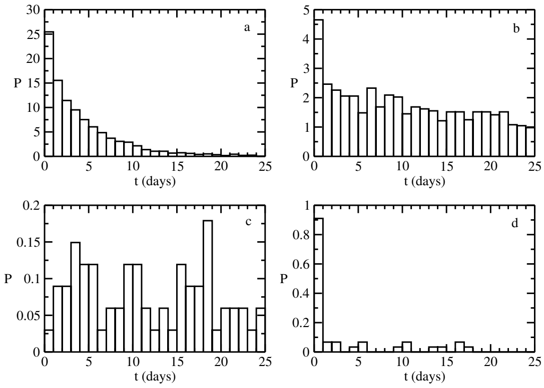

Figure 9: Next-earthquake (conditional) probability $P = P(t, M; M_0)$ (in %, Vrancea) for $3 \leq M_0 < 4$ and $3 \leq M < 4$ (panel a), $4 \leq M < 5$ (panel b), $5 \leq M < 6$ (panel c) and $6 \leq M$ (panel d).

order to include only earthquakes lower in energy than the main shock. The distribution given by equation (24) with a cutoff time t_c can be used for adjacent earthquakes (next-earthquake distribution, or inter-event time distribution, or waiting-time distribution), where in the interval $0 < \tau < t_c$ exists only one succeeding event and only one preceding event.

Similarly, from equation (23) we get the short-term energy distribution

$$P(\varepsilon) = \frac{1}{\ln 2} \frac{1}{E_m/(1+r) + \varepsilon} \tag{25}$$

for $0 < \varepsilon < E_m/(1+r)$. The energy released in the short-term seismic activity is

$$\overline{\varepsilon} = \frac{1 - \ln 2}{\ln 2} \cdot \frac{E_m}{1+r} \ . \tag{26}$$

Making use of the magnitude partition $M = M_m + \mu$, we get

$$P(\mu) = \frac{1}{\ln 2} \frac{1}{1/\beta + \mu} \tag{27}$$

from the magnitude distribution $\beta e^{-\beta M}$ (equation (13)), where M_m is the magnitude of the main shock and μ is the magnitude of the short-term earthquake, $0 < \mu < 1/\beta$. However, the scale of these variables (τ, ε, μ) are very different.

The short-term distributions are applicable to associated earthquakes lower in energy than the main shock. The associated earthquakes comparable in energy with the main shock are interesting. From equation (24), such an earthquake occurs at time $\tau \simeq t_m/2$, as expected. Let us write equation (4) as

$$1 + t/t_0 = (E/E_0)^r (1 + E_0/E)^r \simeq (E/E_0)^r (1 + rE_0/E) \tag{28}$$

for E close to $E_m \gg E_0$. Let us estimate the variation

$$\begin{aligned} &\delta \left[(E/E_0)^r \cdot rE_0/E \right]_{E_m} = \\ &= \left[\frac{r}{E_0} \left(\frac{E_m}{E_0} \right)^{r-1} \frac{rE_0}{E_m} - \left(\frac{E_m}{E_0} \right)^r \frac{rE_0}{E_m^2} \right] \delta E = \\ &= r \left(\frac{E_m^{r-2}}{E_0^{r-1}} \right) (r-1)\, \delta E \ ; \end{aligned} \tag{29}$$

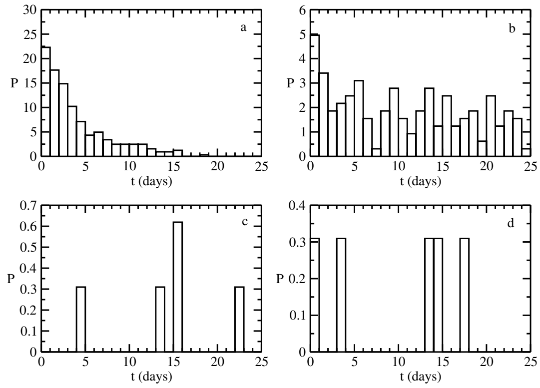

Figure 10: Same as in Fig. 9 for $4 \leq M_0 < 5$.

we can see that for $0 < r \ll 1$ the contribution of the factor $(E/E_0)^r$ to this variation is much smaller than the contribution of the factor rE_0/E. it follows that we may write approximately

$$t_m/t_0 = (E_m/E_0)^r \ , \qquad \frac{\tau}{t_0} = r(E_m/E_0)^r \frac{E_0}{E} \ . \tag{30}$$

For $E \simeq E_m$ we have $\tau/t_0 \simeq r(E_m/E_0)^{r-1} \ll 1$. Comparing with

$$\frac{\tau}{t_0} = r(E_m/E_0)^{r-1}\frac{\varepsilon}{E_0} \tag{31}$$

used above, we get $\varepsilon E = E_0 E_m$. Using the approximation procedure described above in the energy distribution $1/(1 + E/E_0)^{1+r}$, we get the distribution

$$P(E) \simeq \frac{1}{E_m}\frac{1}{(1+r)E_0 + E} \cdot \frac{1}{E} \ , \tag{32}$$

where the integration has been extended from $\simeq E_m$ to ∞. From equation (30) we can see that the rate of released energy is[21]

$$\frac{dE}{d\tau} = rt_0 E_0 (E_m/E_0)^r \frac{1}{\tau^2} \ . \tag{33}$$

The change $E \longrightarrow 1/\varepsilon$ implies that the energy goes as e^{-bM} and the magnitude probability goes as $e^{\beta M} = e^{\beta M_m} e^{\beta(M-M_m)}$. It follows the distribution $\beta e^{-\beta m}$ for the magnitude difference $m = M_m - M > 0$. The approximation procedure described above has a very limited range of validity.

7 Omori law

Let us use a cutoff time t_c in equation (24); we have the series of approximations

$$P(\tau) \sim \frac{1}{t_c + \tau} \sim 1 - \tau/t_c \simeq e^{-\tau/t_c} \tag{34}$$

[21] T. Utsu, "A statistical study on the occurrence of aftershocks", Geophys. Mag. **30** 521 (1961); D. Sornette, C. Vanneste and L. Knopoff, "Statistical model of earthquake foreshocks", Phys. Rev. **A45** 8351 (1992).

for $\tau/t_c < 1$. Since $1/(1+\tau/t_c)$ is larger than $e^{-\tau/t_c}$, it is reasonable to associate the distribution $e^{-\tau/t_c}$ with aftershocks (and foreshocks), which have a causal relation with the main shock. We may extend this distribution to $\tau \longrightarrow \infty$, where it decreases appreciably in comparison with $1/(1+\tau/t_c)$; the latter may include regular earthquakes for longer times. Therefore, the distribution

$$p(\tau) = \alpha e^{-\alpha\tau} \tag{35}$$

with $\alpha = 1/t_c$ may be taken tentatively as the distribution of the accompanying seismic activity, consisting of aftershocks and foreshocks, of a main shock. If the main shock generates aftershocks and foreshocks, the latter may generate their own foreshocks and aftershocks, in a self-replication (or branching, or avalanche) process.[22] Then, we may view the distribution given by equation (35) as a distribution in the variable α, *i.e.*

$$\frac{dP}{d\alpha} \sim e^{-\alpha\tau} \tag{36}$$

and

$$P \sim \frac{1}{\tau} \ . \tag{37}$$

This is known as Omori's distribution (Omori's law).[23] In fact, the integration in equation (36) implies a cutoff, which leads to

$$P(\tau) \sim \frac{1}{t_c + \tau} \ , \tag{38}$$

i.e. we recover the original distribution given by equation (34). This is why short-term distributions are called Omori-type distributions. The exponential distribution given by equation (35) is called generating distribution.

In a self-replication process the probability P generated by a distribution p is given by

$$P = p + spP \ , \tag{39}$$

where s is the rate of continuity of the process; hence,

$$P = \frac{p}{1 - sp} = \frac{1}{1/p - s} \ , \tag{40}$$

which is an Euler's transform between sp and $-sP$ ($p = P/(1+sP)$). If we expand p in a power series, we get the Omori-type law; for instance, with the exponential generating distribution, we get

$$P = \frac{1}{(1-\alpha s)/\alpha + \tau} \ ; \tag{41}$$

if $\alpha s < 1$, the process slows down.

8 Next-earthquake distributions

The probability density of N serial events denoted by i and occurring at time t_i can be written as $N^{-1}\sum_i \delta(t_i - t)$. Similarly, the pair distribution of nearest-neighbours separated by time t is given

[22] B. F. Apostol, "Euler's transform and a generalized Omori's law", Phys. Lett. **A351** 175 (2006).
[23] F. Omori, "On the after-shocks of earthquakes", J. Coll. Sci. Imper. Univ. Tokyo **7** 111 (1894).

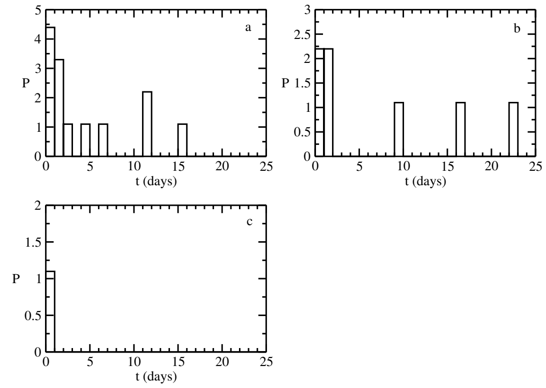

Figure 11: Same as in Figs. 9 and 10 for $5 \leq M_0 < 6$.

$$P(t) = \frac{dN}{Nd\tau} = \frac{1}{N}\sum_i \delta(t_{i+1} - t_i - t) \ . \tag{42}$$

This function is also known as the next-earthquake distribution, recurrence, or waiting time distribution, or inter-event time distribution.

The earthquakes occurred in Vrancea between $1981 - 2018$ with $M \geq 3$ (ROMPLUS Catalog) are distributed in Fig. 7 on the inter-event time (panel a); the corresponding probabilities $P(t)$ (in %) are shown in Fig. 7 panel b (time is measured in days on the abscissa). We can see that the rate of occurrence per day of the next earthquake follows a power-law time dependence (Omori-type law) over a time window of about 25 days. The distribution is fitted with the law $a/(b+t)$, where $a = 1066.45$, $b = 1.15$ and t denotes the time (coefficient of determination $R = 0.96$).[24] The mean time for $P(t)$ is $\simeq 5.89$ days, and the variance is $\sigma = 9.55$ days. We note the presence of the cutoff time b. To check whether this behavior is biased by the aftershock sequences of the strongest seisms of the investigated time period (four earthquakes with magnitude ≥ 6.0), it was considered also two shorter time intervals: $1991 - 2018$, avoiding the aftershock sequences of three events with $M > 6.0$ (occurred on August 30, 1986, $M = 7.1$, May 30, 1990, $M = 6.9$, and May 31, 1990, $M = 6.4$), and $2005 - 2018$, when no earthquake larger than magnitude 5.6 occurred. The results show very similar next-earthquake probabilities of occurrence, in all three cases ($a = 881.85$, $b = 1.25$, $R = 0.94$ for $1991 - 2018$ and $a = 492.6$, $b = 1.16$, $R = 0.93$ for $2005 - 2018$).

From the practical standpoint a relevant question in short-term earthquake forecasting seems to be "what happens next?". Let us assume that an earthquake occurs at time t_0 and the next one occurs at some time t measured with respect to t_0. We can define a distribution $P(t)$ of these next earthquakes, and determine it from a set of relevant statistical data. Once determined, it can be used for estimating the time probability of occurrence of the next earthquake, based on the principle "what happened will happen again". For instance, from Fig. 7, panel b, we can say that the probability for an earthquake with magnitude $M \geq 3$ to occur in the next day after an earthquake with magnitude $M \geq 3$ have occurred is $\simeq 27\%$. Let the earthquakes be labeled by some generic parameter x, like magnitude, location, depth, etc. Then, we may distribute the next earthquakes with respect to x, and introduce the time probability distribution $P(x, t)$ of the next earthquake characterized by parameter x occurring at time t. Another distribution $P(x, t \mid x_0)$ may also be introduced with respect to an earthquake labeled by parameter x_0, which

[24] The coefficient of determination R is defined by $R^2 = 1 - \sum_i (d_i - f_i)^2 / \sum_i (d_i - \overline{d})^2$, where d_i denote the data, f_i denote the fit and $\overline{d}$ is the mean value of the data.

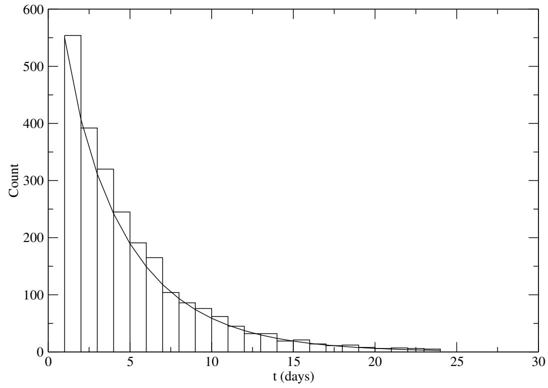

Figure 12: Next-earthquake distribution for Vrancea $1981-2018$, $M \geq 3$ fitted with $ae^{-0.22t}/t^{0.12}$, $a \simeq 683$ $(t \geq 1)$.

is a conditional probability. The procedure may obviously be extended, by introducing, similarly, the probability distributions $P(x, t \mid x_{01}, x_{02}, ...)$, or $P(x_1, x_2, ..., t \mid x_{01}, x_{02}, ...)$, which resemble the hierarchy of n-point correlation functions. Characteristic scale time or size, or correlation range could be identified from the statistical analysis of such functions, providing the statistical set of data is large enough, which may shed light on the statistical patterns of a seismic activity. The statistics is rather poor, in general, precisely for those range of x where the estimation of the seismic hazard and risk is most interesting, like, for instance, for x corresponding to high values of magnitude M. The generic parameter x in the analysis of the inter-event time distributions described here is the magnitude M.

The probabilities $P(M, t)$ for $M_0 \geq 3$ for Vrancea $1981-2018$ are shown in Fig. 8 for $3 \leq M \leq 4$, $4 \leq M \leq 5$, $5 \leq M \leq 6$, and $6 \leq M$ (panels a, b, c and d, respectively). First, we note that the inter-event distributions $P(M, t)$ for Vrancea exhibit a characteristic decrease in time, with the highest probability of next-earthquake occurrence in the same day as the reference earthquake, at least for small magnitudes ($M < 5$). Then, we note the decreasing maximum values of these probabilities ~ 22.7 for $3 \leq M < 4$, ~ 2.75 for $4 \leq M < 5$, while the probability $P(M, t)$ vanishes practically for $M > 5$. Also, it is worth noting that $P(t)$ and $P(3 \leq M < 4, t)$ are similar, obeying, Omori-type power laws, at least for short times, while the distributions become gradually irregular, exhibiting large fluctuations on increasing magnitude above $M = 4-5$. The statistics becomes poor for higher magnitude ($M > 5$), as expected. A correlation time of $20-25$ days can be estimated, after which the probabilities decrease appreciably (below 1), as well as a size correlation of $M = 4-5$, above which the distributions acquire very small values, and are very irregular. The null hypothesis was tested on these distributions, by comparing the results of the first half of data with those derived from the second half of data.

The conditional probabilities including the magnitude M_0 of the former earthquake $\mathrm{P}(M, t \mid M_0)$ are shown in Figs. 9-11 for $3 \leq M_0 < 4$ (Fig. 9), $4 \leq M_0 < 5$ (Fig. 10) and $5 \leq M_0 < 6$ (Fig. 11). The first observation is that distributing the time-magnitude events with respect to the magnitude M_0 does not change practically the characteristic time-decreasing behavior of the next-earthquake activity for small magnitudes. It can be seen in Fig. 2 (panel a) and Fig. 9 that $P(M, t)$ and $P(M, t \mid 3 \leq M_0 < 4)$ are very similar, while considerable differences appear for $P(M, t \mid 4 \leq M_0 < 5)$, even for small magnitudes $3 \leq M < 4$. This reflects again the size correlation $M = 4-5$, and makes useless the estimation of the confidence levels for higher magnitudes, as the corresponding distributions are affected by large fluctuations. Higher-order correlation functions (as well as higher-magnitude analysis, or enlarging the magnitude gap $\Delta M = 1$) reduce considerably the statistical set, thus exhibiting a poor confidence.

9 "Scaling" and cutoff parameters

The next-earthquake distribution given by equation (42) have an interesting scaling property; it may be written as

$$P(t) = \frac{dN}{Nd\tau} = \frac{R}{N}\sum_i \delta(Rt_{i+1} - Rt_i - Rt) \ , \tag{43}$$

where R is a scaling factor. This equation seems to show that $P(t) = R \cdot P(Rt)$. The only function which has this property is the Omori distribution $P(t) = 1/t$. Such a representation would be true if $t_{i+1} - t_i = t$ would go from zero to infinity, because R takes any value in this range. Unfortunately, in analyzing the experimental data there is always a lower cutoff time t_c (an upper cutoff time is not important since $P(t)$ vanishes at infinity). In addition, a cutoff time t_0 is needed in a theoretical description.

The substantial behaviour of the function $P(t)$ pertains to the short-time seismic activity. The short-time distributions are Omori-type laws $\sim 1/(t_c + t)$. The cutoff time t_c depends on the main shock. We may use a distribution averaged over a short time Δt from a lower cutoff time t_l; it is given by

$$\begin{gathered} \tfrac{1}{\Delta t}\int_{t_l}^{t_l+\Delta t} dt_c \tfrac{1}{t_c+t} = \tfrac{1}{\Delta t}\ln\tfrac{t_l+\Delta t+t}{t_l+t} = \\ = \tfrac{1}{\Delta t}\ln\left(1+\tfrac{\Delta t}{t_l+t}\right) \simeq \tfrac{1}{t_l+t} \ ; \end{gathered} \tag{44}$$

we can see that such an averaged distribution depends only on the lower cutoff time. Usually, this cutoff time is associated with a lower cutoff magnitude M_l, through

$$t_l/t_0 \simeq e^{\beta M_l} \ . \tag{45}$$

Using the fitting parameters $-\ln t_0 = 11.32$ (in years) and $\beta = 2.26$ obtained before for Vrancea seismic activity between $1981 - 2018$ with $M_l = 3$, we get $t_0 \simeq 4.6 \times 10^{-3}$ days and $t_l \simeq 5.8$ days. The probability given by equation (43) is

$$\frac{\Delta N}{N} = \frac{\Delta t}{t_l + t} \ , \tag{46}$$

where Δt is the time step. If we take $R = t_l^{-1}$, we get $\Delta N/N = R\Delta t/(1 + Rt)$; since $R \ll 1$, the data colapse and the quality of the fit is better. Equation (46) can be written as

$$\begin{gathered} \tfrac{\Delta N}{N} = \tfrac{1}{t_l/\Delta t + t/\Delta t} \sim \tfrac{1}{1+\frac{\Delta t}{t_l}\frac{t}{\Delta t}} \simeq \\ \simeq \tfrac{1}{(1+t/\Delta t)^{\Delta t/t_l}} \sim \tfrac{1}{t^{\Delta t/t_l}} \ , \end{gathered} \tag{47}$$

since $\alpha = \Delta t/t_l \ll 1$ (usually $\Delta t = 1$ day). Power laws $1/t^{\alpha}$ with $0 < \alpha < 1$ are used for fitting next-earthquake distributions, multiplied by Poisson laws $e^{-\gamma t}$ in order to account for the rapid decaying of these distributions to infinity (the probability density $t^{-\alpha}e^{-\gamma t}$ is called gamma distribution). The fits are used in the form $\Delta N/N \sim (Rt)^{-\alpha}e^{-\gamma Rt}$. The property $P(t) = A \cdot P(Bt)$ for functions $P(t)$, where A, B are constants, is called scaling property. For a global seismic activity $\alpha \simeq 0.33$,[25] while $\alpha \simeq 0.25$ for the seismic activity in Vrancea, $1974 - 2004$ ($M \geq 3$).[26]

[25] A. Corral, "Long-term clustering, scaling, and universality in the temporal occurrence of earthquakes", Phys. Rev. Lett. **92** 108501 (2004); "Renormalization-group transformations and correlations of seismicity", Phys. Rev. Lett. **95** 028501 (2005).

[26] B. F. Apostol, L. C. Cune and M. Apostol, "Scaling and universal power laws in time series of seismic events", Roum. J. Phys. **53** 593 (2008).

The same function is used in Fig. 12 with $\alpha \simeq 0.12$ and $\gamma \simeq 0.22$ to fit the next-earthquake distribution for Vrancea 1981 − 2018 ($M \geq 3$, $t \geq 1$). The value $\alpha \simeq 0.12$ agrees well with $\alpha = \Delta t/t_l = 1/5.8 \simeq 0.17$.

10 Concluding remarks

It seems that Statistical Seismology begun with Wadati in 1932, who viewed the earthquakes as probabilistic events, distributed by a power law $E^{-\alpha}$ in energy, where $1.7 < \alpha < 2.3$ for earthquakes in Japan.[27] Later,[28] power (and exponential) laws for earthquake distributions suggested a connection between earthquakes and the critical phenomena in statistical mechanics of the phase transitions.[29]

This association is of a very limited validity. First, the phase transitions exhibit a driving (tuning) parameter which approaches a critical value (like temperature, density), while the earthquakes do not exhibit such a parameter; so, they were called "self-organized" critical phenomena. Second, the laws of the critical phenomena exhibit the scaling property, *i.e.* the absence of a characteristic scale, like the power laws. The laws of the earthquakes do not exhibit such scaling laws, because of the cutoff parameters, like t_0, E_0, etc. The cutoff parameters are essential for earthquakes, because they ensure the finite values of the distributions for small values of the variables (*e.g.*, the energy). Most of the seismic events occur with small values of energy, such that the distribution value in this limit is extremely important. The threshold values of the parameters are critical in fitting the experimental data for earthquakes. The scaling property may make the power laws to look universal, a property which is not fully enjoyed by earthquakes. Moreover, the critical phenomena are cooperative phenomena, which imply interaction and correlations between many constituents of the macroscopic bodies, a property which is difficult to be recognized for earthquakes. In order to sustain the self-organized criticality the statistical mechanics has been pushed to the critical behaviour of dynamical, *i.e.* mechanical, systems. To this end several models have been devised, like the Burridge-Knopoff model,[30] the Carlson-Langer model,[31] the Olami-Feder-Christensen model.[32] So, the self-organized criticality has been replaced gradually by stochastic models of earthquakes (the so called point processes[33]), which view the earthquakes as phenomena which occur with probabilities. (This general view is adopted here).

With the advent of the stochastic approach the temporal distributions for earthquakes have been brought to attention, especially the Omori law. The accompanying seismic activity of foreshocks and aftershocks is related and, probably, to some extent even caused, by the main shock, but still it has a stochastic character. It indicates correlations in the seismic activity, to some extent (which brings again in discussion a self-organized criticality).

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[27]K. Wadati, "On the frequency distribution of earthquakes", J. Meteorol. Soc. Japan **10** 559 (1932).

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Omori's law brought to the forefront of research the next-earthquake distributions (inter-event time distributions). These are conditional probabilities of occurrence of an adjacent earthquake at the time τ ellapsed from the occurrence of another earthquake, both earthquakes with various characteristics. The exponential distribution $\sim e^{-\gamma\tau}$ may subsist for long times, when replication diminished and the events are rare. At the same time, it may be connected with $1-\gamma\tau = 1-\gamma\tau_0(\tau/\tau_0) \simeq (1+\tau/\tau_0)^{-\gamma\tau_0}$ for small times, where τ_0 is a characteristic time. Leaving aside this time, we may use the function $\tau^{-\alpha}e^{-\gamma\tau}$ as a fit function for the next earthquake distributions, where α and γ are fitting parameters. This is the gamma distribution introduced by Corral.[34] It is worth noting the equation

$$\begin{gathered}(1+\tau)^{-\alpha}e^{-\gamma\tau} = (1+\tau)^{-1}e^{(1-\alpha)\ln(1+\tau)-\gamma\tau} \simeq \\ \simeq (1+\tau)^{-1}e^{(1-\alpha-\gamma)\tau}\end{gathered} \tag{48}$$

for small values of τ, which leads to Omori-type law $1/(1+\tau)$ ($i.e.$ $1/(1+\tau/\tau_0)$) for $\alpha+\gamma \simeq 1$. Indeed, it is reasonable to assume that the short-term seismic activity involves correlations and self-replication, reflected by Omori-type laws, where the aftershocks and foreshocks are only a particular way of viewing the short-term inter-event temporal seismic activity.

The functions used to fit the short-term seismic activity involve scaling parameters. In general, a function $f(x,y,...)$ which enjoys the scaling property obeys an equation $f(x,y,...) = Af(\Lambda_x x, \Lambda_y y,...)$, where $\Lambda_{x,y,...}$ are scale factors and A is a constant. If the scale is reduced, the spread of the data is reduced (the data "collapse"), and the fit quality increases; in addition, the data distribution look the same, such that the fitting function looks universal. Fitting the gamma distribution to experimental data for space-time-energy (magnitude), etc variables (the so-called dynamical scaling models) and the deviations of these fits is the substance of the main line of research in present Statistical Seismology.[35] The deviations occur especially for small values of the data, where the statistics is already poor, and are very sensitive to small values of the cutoffs (threshold values), where a non-stochastic behaviour may be expected.

11 Point processes

A special type of statistical analysis ($i.e.$, fitting approach) is of much interest today; it is called the point-process approach. First, we recall that probabilities imply, to some extent, an "insufficient reason", or a "principle of indifference", like in a perfect molecular chaos.[36] On the other hand, we would prefer the probabilities to be meaningful to some extent. Such probabilities are conditional probabilities.[37] Let us assume that N_0 events happen in an interval from 0 to T; then the probability for an event to occur in the range N to $N+dN$ after N events occurred already is

$$dP = \frac{dN}{N_0 - N} \; ; \tag{49}$$

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the number of events N is

$$N = N_0\left(1 - e^{-P}\right) \ . \tag{50}$$

Let us define the density of probability $\lambda = dP/dt$ and the frequency density $f = dN/N_0dt$ with respect to a variable t; equation (49) reads[38]

$$\lambda = \frac{f}{1 - \int_0^t f dt'} \tag{51}$$

and equation (50) gives

$$f = \lambda e^{-\int_0^t \lambda dt'} \ ; \tag{52}$$

for a constant λ the frequency density is the Poisson distribution $f = \lambda e^{-\lambda t}$.

The density f and the cummulative probability $F = \int_0^t f dt'$ are known experimentally; then, we may derive the conditional probability density λ; it may be used for inferrence. However, the events are discrete, not continuous. We can divide the interval T in small segments Δt_i, such that in every segment there exists one event or none. Such a sequence is a point process. The probability density λ_i becomes a random variable, which we can compute from data. The frequency density for each segment Δt_i is

$$f_i = \lambda_i e^{-\lambda_i \Delta t_i} \ ; \tag{53}$$

the total "probability"

$$\prod_i f_i = \prod_i \lambda_i \cdot e^{-\sum_i \lambda_i \Delta t_i} \ , \tag{54}$$

or its logarithm

$$L = \sum_i \ln \lambda_i - \sum_i \lambda_i \Delta t_i = \sum_i \ln \lambda_i - \int_0^t \lambda dt' \ , \tag{55}$$

should be maximal, with respect to the parameters defining a theoretical (fitting model) λ; L is called likelihood. Various model λ are used to fit seismological data.[39]

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