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On Quantum Electrodynamics

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Abstract

We give here a reformulation of the Quantum Electrodynamics (QED). It is inspired by the original Bethe treatment of the Lamb shift (Phys. Rev. **72** 339 (1947)), of using the Compton-wavelength cutoff. Unfortunately, this reformulation lacks the accurate technical means of disentangling the physical effects of the charge self-energy and photon fluctuations. Therefore, a comparison with the experimental data of specific, relevant instances like the anomalous magnetic moment of the electron or the Lamb shift, is limited to order-of-magnitude estimates. Specifically, many points of the formulation of the QED are critically discussed. First, it is shown that the insistence of keeping the longitudinal degrees of freedom of the electromagnetic interaction in the manifestly covariant formulation of the QED is not warranted. This technical point leads to infrared divergences (infrared catastrophe), which, as it is well known, are discarded in the QED, on physical grounds. So, their preservation in the technical treatment of the QED is not, actually, harmful. Then, the insistence on the relativistically covariant fields and their interaction is shown to be impossible and not necessary, nor warranted by experiment. The quantum-mechanical evolution equation is not relativistically invariant. The quantum fields are global, so their relativistic invariance is meaningless. From the perspective of the comparison with the experimental data this point is also not harmful, as the experiments are not, indeed, amenable to a relativistically invariant formulation. Thereafter, it is shown that the Dirac equation describes an undetermined Zitterbewegung, which leads to ultraviolet divergences, as a consequence of including self-interaction for (delocalized) point charges. We show that the physical motion belongs to a boson field of charged particles with spin 1/2 (electrons). The equation of motion, the lagrangian and the hamiltonian of this field are derived, and the quantum-mechanical series of perturbation of this field is analyzed. The boson motion is averaged over the local Zitterbewegung, the latter being limited by the Compton wavelength. Unfortunately, the exact separation of this physical motion from the undetermined Zitterbewegung of the Dirac equation can be done only by using approximately the Compton-wavelength cutoff, which does not lead to accurate numerical results, comparable with experimental data. The accurate technical disentangling of the undetermined Zitterbewegung from the averaged motion of the boson field is done by the renormalization and regularization technique, which imply exact rules. Although the unicity of the latter can be discussed, a standard version of regularization has emerged, such that the numerical results of the QED can be compared with experiment. Being a proper and exact technique (though unsafe and risky), it is no surprise that the agreement with experiment is excellent. We hope to help clarify herein the strange and excellent technical success of this lame duck which is the standard QED. It is obvious that a wavelength cutoff avoids divergences; the problem is to render convincing such a cutoff.

Introduction. The renormalization and regularization techniques of the Quantum Electrodynamics have been seriously criticized, by the very founders of this discipline. Dirac said that they have an "illogical character",[1] and Feynman said that they are a "dippy shell game"[2]. In the light of such a criticism it may come as a surprise the excellent agreement of the numerical results of the Quantum Electrodynamics with experiment. Nobody has explained this agreement in these circumstances. Dirac said that, very likely, it is a "fluke" (same Reference). We give here arguments to help understand this agreement.

The renormalization procedure[3] is reasonable and may be acceptable, if it could be applied. Unfortunately, it depends on the regularization techniques, which are not unique.[4] The identification of the infinities with the same formal structure, as done originally by Schwinger, for instance, though a very unsafe and risky procedure, may be accepted as a working one, at least. However, it works in finite orders of the perturbation theory, and fails for the entire perturbation series, where the theory becomes inconsistent.[5]

We show herein that the formulation of the Quantum Electrodynamics exhibits many other inconsistencies, like the use of a manifestly covariant formalism in spite of the non-relativist character of the quantum-mechanical evolution equation, inclusion of the longitudinal degrees of freedom of the electromagnetic field (which do not participate in dynamics), or the use of the Dirac equation for the undetermined Zitterbewegung.[6] However, these inconsistencies turn out to be ineffective, mainly because they are applied to the undetermined Zitterbewegung which is removed by renormalization and regularization (except for the infrared divergences, associated with the longitudinal electromagnetic field, which are simply discarded, on sound physical grounds[7]).

We show herein that the undetermined Zitterbewegung is the origin of the ultraviolet divergences. Indeed, it implies the charge and fields infinite self-interaction, extended over small distances, of the order of the Compton wavelength. Also, we show that an additional, physical motion is superposed over the Zitterbewegung, governed by a boson field derived from the Dirac equation. The disentanglement of the boson motion from the Zitterbewegung is made by a cutoff of the order of the Compton wavelength. Thus, the boson theory provides a quantum electrodynamics free of divergences, though only with order-of-magnitude estimates of the numerical results. A similar disentanglement is performed by the renormalization and regularization technique of the Quantum Electrodynamics, which, however, being proper and exact, leads to accurate numerical results.

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[1] P. A. M. Dirac, "The evolution of the physicist's picture of Nature", Am. Sci. **208** 45 (1963).

[2] R. P. Feynman, *QED. The Strange Theory of Light and Matter*, Princeton University Press, NJ (1986).

[3] W. Pauli and M. Fierz, "Zur Theorie der Emission langwelliger Lichtquanten", Nuovo Cim. **15** 167 (1938); H. Kramers, "Quantentheorie des Elektrons und der Strahlung", in *Hand. und Jahrbuch der Chemische Physik*, I, part 2, Leipzig (1938); "Die Wechselwirkung zwischen geladenen Teilchen und Strahlungsfeld", Nuovo Cim. **15** 108 (1938); "Non-relativistic quantum electrodynamics and correspondence principle", in *Rapports et Discussions du 8e Congres Solvay*, 1948, Brussels, R. Stoop (1950); *Collected Scientific Papers*, North Holland, Amsterdam (1956).

[4] W. Pauli and F. Villars, "On the invariant regularization in relativistic quantum theory", Revs. Mod. Phys. **21** 434 (1949).

[5] L. Landau, "On the Quantum Theory of Fields", in *Niels Bohr and the Development of Physics*, ed. W. Pauli, Pergamon Press (1955) and references therein; L. Landau and E. Lifshitz, Course of Theoretical Physics, vol. 4, Quantum Electrodynamics (V. Berestetskii, E. Lifshitz, l. Pitaevski), Butterworth-Heinemann (1971); M. Gell-Mann and F. E. Low, "Quantum Electrodynamics at small distances", Phys. Rev. **95** 1300 (1954).

[6] E. Schroedinger, "Ueber die kraeftefreie Bewegung in der relativistischen Quantenmechanik", Sitzungsberichte der Preussischen Akademie der Wissenschaften, Berlin, 418 (1930); "Zur Quantendynamik des Elektrons", Sitzungsberichte der Preussischen Akademie der Wissenschaften, Berlin, 63 (1931)

[7] F. Bloch and A. Nordsieck, "Note on the radiation field of the electron", Phys. Rev. **52** 54 (1937).

Interaction. The classical electromagnetic field in vacuum is governed by the Maxwell equations

$$\partial_\mu \partial^\mu A^\nu = \frac{1}{c^2}\frac{\partial^2 A^\nu}{\partial t^2} - \Delta A^\nu = \frac{4\pi}{c} j^\nu ed \tag{1}$$

(with usual notations), where $A^\mu = (\Phi, \mathbf{A})$ are the electromagnetic potentials, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ are the components of the electromagnetic field, $j^\mu = (c\rho, \rho\mathbf{v})$ is the current density, ρ is the charge density and c is the speed of light in vacuum; Φ is the scalar potential, $\mathbf{A}$ is the vector potential, $\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - grad\Phi$ is the electric field, $\mathbf{H} = curl\mathbf{A}$ is the magnetic field and $\mathbf{v} = d\mathbf{r}/dt$ is the charge velocity. The coordinates are $x^\mu = (ct, \mathbf{r})$, where t is the time and $\mathbf{r}$ is the position ($\partial_\mu = \partial/\partial x^\mu$); the metric tensor is $g^{\mu\nu} = g_{\mu\nu} = (+, -, -, -)$. The potentials satisfy the Lorenz-gauge condition $\partial_\mu A^\mu = 0$ ($\frac{1}{c}\frac{\partial \Phi}{\partial t} + div\mathbf{A} = 0$), such that the Maxwell equations can also be written as $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu$; this condition reflects the charge conservation $\frac{\partial \rho}{\partial t} + div\mathbf{j} = 0$. The classical equations of motion of a mass density μ with a charge density ρ in an electromagnetic field are

$$\mu c \frac{du^\mu}{ds} = \frac{\rho}{c} F^{\mu\nu} u_\nu \ , \tag{2}$$

where $u^\mu = dx^\mu/ds$ are the four-velocities, or, since $j_\mu = \rho u_\mu \frac{ds}{dt}$,

$$\mu c \frac{du^\mu}{dt} = \frac{1}{c} F^{\mu\nu} j_\nu \ , \tag{3}$$

where $s^2 = x^\mu x_\mu = c^2 t^2 - \mathbf{r}^2$ is the squared relativistic distance.

The problem of the classical Electrodynamics is to solve the coupled equations (1) and (2). Obviously, this problem leads to self-interaction, which is unphysical. The standard approach to this problem is a perturbation-theory series, where higher-order terms (starting even with the second order) should be discarded as unphysical. As it is well known,[8] a point charge has an infinite self-energy, and the reaction force (*i.e.* the force generated upon a charge by its own radiated field) leads to ambiguities and divergences; it was recognized that the reaction force is only a small uncertainty (a damping) in the motion of the charge.[9] The divergences occur for small distances, of the order of the classical electromagnetic radius of the charge; for an electron this radius is $r_0 = e^2/mc^2 \simeq 2.8 \times 10^{-13} cm$, where e is the electron charge and m is the electron mass. It follows that the changes in wavelengths λ should be larger than r_0 ($\Delta\lambda \gg r_0$) and the fields should be limited by $\mid e \mid (E, H) r_0 \lesssim mc^2$. Therefore, the problem of the interaction of the electrical charges with the electromagnetic field exhibits a basic limitation. However, it is worth noting that this limitation is associated with a classical point charge with a structureless motion. The Dirac equation endows the motion with a special structure, which may give a sense to a part of the self-interaction. This problem, which is termed below the boson theory of the Dirac equation, is discussed herein.

It is worth noting in this context that the proper formulation of the interaction problem requires a first-order contribution only (in a perturbation-theory approach), which treats the particles and the field (the interaction) as separate entities. However, in many-particle ensembles the effect of the rest of the particles may look, formally, as a self-interacting field (interaction), mediating the interaction between any pair of particles, as, for instance, in the well-known random-phase approximation. Similary, the same effect of the rest of particles leads to a quasiparticle self-energy (effective mass), as if the quasiparticle would interact with itself.

[8]See, for instance, F. Rohrlich, *Classical Charged Particles*, World Scientific (2007).

[9]L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 2, *The Classical Theory of Fields*, Elsevier (1975); M. Apostol, "Damping (reaction) force for a charge in high-intensity radiation field", J. Theor. Phys. **262** (2017).

Quantization. The photon. The Maxwell equations (1) can be derived from a classical lagrangian and a hamiltonian formalism can be set up for the electromagnetic field (the same is true for charges). The electromagnetic field has an energy

$$W_{em}=\frac{1}{8\pi}\int d\mathbf{r}\left(E^2+H^2\right) \quad , \tag{4}$$

a momentum

$$\mathbf{G}=\frac{1}{4\pi c}\int d\mathbf{r}\,(\mathbf{E}\times\mathbf{H}) \tag{5}$$

and an interaction with charges given by

$$V=\frac{1}{c}\int d\mathbf{r}j_\mu A^\mu=\int d\mathbf{r}\left(\rho\Phi-\frac{1}{c}\mathbf{jA}\right) \quad . \tag{6}$$

For charges and currents with their own field the total energy is conserved (it does not depend on time); external fields may be included. The effects of the interaction are viewed in a perturbation-theory scheme. Let us introduce the Fourier decompositions

$$\begin{gathered}\mathbf{A}_t=\sum_{\mathbf{k}}c\sqrt{\tfrac{2\pi\hbar}{\omega_k}}\left(\mathbf{e}_{\mathbf{k}}a_{\mathbf{k}}+\mathbf{e}_{-\mathbf{k}}a^*_{-\mathbf{k}}\right)e^{i\mathbf{kr}} \ , \\ \mathbf{A}_l=\sum_{\mathbf{k}}\tfrac{i\mathbf{k}}{\omega_k k}\left(\dot{f}_{\mathbf{k}}+\dot{f}^*_{-\mathbf{k}}\right)e^{i\mathbf{kr}} \ , \\ \Phi=\sum_{\mathbf{k}}\left(f_{\mathbf{k}}+f^*_{-\mathbf{k}}\right)e^{i\mathbf{kr}} \ ,\end{gathered} \tag{7}$$

where $\mathbf{A}=\mathbf{A}_t+\mathbf{A}_l$, $\omega_k=ck$ and $\mathbf{e}_{\mathbf{k}}$ are real vectors, each perpendicular to its own $\mathbf{k}$; $\hbar$ is Planck's constant. We can see that the vector potential is decomposed into a transverse part ($\mathbf{A}_t$) and a longitudinal part ($\mathbf{A}_l$), and the gauge condition $\frac{1}{c}\frac{\partial\Phi}{\partial t}+div\mathbf{A}=0$ is satisfied. Polarization labels can be included in the coefficients. According to Maxwell equations $\mathbf{A}$ is a polar vector, and changes sign under the temporal inversion. These symmetries are not incorporated in equations (7). Since we use these equations for quantization, these symmetries are transferred upon the wavefunctions. (In the original Dirac theory of radiation[10] these symmetries are preserved).

It is convenient to introduce the notations

$$\mathbf{A}_{\mathbf{k}}=\mathbf{e}_{\mathbf{k}}a_{\mathbf{k}}+\mathbf{e}_{-\mathbf{k}}a^*_{-\mathbf{k}} \ , \ \Phi_{\mathbf{k}}=f_{\mathbf{k}}+f^*_{-\mathbf{k}} \tag{8}$$

($\mathbf{A}^*_{-\mathbf{k}}=\mathbf{A}_{\mathbf{k}}$, $\Phi^*_{-\mathbf{k}}=\Phi_{\mathbf{k}}$); according to the Maxwell equation for the scalar potential, we have

$$\frac{1}{c^2}\ddot{\Phi}_{\mathbf{k}}+k^2\Phi_{\mathbf{k}}=4\pi\rho_{\mathbf{k}} \quad , \tag{9}$$

where $\rho_{\mathbf{k}}$ is the Fourier transform of the charge density ($\rho^*_{-\mathbf{k}}=\rho_{\mathbf{k}}$). The longitudinal electric field is

$$\mathbf{E}_l=-\sum_{\mathbf{k}}\frac{4\pi i\mathbf{k}}{k^2}\rho_{\mathbf{k}}e^{i\mathbf{kr}} \tag{10}$$

(from $\mathbf{E}_l=-\frac{1}{c}\frac{\partial\mathbf{A}_l}{\partial t}-grad\Phi$) and the longitudinal part of the electromagnetic energy is

$$W^l_{em}=\frac{1}{8\pi}\int d\mathbf{r}E_l^2=\sum_{\mathbf{k}}\frac{2\pi}{k^2}\rho_{\mathbf{k}}\rho_{-\mathbf{k}} \tag{11}$$

[10] P. A. M. Dirac, "The quantum theory of the emission and absorption of radiation", Proc. Roy. Soc. **A114** 243 (1927); "The quantum theory of dispersion", Proc. Roy. Soc. **A114** 710 (1927). See also E. Fermi, "Quantum theory of radiation", Revs. Mod. Phys. **4** 87 (1932); W. Heitler, *The Quantum Theory of Radiation*, Dover (1984).

(we assume the integration volume equal to unity); this is the Coulomb interaction V_c. The longitudinal part of the interaction V given by equation (6) can be written as

$$V_l = \int dtd\mathbf{r}\mathbf{j}\mathbf{E}_l = \sum_{\mathbf{k}} \int dt \tfrac{4\pi i \mathbf{k}\mathbf{j}_{\mathbf{k}}}{k^2}\rho_{-\mathbf{k}} = -\sum_{\mathbf{k}} \int dt \tfrac{4\pi}{k^2}\dot{\rho}_{\mathbf{k}}\rho_{-\mathbf{k}} = \\ = -\sum_{\mathbf{k}} \tfrac{2\pi}{k^2}\rho_{\mathbf{k}}\rho_{-\mathbf{k}} = -V_c \ , \tag{12}$$

where $\mathbf{j}_{\mathbf{k}}$ is the Fourier transform of the current density ($\mathbf{j}^*_{-\mathbf{k}} = \mathbf{j}_{\mathbf{k}}$); we can see that the longitudinal degrees of freedom are eliminated from the problem.

Let us assume for the moment that the interaction is left aside, *i.e.* we put $\rho = 0$ and $\mathbf{j} = 0$. Then, $a_{\mathbf{k}}$ and $f_{\mathbf{k}}$ have a time factor $e^{-i\omega_k t}$. In addition, the longitudinal field $\mathbf{E}_l$ is zero, and so is the longitudinal part of the electromagnetic energy $W^l_{em} = 0$. We get immediately

$$W^t_{em} = \sum_{\mathbf{k}} \frac{1}{2}\hbar\omega_k \left(a^*_{\mathbf{k}}a_{\mathbf{k}} + a_{\mathbf{k}}a^*_{\mathbf{k}}\right) \ , \tag{13}$$

where the coefficients a_k do not depend on time. This is the total energy of the electromagnetic field. If we introduce the conbinations

$$q_{\mathbf{k}} = \sqrt{\frac{\hbar}{2\omega_k}}\left(a^*_{\mathbf{k}} + a_{\mathbf{k}}\right) \ , \ p_{\mathbf{k}} = i\sqrt{\frac{\hbar\omega_k}{2}}\left(a^*_{\mathbf{k}} - a_{\mathbf{k}}\right) \ , \tag{14}$$

equation (13) becomes the energy of a set of harmonic oscillators with frequencies ω_k,

$$W^t_{em} = \sum_{\mathbf{k}} \frac{1}{2}\left(p^2_{\mathbf{k}} + \omega^2_k q^2_{\mathbf{k}}\right) \ . \tag{15}$$

The quantization of these oscillators proceeds in the usual way. W^t_{em} is viewed as the hamiltonian H_{em} of the electromagnetic field, there exist harmonic-oscillator wavefunctions of the coordinates q_k and $[p_{\mathbf{k}}, q_{\mathbf{k}'}] = -i\hbar\delta_{\mathbf{k}\mathbf{k}'}$. It is convenient to view $a_{\mathbf{k}}$ and $a^*_{\mathbf{k}}$ as operators, with boson commutation relations $[a_{\mathbf{k}}, a^*_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'}$, $[a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0$; the energy becomes

$$H_{em} = W^t_{em} = \sum_{\mathbf{k}} \hbar\omega_k \left(a^*_{\mathbf{k}}a_{\mathbf{k}} + 1/2\right) \tag{16}$$

and the momentum (equation (5)) is

$$\mathbf{G} = \sum_{\mathbf{k}} \hbar\mathbf{k}\left(a^*_{\mathbf{k}}a_{\mathbf{k}} + 1/2\right) \ ; \tag{17}$$

this is the second quantization (or quantization of the field);[11] it provides the photon (quanta of light) picture.[12] The creation and destruction operators $a^*_{\mathbf{k}}$ and $a_{\mathbf{k}}$ act upon states of occupation numbers $n_{\mathbf{k}} = 0, 1, 2...$, denoted $\mid n_{\mathbf{k}} >$, with well-known matrix elements

$$a_{\mathbf{k}} \mid n_{\mathbf{k}} >= \sqrt{n_{\mathbf{k}}} \mid n_{\mathbf{k}} - 1 > \ , \\ a^*_{\mathbf{k}} \mid n_{\mathbf{k}} >= \sqrt{n_{\mathbf{k}} + 1} \mid n_{\mathbf{k}\alpha} + 1 > \tag{18}$$

[11]P. A. M. Dirac, "The quantum theory of the emission and absorption of radiation", Proc. Roy. Soc. **A114** 243 (1927).

[12]A. Einstein, "Uber einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt", Ann. Physik **17** 132 (1905).

and

$$n_{\mathbf{k}} \mid n_{\mathbf{k}} >= a_{\mathbf{k}}^{*} a_{\mathbf{k}} \mid n_{\mathbf{k}} >= n_{\mathbf{k}\alpha} \mid n_{\mathbf{k}\alpha} > \quad . \tag{19}$$

The time dependence is given by

$$\begin{gathered} i\hbar\tfrac{d}{dt}a_{\mathbf{k}} = [a_{\mathbf{k}}, H_{em}] \ , \ a(t) = e^{-\frac{i}{\hbar}H_{em}t} a_{\mathbf{k}} e^{\frac{i}{\hbar}H_{em}t} \ , \\ a_{\mathbf{k}}(t) = a_{\mathbf{k}} e^{-i\omega_k t} \ , \end{gathered} \tag{20}$$

in agreement with the (free) field equations (1) (Heisenberg representation). The states obey the Schroedinger equation

$$i\hbar\frac{d}{dt} \mid n_k >= H_{em} \mid n_{\mathbf{k}} > \quad . \tag{21}$$

If the interaction

$$V = \int d\mathbf{r} \left(\rho\Phi - \frac{1}{c}\mathbf{jA}\right)_t = -\frac{1}{c}\int d\mathbf{r}\mathbf{j}\mathbf{A}_t \tag{22}$$

is present, the states are superpositions of photon states, including the degrees of freedom of the charges; such a state, $\mid s >$, obeys the Schroedinger equation

$$i\hbar\frac{d}{dt} \mid s >= (H_{em} + H_p + V) \mid s > \ , \tag{23}$$

where H_p is the (free) hamiltonian of the charges; with $H_0 = H_{em} + H_p$ and $\mid s >= e^{-\frac{i}{\hbar}H_0 t} \mid v >$, equation (23) becomes

$$i\hbar\frac{d}{dt} \mid v >= e^{\frac{i}{\hbar}H_0 t} V e^{-\frac{i}{\hbar}H_0 t} \mid v > \tag{24}$$

(interaction representation). The solution of this equation is

$$\mid v >=\mid v_0 > -\frac{i}{\hbar}\int_0^t dt_1 V(t_1) \mid v > \ , \tag{25}$$

or, the perturbation-theory series

$$\begin{gathered} \mid v >=\mid v_0 > -\tfrac{i}{\hbar}\int_0^t dt_1 V(t_1) \mid v_0 > + \\ + \left(-\tfrac{i}{\hbar}\right)^2 \int_0^t dt_1 V(t_1) \int_0^{t_1} dt_2 V(t_2) \mid v_0 > + ... \ , \end{gathered} \tag{26}$$

where $\mid v_0 >=\mid v >_{t=0}$ (initial state, also denoted $\mid v_i >$) and $V(t) = e^{\frac{i}{\hbar}H_0 t} V e^{-\frac{i}{\hbar}H_0 t}$. Equation (26) gives a formal solution to the problem of interaction. The second-order term in this equation includes already self-interaction. The first-order term provides the Dirac theory of radiation interacting with (non-relativistic) matter (an additional interaction $\sim A^2$ occurs in this case from the hamiltonian of the charges).[13]

Relativistic quantum fields. Quantum Electrodynamics and the theory of relativistic quantum fields insist upon the relativistic invariance. This point occurred already from the early attempts of quantizing the fields.[14] Such an insistence is inappropriate, since the relativistic invariance is

[13]P. A. M. Dirac, "The quantum theory of the emission and absorption of radiation", Proc. Roy. Soc. **A114** 243 (1927); "The quantum theory of dispersion", Proc. Roy. Soc. **A114** 710 (1927). See also E. Fermi, "Quantum theory of radiation", Revs. Mod. Phys. **4** 87 (1932); W. Heitler, *The Quantum Theory of Radiation*, Dover (1984).

[14]P. Jordan and W. Pauli, "Zur Quantenelektrodynamik ladungsfreier Felder", Z. Phys. **47** 151 (1928); W. Heisenberg and W. Pauli, "Zur Quantendynamik der Wellenfelder", Z. Phys. **56** 1 (1929); "Zur Quantentheorie der Wellenfelder. II", **59** 168 (1930); W. Pauli, "Relativistic field theories of elementary particles", Revs. Mod. Phys. **13** 203 (1941).

both impossible and unnecessary in the quantum-mechanical context. This circumstance occurs already for the electromagnetic field.

First we note that the problem of the quantum-mechanical interaction of the electromagnetic field and electrical charges is tackled in terms of photons. The existence of the photon requires the spatial Fourier decomposition of the electromagnetic potentials (equations (7)). The Fourier coefficients $\mathbf{e}_{\mathbf{k}}a_{\mathbf{k}}$ are not relativistically invariant. Second, the elimination of the longitudinal degrees of freedom makes the remaining transverse field relativistically non-invariant. Third, the quantum-mechanical evolution equations (Schroedinger, Heisenberg equations (20), (21), (23), (24)) are not relativistically invariant (due to the distinct position of time; the usual space-like parameter technique[15] does not solve this problem). The energy (hamiltonians) are global quantities, which do not depend on position (they depend on the Fourier coefficients $q_{\mathbf{k}}$, $a_{\mathbf{k}}$), while the evolution implies each moment of time. The quantum-mechanical interaction of the electromagnetic field with charges is delocalized, in a space region, due to the wave-like extension of the photon (and of charges); like in any other quantum-mechanical problem the spatial coordinates (as well as the time) are parameters which can be measured only with uncertainty; consequently, the space-time local coordinates cannot be subject to Lorentz transformations. The relativistic invariance implies a change of reference frame, but in quantum-mechanical interaction there is no local reference frame; the measured quantities are not local, they are global.

The Quantum Electrodynamics is built in a few versions, all equivalent with the interaction picture described above. All assume, inappropriately, local fields (wavefunctions) governed by local densities of hamiltonians (lagrangians). This is explicitly shown in Tomonaga's[16] and Schwinger's[17] papers, while Feynman attributes trajectories to photons and charges and assumes local scattering processes.[18] (This was Bohr's objection to Feynman in Pocono Conference in 1948). The same procedure is present in the precursory formulation by Stueckelberg.[19] It is difficult to assess to what extent the subsequent space-time integration corrects such inappropriate procedures. It is worth stressing that the photon wavefunctions are functions of the coordinates $q_{\mathbf{k}}$ (equation (14)), which are Fourier coefficients of the fields, or, more generally, functions of the $a_{\mathbf{k}}$ (occupation numbers; including the degrees of fredom of the charges); therefore, it is the field $\mathbf{A}_t$ which moves (for photons, similar for charges), and its motion affects all the $\mathbf{r}, t$-points, not only a specified local point $\mathbf{r}, t$. It is worth noting that Feynman's propagators for space-time fields disappear if the space integration is performed, because this integration leaves behind momentum operators which carry only an exponential time factor. By the same procedure most of Schwinger's technical manipulations become void.

[15] S. Tomonaga, "On a relativistically invariant formulation of the quantum theory of wave fields", Progr. Theor. Phys. **1** 27 (1946); J. Schwinger, "Quantum Electrodynamics. I. A covariant formulation", Phys. Rev. **74** 1439 (1948).

[16] S. Tomonaga, "On a relativistically invariant formulation of the quantum theory of wave fields", Progr. Theor. Phys. **1** 27 (1946); "On infinite field reactions in quantum field theory", Phys. Rev. **74** 224 (1948).

[17] J. Schwinger, "On Quantum-Electrodynamics and the magnetic moment of the electron", Phys. Rev. **73** 416 (1948); "Quantum Electrodynamics. I. A covariant formulation", Phys. Rev. **74** 1439 (1948); "Quantum electrodynamics. II. Vacuum polarization and self-energy", Phys. Rev. **75** 651 (1949); "On radiative corrections to electron scattering", Phys. Rev. **75** 898 (1949); "Quantum Electrodynamics, III: The electromagnetic properties of the electron-radiative corrections to scattering", Phys. Rev. **76** 790 (1949); "On gauge invariance and vacuum polarization", Phys. Rev. **82** 664 (1951).

[18] R. P. Feynman, "Relativistic cut-off for Quantum Electrodynamics", Phys. Rev. **74** 1430 (1948); "The theory of positrons", Phys. Rev. **76** 749 (1949); "Space-time approach to Quantum Electrodynamics", Phys. Rev. **76** 769 (1949); "Mathematical formulation of the quantum theory of electromagnetic interaction", Phys. Rev. **80** 440 (1950). See also

[19] E. C. G. Stueckelberg, "Relativistisch invariante Stoerungstheorie des Diracschen Elektrons. I. Teil: Streustrahlung und Bremsstrahlung", Ann. Phys. 5. Folge, **21** 367 (1934) (Berichtigung, p. 744).

The insistence to formulate a relativistically invariant (covariant) theory leads to maintaining all the components A^{μ} of the electromagnetic potential in the hamiltonian, scalar and longitudinal components included, although the latter do not participate in dynamics. The connection provided by the gauge condition between the latter is ensured as an average, likewise the Maxwell equations; the Maxwell equations are not satisfied anymore, such that we have virtual photons. Moreover, the scalar and longitudinal components of the field cannot be quantized, because there does not exist a free hamiltonian for the scalar components (the coefficients $f_{\mathbf{k}}$); the electromagnetic energy for the longitudinal field is given entirely in terms of the charge density (equation (11)). Keeping the quantized longitudinal (scalar) field leads to infrared divergences in standard Quantum Electrodynamics, which are subsequently removed, on sound physical grounds.[20]

The potential A^{μ} is formally relativistically invariant, and so are the field commutators, at different times (through Jordan-Pauli function). This shows that quantization is the same in any inertial reference frame, but the quantum-mechanical motion is different from the space-time relativistic motion. These two motions cannot be unified, it is not necessary to be unified, nor desirable, and, in fact, they have not been in Quantum Electrodynamics, in spite of the claims and the efforts of the people who built this discipline.[21]

All these inconsistencies occurring in formulating the Quantum Electrodynamics are ineffective, in fact, because they are associated with an undetermined motion which is removed by renormalization.

Classically, a (point) particle placed at x ($x = (x^{\mu})$) is a well-defined concept. Quantum-mechanically it is acceptabe to say that the position x has an uncertainty. This concept is formulated mathematically by the existence of a wavefunction $\varphi(x)$, whose square $\mid \varphi(x) \mid^2$ is the probability of localization of the particle, *i.e.* it is related to the density of particles obtained by repeated measurements. This means that we count the particles localized in a volume Δv which encloses the position of the particle at a moment of time; their number is Δn. Then, $\mid \varphi(x) \mid^2 = \Delta n/\Delta v$ in the limit $\Delta n, \Delta v \longrightarrow 0$ for any time; it is assumed that this limit exists. The so-called field-theoretical methods of second quantization in ensembles of many identical particles in condensed matter assign the same wavefunction meaning to the "fields". However, if we view the particles as quanta of a field $\psi(x)$, or if we view the quantum-mechanical motion of these particles (like the change in density, for instance), then we encounter a contradiction: on one hand, $\psi(x)$ is a well-defined generalized coordinate of motion and, on the other hand, it is not well-defined since the position x has uncertainties. The way out of this contradiction is to view the motion of ψ globally, irrespective of x. This is done formally by using the coefficients of the spatial Fourier transform of $\psi(x)$. The quantum-mechanical evolution equation implies the time t and a hamiltonian (energy) which depends on these Fourier coefficients. In this context the relativistic invariance is meaningless. A quantum-mechanical field is a global quantity. In condensed matter an interesting example is provided by phonons, which in the harmonic approximation are quanta of a global field (atom displacement), while, with anharmonic interactions, they become local quantum-mechanical displacements of the atoms, described by a wavefunction.

Dirac equation. A quantum-mechanical equation (Schroedinger equation) for the motion of a particle is achieved formally by using $i\hbar\frac{\partial}{\partial t}$ for energy and $-i\hbar\frac{\partial}{\partial \mathbf{r}}$ for momentum in the energy-momentum relationship. The equation should describe the motion both in the quantum-mechanical limit, *i.e.* for amounts of mechanical action comparable with $\hbar$, and in the (quasi-) classical limit, where the mechanical action changes by amounts much larger than $\hbar$ (*i.e.*, for the formal limit

[20] F. Bloch and A. Nordsieck, "Note on the radiation field of the electron", Phys. Rev. **52** 54 (1937).

[21] J. Schwinger, *Selected Papers in Quantum Electrodynamics*, Dover, NY (1958); S. Schweber, *QED and the Men who Made it*, Princeton University Press, NJ (1994).

$\hbar \longrightarrow 0$). Obviously, there is no (consistent) Schroedinger equation for the classical limit of a relativistic particle with mass m (*e.g.*, an electron), since the classical energy-momentum relationship in this case is $\mathcal{E}^2 = p^2c^2 + m^2c^4$, and a consistent Schroedinger equation requires a term linear in $\mathcal{E}$ (for the conservation of the probability). However, the above equation can be quantized as

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + \frac{m^2c^2}{\hbar^2}\psi = 0 \ , \tag{27}$$

an equation which, obviously, describes the motion of a relativistic quantum field ψ, which may be associated to a classical electron. Its motion is meaningful in the quantum-mechanical limit. Noteworthy, equation (27) exhibits a characteristic wavelength $\lambda_c = \frac{\hbar}{mc}$ (Compton wavelength) and a characteristic energy $\mathcal{E}_c = mc^2$ (rest energy). In addition, an electromagnetic field with potentials $(\Phi, \mathbf{A})$ can be introduced for the electron with charge e, by

$$(\mathcal{E} - e\Phi)^2 = \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + m^2c^4 \tag{28}$$

(Hamilton-Jacobi equation). A covariant form of equation $\mathcal{E}^2 = p^2c^2 + m^2c^4$ is $p_\mu p^\mu = m^2c^2$, which is quantized by $p_\mu = i\hbar\partial_\mu$ (and the electromagnetic field is introduced by $p_\mu \longrightarrow p_\mu - \frac{e}{c}A_\mu$). This is the well-known Klein-Gordon equation.[22]

A consistent Schroedinger equation for the relativistic electron is the Dirac equation[23]

$$\gamma^\mu p_\mu \varphi = mc\varphi \ , \tag{29}$$

where γ^μ are the Dirac matrices and φ is a bispinor with the components labelled by two spin indices and two indices corresponding to $\pm\mathcal{E}$. The Dirac equation shows that the electron has spin 1/2 and it may have negative energies. An electron field can be constructed by a Fourier superposition of bispinors, with coefficients $c_{\mathbf{k}\sigma}$, $b^*_{\mathbf{k}\sigma}$ ($\mathbf{p} = \hbar\mathbf{k}$, $\sigma = \pm$) which obey anticommutation relations; the electrons are fermions. The field equation is the Dirac equation. The electromagnetic field is introduced through $p_\mu \longrightarrow p_\mu - \frac{e}{c}A_\mu$, leading to an electron-photon interaction $V = \frac{e}{c}j^\mu A_\mu$, where the current density is $j^\mu = \overline{\psi}\gamma^\mu\psi$ ($\overline{\psi}$ is the Dirac conjugate field). The creation and destruction operators imply that the negative energies (associated with the coefficients $b_{\mathbf{k}\sigma}$) indicate a distinct type of electrons ($c_{\mathbf{k}\sigma}$ are associated with positive energies), with opposite charge $-e$, called positrons (the antiparticle of the electron), since the charge is conserved. The Quantum Electrodynamics is constructed by using the Dirac electron field. Logarithmic divergences occur, since the Dirac equation is linear in momentum-energy (these divergences are associated with vacuum polarization (electron-positron pairs) and photon fluctuations). Consequently, mass and charge renormaliztion can be done, and regularization procedures can be applied. This way, finite results are obtained.

Obviously, the Dirac equation (29) has not a classical limit: in the (quasi-) classical limit $\hbar \longrightarrow 0$ the Dirac equation becomes $\gamma^\mu p_\mu = mc$, where $p_\mu = (\mathcal{E}/c, -\mathbf{p})$; this is not a valid equation. It follows that the Dirac equation cannot be used to describe motion which implies large amounts of mechanical action. For instance, it cannot be used for a typical scattering experiment. Applying twice the Dirac equation we get the Klein-Gordon equation

$$\gamma^\mu\gamma^\nu p_\mu p_\nu \varphi = m^2c^2\varphi \ , \ \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi + \frac{m^2c^2}{\hbar^2}\varphi = 0 \tag{30}$$

[22] O. Klein, "Quantentheorie und fuenfdimensionale Relativitaetstheorie", Z. Phys. **37** 895 (1926); W. Gordon, "Der Comptoneffekt nach der Schroedingerschen Theorie", Z. Phys. **40** 117 (1926).

[23] P. A. M. Dirac, "The quantum theory of the electron", Proc. Roy. Soc. **A117** 610 (1928); "A theory of electrons and protons", Proc. Roy. Soc. **A126** 360 (1930).

(since $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$), or, in the presence of the electromagnetic field,

$$\left(p_\mu - \frac{e}{c}A_\mu\right)\left(p^\mu - \frac{e}{c}A^\mu\right)\varphi - \frac{ie\hbar}{2c}\sigma^{\mu\nu}F_{\mu\nu}\varphi = m^2c^2\varphi \ , \tag{31}$$

where $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field; an equation which can also be written as

$$\left(p_\mu - \frac{e}{c}A_\mu\right)\left(p^\mu - \frac{e}{c}A^\mu\right)\varphi + \frac{e\hbar}{c}\sum\mathbf{H} - \frac{ie\hbar}{c}\boldsymbol{\alpha}\mathbf{E} = m^2c^2\varphi \ , \tag{32}$$

where

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \ , \ \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \ , \tag{33}$$

$\boldsymbol{\sigma}$ are the Pauli matrices, $\mathbf{E}$ is the electric field and $\mathbf{H}$ is the magnetic field ($\sigma^{\mu\nu} = (\boldsymbol{\alpha}, i\boldsymbol{\Sigma})$). In the classical limit ($\hbar \longrightarrow 0$) the spin terms in equation (32) disappear, and we recover the classical Klein-Gordon equation with electromagnetic field. However, it is worth noting that equations (30) and (31) are different from the Dirac equation.

Let us take a closer look at the Dirac equation (29), either for wavefunctions or fermion fields. First, we note that this equation describes quantum-mechanically delocalized particles over the Compton wavelength λ_c. Consequently, in the presence of the electromagnetic field (*i.e.* for charges), we expect an infinite self-interaction (ultraviolet divergences), though the singularity is softened to a logarithmic one (due to the delocalization). Next, we note that the presence of the matrices γ^μ makes the motion undetermined (the Zitterbewegung[24]). This means that the Dirac equation cannot describe physical motion (produced by interaction), it only describes the existence of particles. On the other hand, it is precisely the undetermined motion over the Compton wavelength which makes possible a physical motion. This physical motion is the motion of the undetermined motion; parts of the Zitterbewegung may move (and interact), and distinct Zitterbewegungs may move and interact with respect to one another. This latter motion implies a finite self-interaction; it is not a motion of a particle, it is a motion of a motion of a particle.

A quantum-mechanical particle with mass m has an intrinsic maximal momentum uncertainty mc; consequently, it has a minimal position uncertainty $\lambda_c = \hbar/mc \simeq 3.8 \times 10^{-11}cm$ (Compton wavelength). It cannot be localized more accurately than the wavelength λ_c; over this distance its motion is undetermined. Similarly, it cannot be measured in times shorter than $\tau_c = \hbar/mc^2$.[25] The Dirac equation includes this basic limitation, *i.e.* it describes an undetermined motion over short distances and short durations of time. This undetermined motion is known as the Zitterbewegung. The presence of the matrices γ^μ in the Dirac equation (29) implies an undetermined velocity $v^\mu = c\gamma^\mu$, which indicates an undetermined motion. The relativistic limitation of the Dirac equation is seen in an external electromagnetic field. For instance, a Bohr radius in a heavy atom with atomic number Z should be larger than the Compton wavelength, *i.e.* $\frac{\hbar^2}{mZe^2} > \frac{\hbar}{mc}$, or $Z < \frac{\hbar c}{e^2} \simeq 137$ (the inverse of the fine-structure constant $\alpha = e^2/\hbar c$). Similarly, in tunneling experiments the potential barrier should be sufficiently low (Klein paradox[26]); in an external magnetic field the Dirac equation may give corrections to the magnetic moment (the anomalous magnetic moment of the electron, after discarding some infinities).[27] However, in the problem of the interaction of

[24]E. Schroedinger, "Ueber die kraeftefreie Bewegung in der relativistischen Quantenmechanik", Sitzungsberichte der Preussischen Akademie der Wissenschaften, Berlin, 418 (1930); "Zur Quantendynamik des Elektrons", Sitzungsberichte der Preussischen Akademie der Wissenschaften, Berlin, 63 (1931)

[25]L. Landau and R. Peierls, "Erweiterung des Unbestimmtheitsprinzips fur die relativistische Quantentheorie", Z. Phys. **69** 56 (1931); L. Landau and E. Lifshitz, Course of Theoretical Physics, vol. 4, Quantum Electrodynamics (V. Berestetskii, E. Lifshitz, l. Pitaevski), Butterworth-Heinemann (1971).

[26]O.Klein, "Die Reflexion von Elektronen an einem Potentialsprung nach der relativistischen Dynamik von Dirac", Z. Phys. **53** 157 (1929).

[27]J. M. Luttinger, "A note on the magnetic moment of the electron", Phys. Rev. **74** 893 (1848).

the charges with their own electromagnetic field, where the self-interaction may appear, *i.e.* the motion over short distances and durations is effective, the relativistically undetermined motion reflects itself in divergences; noteworthy, these divergences are of a special kind: they are generated by undetermined functions, since they are produced by an undetermined motion. Consequently, techniques of rendering these functions meaningful should be looked for elsewhere.[28] This is the basic problem (of regularizing divergences) of the Quantum Electrodynamics, which, in this respect, is, at least, incomplete.[29] It is worth noting that the limitation discussed here for the Dirac equation occurs in the non-relativistic Schroedinger equation for charges too.[30] However, there, the Compton cutoff is effective naturally, by the requirement of preserving the non-relativistic character of the motion of the charges.

The relativistic invariance is capable of generating its own motion, by the modifications it brings in the equation of motion. In the Dirac equation this special motion is associated with the matrices γ^μ. Apart from the space-time motion associated with the momentum p_μ, the relativistic motion described by the matrices γ^μ is localized over distances of the order of the characteristic length λ_c and durations of the order τ_c; these parameters are characteristic to the Dirac equation. By the matriceal nature of the relativistic entities γ^μ, this motion is undetermined; it is the Zitterbewegung. It implies not only a mixture of the components of the bispinor, but also an undetermined mixture of the coordinates x^μ. A change of these coordinates over the regions where the Zitterbewegung is localized will correspond to a superposed motion, averaged over λ_c and τ_c. Obviously, this emergent motion is limited by regions of size λ_c and τ_c.

Charged 1/2-spin bosons. Let us give a variation $\delta x^\mu = u^\mu$ to the coordinates x^μ, according to the scheme $x^\mu \longrightarrow x^\mu + \delta x^\mu$, $\delta x^\mu = u^\mu$, $x^\mu \longrightarrow x^\mu + u^\mu$. We will take the first-order variations with respect to u^μ of the Dirac equation

$$\gamma^\mu \partial_\mu \psi = \frac{mc}{i\hbar}\psi \ . \tag{34}$$

According to the Dirac equation, the Zitterbewegung implies that the coordinates may be viewed as matrices (like γ^μ). We write $x^\mu = s^\mu \cdot 1$, where $s^0 = ct$, $\mathbf{s} = \mathbf{r}$ and 1 denotes the unit matrix; we have $x_\mu x^\mu = s_\mu s^\mu \cdot 1 = s^2 \cdot 1$, where $s^2 = c^2t^2 - \mathbf{r}^2$. For δx^μ we need $\delta x_\mu \delta x^\mu = u_\mu u^\mu = u^2 = ds^2$; the (non-trivial) solution of this equation is

$$\delta x^\mu = u^\mu = \frac{1}{2} u \gamma^\mu \tag{35}$$

(since $\gamma_\mu \gamma^\mu = 4$); we can see the connection with the velocities $v^\mu = c\gamma^\mu$. The first-order expansion of the Dirac equation is

$$\gamma^\mu \left(\partial_\mu \psi + u^\nu \partial_\nu \partial_\mu \psi\right) = \frac{mc}{i\hbar}\left(\psi + u^\nu \partial_\nu \psi\right) \ \ , \tag{36}$$

or

$$\partial^\mu \partial_\mu \left(u\psi\right) = -\frac{m^2c^2}{\hbar^2}\left(u\psi\right) \ \ , \tag{37}$$

[28] W. Pauli and F. Villars, "On the invariant regularization in relativistic quantum theory", Revs. Mod. Phys. **21** 434 (1949).

[29] L. Landau and E. Lifshitz, Course of Theoretical Physics, vol. 4, Quantum Electrodynamics (V. Berestetskii, E. Lifshitz, l. Pitaevski), Butterworth-Heinemann (1971).

[30] W. Pauli and M. Fierz, "Zur Theorie der Emission langwelliger Lichtquanten", Nuovo Cim. **15** 167 (1938); H. Kramers, "Quantentheorie des Elektrons und der Strahlung", in *Hand. und Jahrbuch der Chemische Physik*, I, part 2, Leipzig (1938); "Die Wechselwirkung zwischen geladenen Teilchen und Strahlungsfeld", Nuovo Cim. **15** 108 (1938); "Non-relativistic quantum electrodynamics and correspondence principle", in *Rapports et Discussions du 8e Congres Solvay*, 1948, Brussels, R. Stoop (1950); *Collected Scientific Papers*, North Holland, Amsterdam (1956).

which is the Klein-Gordon equation $p_\mu p^\mu (u\psi) = m^2c^2 (u\psi)$. This equation describes the motion averaged over the Zitterbewegung. We note that we transferred the motion of ψ to u. It is the motion of this u which is described by the Klein-Gordon equation. We can see that the derivation of the Klein-Gordon equation given here amounts to applying twice the Dirac equation for the new field $u\psi$. The Dirac spinors are superfluous, and the new field $u\psi$ retains only the bispinor labels; we denote this new field by ψ. The second quantization of the Klein-Gordon equation is made by bosons. These bosons represent the changes in motion of each fermion state labelled by spin and negative energy. Therefore, we may represent the new field ψ as

$$\psi = \sum_{\mathbf{k}} c\sqrt{\frac{\hbar}{2\varepsilon_k}} (\psi_{\mathbf{k}\alpha}) e^{i\mathbf{kr}} \ , \tag{38}$$

where $\varepsilon_k = \varepsilon_{\mathbf{k}} = c\sqrt{k^2 + m^2c^2/\hbar^2}$,

$$(\psi_{\mathbf{k}\alpha}) = \begin{pmatrix} c_{\mathbf{k},+} \\ c_{\mathbf{k},-} \\ b^*_{-\mathbf{k},-} \\ b^*_{-\mathbf{k},+} \end{pmatrix} \ , \tag{39}$$

and c, b's satisfy usual boson commutation relations for four distinct types of bosons, corresponding to $c_{\mathbf{k}\sigma}$ and $b_{\mathbf{k}\sigma}$; $\sigma = \pm$ is the spin label and the c's and the b's correspond to positive and negative energies (frequencies), respectively.

The linear approximation used in the expansion above spoils the effects of the electromagnetic field, such that, in the presence of the electromagnetic field, we need to use the covariant derivative $D_\mu = \partial_\mu - \frac{e}{ic\hbar}A_\mu$, according to $\psi \longrightarrow \psi + u^\mu D_\mu \psi$; we get the Klein-Gordon equations (31) or (32) with electromagnetic field and spin. We note that the average of the motion of the electron over small distances implies a lower bound upon the electron wavelength, as well as upon the interaction (photons) wavelength, of the order of the Compton wavelength λ_c.

Free bosons. The free equation of motion (32) reads

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + \frac{m^2c^2}{\hbar^2}\psi = 0 \ . \tag{40}$$

If we multiply this equation by $\dot{\overline{\psi}}$, the equation for $\overline{\psi}$ by $\dot{\psi}$, where $\overline{\psi}$ is the adjoint of ψ (transposed conjugate), and add the two equations we get

$$\frac{\partial}{\partial t}\left(\frac{1}{c^2}\dot{\overline{\psi}}\dot{\psi} + \partial_i\overline{\psi}\partial_i\psi + \frac{m^2c^2}{\hbar^2}\overline{\psi}\psi\right) - \partial_i\left(\dot{\overline{\psi}}\partial_i\psi + \partial_i\overline{\psi}\dot{\psi}\right) = 0 \ , \tag{41}$$

where $i = 1, 2, 3$; hence, we can see that

$$w_e = \frac{1}{c^2}\dot{\overline{\psi}}\dot{\psi} + \partial_i\overline{\psi}\partial_i\psi + \frac{m^2c^2}{\hbar^2}\overline{\psi}\psi \tag{42}$$

is the energy density and

$$\mathbf{g}_e = -\frac{1}{c^2}\left(\dot{\overline{\psi}} grad\psi + grad\overline{\psi}\dot{\psi}\right) \tag{43}$$

is the momentum density. Making use of equations (38) and (39), the total energy is

$$\begin{aligned} W_e = \int d\mathbf{r} w_e = \textstyle\sum_{\mathbf{k}\sigma} \hbar\varepsilon_k \left(c^*_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma} + b_{\mathbf{k}\sigma}b^*_{\mathbf{k}\sigma}\right) = \\ = \textstyle\sum_{\mathbf{k}\sigma} \hbar\varepsilon_k \left(c^*_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma} + b^*_{\mathbf{k}\sigma}b_{\mathbf{k}\sigma} + 1\right) \ ; \end{aligned} \tag{44}$$

similarly, the total momentum is

$$\begin{gathered}\mathbf{G}_e=\int d\mathbf{r}\mathbf{g}_e=\sum_{\mathbf{k}\sigma}\hbar\mathbf{k}\left(c^*_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}+b_{\mathbf{k}\sigma}b^*_{\mathbf{k}\sigma}\right)=\\ =\sum_{\mathbf{k}\sigma}\hbar\mathbf{k}\left(c^*_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}+b^*_{\mathbf{k}\sigma}b_{\mathbf{k}\sigma}+1\right)\ .\end{gathered}\tag{45}$$

If we multiply the equations for ψ and $\overline{\psi}$ by $\overline{\psi}$ and ψ, respectively, and subtract the two equations from one another, we get another law of conservation, which reads

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left(\overline{\psi}\dot{\psi}-\dot{\overline{\psi}}\psi\right)-\partial_i\left(\overline{\psi}\partial_i\psi-\partial_i\overline{\psi}\psi\right)=0\ ;\tag{46}$$

hence,

$$\begin{gathered}Q=\frac{ie}{\hbar c^2}\int d\mathbf{r}\left(\overline{\psi}\dot{\psi}-\dot{\overline{\psi}}\psi\right)=e\sum_{\mathbf{k}\sigma}\left(c^*_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}-b_{\mathbf{k}\sigma}b^*_{\mathbf{k}\sigma}\right)=\\ =e\sum_{\mathbf{k}\sigma}\left(c^*_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}-b^*_{\mathbf{k}\sigma}b_{\mathbf{k}\sigma}-1\right)\end{gathered}\tag{47}$$

is the electric charge (where e is the electron charge) and

$$\begin{gathered}\mathbf{J}=-\frac{ie}{\hbar}\int d\mathbf{r}\left(\overline{\psi}grad\psi-grad\overline{\psi}\psi\right)=\\ =e\sum_{\mathbf{k}\sigma}\frac{c^2\mathbf{k}}{\varepsilon_k}\left(c^*_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}-b_{\mathbf{k}\sigma}b^*_{\mathbf{k}\sigma}\right)=\\ ==e\sum_{\mathbf{k}\sigma}\frac{c^2\mathbf{k}}{\varepsilon_k}\left(c^*_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}-b^*_{\mathbf{k}\sigma}b_{\mathbf{k}\sigma}-1\right)\end{gathered}\tag{48}$$

is the electric current. With the notation

$$\rho=\frac{ie}{\hbar c^2}\left(\overline{\psi}\dot{\psi}-\dot{\overline{\psi}}\psi\right)\ ,\ j_i=-\frac{ie}{\hbar}\left(\overline{\psi}\partial_i\psi-\partial_i\overline{\psi}\psi\right)\ ,\tag{49}$$

$$j^\mu=(c\rho,\mathbf{j})=\frac{ie}{\hbar}\left(\overline{\psi}\partial^\mu\psi-(\partial^\mu\overline{\psi})\psi\right)\ \ ,\tag{50}$$

equation (46) is the continuity equation $\partial_\mu j^\mu=0$.

Interaction. We adopt the energy W_e given by equation (44) as the free electron hamiltonian

$$H_e=\sum_{\mathbf{k}\sigma}\hbar\varepsilon_k\left(c^*_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}+b^*_{\mathbf{k}\sigma}b_{\mathbf{k}\sigma}\right)\tag{51}$$

(leaving aside the zero-point energy), where $\varepsilon_k=c\sqrt{k^2+k_0^2}$, $k_0=mc/\hbar$ being the inverse of the Compton wavelength. Similarly, we adopt

$$H_{ph}=H_{em}=\sum_{\mathbf{k}}\hbar\omega_k a^*_{\mathbf{k}}a_{\mathbf{k}}\tag{52}$$

(equation (16)) as the free hamiltonian of the photons, where $\omega_k=ck$; noteworthy, this hamiltonian implies only the transverse electromagnetic field. The electromagnetic energy includes the Coulomb contribution V_c arising from the longitudinal field $\mathbf{E}_l$,

$$\mathbf{E}_l=-\sum_{\mathbf{k}}\frac{i\mathbf{k}}{\omega_k^2}\left(\ddot{\Phi}_{\mathbf{k}}+\omega_k^2\Phi_k\right)e^{i\mathbf{k}\mathbf{r}}\ ,\ V_c=\frac{1}{8\pi}\int d\mathbf{r}E_l^2\ ,\tag{53}$$

where $\Phi = \sum_{\mathbf{k}} \Phi_{\mathbf{k}} e^{i\mathbf{kr}}$ (equations (10) and (11)). The contribution of the longitudinal field to the electromagnetic energy is vanishing for free fields. If we include the interaction for the longitudinal field, we get $\mathbf{E}_{l\mathbf{k}} = -(4\pi i\mathbf{k}/k^2)\rho_{\mathbf{k}}$ and $V_c = \sum_{\mathbf{k}} \frac{2\pi}{k^2}\rho_{\mathbf{k}}\rho_{-\mathbf{k}}$, where $\rho_{\mathbf{k}}$ is the Fourier component of the charge density.

The interaction of the electrons with the photons can be derived from equation (32); leaving aside for the moment the spin part of the interaction, this equation can be written as

$$(p_\mu p^\mu - \tfrac{e}{c} p_\mu A^\mu - \tfrac{e}{c} A_\mu p^\mu + \tfrac{e^2}{c^2} A_\mu A^\mu = 0 \ , \tag{54}$$

i.e.

$$\begin{aligned} \tfrac{1}{c^2}\tfrac{\partial^2\psi}{\partial t^2} - \Delta\psi + \tfrac{m^2c^2}{\hbar^2}\psi + \tfrac{e}{c\hbar^2} p_\mu A^\mu \psi + \tfrac{e}{c\hbar^2} A_\mu p^\mu \psi - \\ - \tfrac{e^2}{c^2\hbar^2} A_\mu A^\mu \psi = 0 \ . \end{aligned} \tag{55}$$

The interaction can be obtained from the equation of motion by means of the work done by the field upon charges. It is worth noting that, in contrast with equation (6), the full interaction includes now a field self-interaction energy (the term $\sim A_\mu A^\mu$) (this is a consequence of the hamiltonian formalism for the boson field).

For the motion of the boson field ψ a direct way of identifying the interaction is provided by the lagrangian. The free lagrangian densities are

$$\begin{aligned} L_e &= \tfrac{1}{c^2}\dot{\overline{\psi}}\dot{\psi} - \left(\partial_i\overline{\psi}\right)\left(\partial_i\psi\right) - \tfrac{m^2c^2}{\hbar^2}\overline{\psi}\psi \ , \\ L_{em} &= -\tfrac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} \ , \end{aligned} \tag{56}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field. The interaction energy densities are

$$v_p = \frac{1}{c} j_\mu A^\mu = \frac{ie}{c\hbar}\left[\overline{\psi}\left(\partial_\mu\psi\right) - \left(\partial_\mu\overline{\psi}\right)\psi\right] A^\mu \tag{57}$$

(where the gauge condition $\partial_\mu A^\mu = 0$ is used) and

$$v_d = -\frac{e^2}{c^2\hbar^2}\left(\overline{\psi}\psi\right) A_\mu A^\mu \tag{58}$$

(the labels p and d are chosen by analogy with the non-relativistic "paramagnetic" and "diamagnetic" contributions). Now we can derive the spin interaction too. Equation (32) is derived by using the Dirac conjugate. We should retain from this equation that part which is valid with the bosonic adjoint. It is easy to see that this is done by the spin interactions

$$\begin{aligned} v_H &= -\tfrac{e}{2c\hbar}\overline{\psi}\left(\sum + \sum^*\right)\psi\mathbf{H} \ , \\ v_E &= \tfrac{ie}{2c\hbar}\overline{\psi}\left(\boldsymbol{\alpha} - \boldsymbol{\alpha}^*\right)\psi\mathbf{E} \ . \end{aligned} \tag{59}$$

The symmetrization and antisymmetrization in equations (59) ensure the consistency of the equations of motion for ψ and $\overline{\psi}$ (boson fields) and real energy. It means that in the first equation (59) only the matrices $\sigma_{x,z}$ remain, while in the second equation (59) only the matrix σ_y remains. It is worth noting that $v_{H,E}$ give spin and pair currents, while v_d gives mass to the photon.

The interaction is given by integrating these densities over the whole space,

$$V_{p,d,H,E} = \int d\mathbf{r} v_{p,d,H,E} \tag{60}$$

It is worth noting that the interaction v_p (equation (57)) is the same as the interaction derived in equation (6).

The electromagnetic field in the above equations includes both the scalar potential Φ and the longitudinal component $\mathbf{A}_l$ (corresponding to the coefficients $\Phi_{\mathbf{k}}$). Since the longitudinal field has a vanishing free hamiltonian ($V_c = \frac{1}{8\pi}\int d\mathbf{r}E_l^2 = 0$ for $\ddot{\Phi}_{\mathbf{k}} + \omega_k^2\Phi_k = 0$ in equation (53)), this field cannot be quantized as an independent kind of photons (there is no scalar or longitudinal "photon"). The longitudinal field (scalar potential) moves according to the equation

$$\frac{1}{c^2}\ddot{\Phi} - \Delta\Phi = 4\pi\rho = \frac{4\pi ie}{\hbar c^2}\left(\overline{\psi}\dot{\psi} - \dot{\overline{\psi}}\psi\right) \; ; \tag{61}$$

(this is the equation $div\mathbf{E}_l = 4\pi\rho$, where $\mathbf{E}_l$ is given by equation (53)); we can see that the coefficients $\Phi_{\mathbf{k}}$ consist of bilinear forms of operators $c_{\mathbf{k}\sigma}$, $b_{\mathbf{k}\sigma}$; they correspond to the longitudinal degrees of freedom. The energy V_c becomes the Coulomb interaction energy

$$V_c = \sum_{\mathbf{k}} \frac{2\pi}{k^2}\rho_{\mathbf{k}}\rho_{-\mathbf{k}} \; . \tag{62}$$

The scalar potential Φ is obtained from equation (61) and introduced in the interaction hamiltonian. We get

$$\begin{array}{c}\rho_{\mathbf{k}} = \frac{e}{\hbar c^2}\sum_{\mathbf{k}'\sigma}\beta_{\mathbf{k}'+\mathbf{k}}\beta_{\mathbf{k}'}\left(\varepsilon_{\mathbf{k}'+\mathbf{k}} + \varepsilon_{\mathbf{k}'}\right)\cdot \\ \cdot\left(c^*_{\mathbf{k}'+\mathbf{k}\sigma}c_{\mathbf{k}'\sigma} - b_{-\mathbf{k}'-\mathbf{k}-\sigma}b^*_{-\mathbf{k}'-\sigma}\right) \; , \\ \Phi_{\mathbf{k}} = -\frac{4\pi e}{\hbar}\sum_{\mathbf{k}'\sigma}\frac{\beta_{\mathbf{k}'+\mathbf{k}}\beta_{\mathbf{k}'}\left(\varepsilon_{\mathbf{k}'+\mathbf{k}}+\varepsilon_{\mathbf{k}'}\right)}{\left(\varepsilon_{\mathbf{k}'+\mathbf{k}}-\varepsilon_{\mathbf{k}'}\right)^2-\omega_k^2}\cdot \\ \cdot\left(c^*_{\mathbf{k}'+\mathbf{k}\sigma}c_{\mathbf{k}'\sigma} - b_{-\mathbf{k}'-\mathbf{k}-\sigma}b^*_{-\mathbf{k}'-\sigma}\right) \; ,\end{array} \tag{63}$$

where $\beta_{\mathbf{k}} = \beta_k = c\sqrt{\frac{\hbar}{2\varepsilon_k}}$. We write $\mathbf{A} = \mathbf{A}_t + \mathbf{A}_l$, where

$$\begin{array}{c}\mathbf{A}_{t\mathbf{k}} = \alpha_k\left(\mathbf{e}_{\mathbf{k}}a_{\mathbf{k}} + \mathbf{e}_{-\mathbf{k}}a^*_{-\mathbf{k}}\right) \; , \; \alpha_k = c\sqrt{\frac{2\pi\hbar}{\omega_k}} \; , \\ \mathbf{A}_{l\mathbf{k}} = \frac{4\pi e}{\hbar}\sum_{\mathbf{k}'\sigma}\frac{i\mathbf{k}}{k\omega_k}\frac{\beta_{\mathbf{k}'+\mathbf{k}}\beta_{\mathbf{k}'}\left(\varepsilon^2_{\mathbf{k}'+\mathbf{k}}-\varepsilon^2_{\mathbf{k}'}\right)}{\left(\varepsilon_{\mathbf{k}'+\mathbf{k}}-\varepsilon_{\mathbf{k}'}\right)^2-\omega_k^2}\cdot \\ \cdot\left(c^*_{\mathbf{k}'+\mathbf{k}\sigma}c_{\mathbf{k}'\sigma} - b_{-\mathbf{k}'-\mathbf{k}-\sigma}b^*_{-\mathbf{k}'-\sigma}\right)\end{array} \tag{64}$$

and

$$\mathbf{E}_{l\mathbf{k}} = -\frac{4\pi i\mathbf{k}}{k^2}\rho_{\mathbf{k}} \; . \tag{65}$$

In addition, from the continuity equation we get the Fourier component of the longitudinal current

$$\begin{array}{c}\mathbf{j}_{l\mathbf{k}} = \frac{e}{\hbar c^2}\sum_{\mathbf{k}'\sigma}\frac{\mathbf{k}}{k^2}\beta_{\mathbf{k}'+\mathbf{k}}\beta_{\mathbf{k}'}\left(\varepsilon^2_{\mathbf{k}'+\mathbf{k}} - \varepsilon^2_{\mathbf{k}'}\right)\cdot \\ \cdot\left(c^*_{\mathbf{k}'+\mathbf{k}\sigma}c_{\mathbf{k}'\sigma} - b_{-\mathbf{k}'-\mathbf{k}-\sigma}b^*_{-\mathbf{k}'-\sigma}\right) \; .\end{array} \tag{66}$$

The longitudinal part $\rho\Phi - \frac{1}{c}\mathbf{j}_l\mathbf{A}_l$ of the interaction v_p can be estimated as in equation (12), or by using its contribution to the mechanical action. The result is $-V_c$ (as in equation (12)).

Therefore, the Coulomb interaction disappears from the problem and the interaction v_p is reduced to its transverse part

$$v_p = -\frac{1}{c}\mathbf{j}\mathbf{A}_t \ . \tag{67}$$

However, in contrast with the interaction derived in equation (6), for the field ψ there is an additional interaction v_d (equation (58)). The longitudinal contribution $\Phi^2 - \frac{1}{c^2}(\mathbf{A}_l^2 + 2\mathbf{A}_l\mathbf{A}_t)$ to this interaction brings a term of the order e^3, at least; if we limit ourselves to e^2-orders at most, we may use

$$v_d \simeq \frac{e^2}{c^2\hbar^2}\left(\overline{\psi}\psi\right)\mathbf{A}_t\mathbf{A}_t \ . \tag{68}$$

Similarly, only the transverse field contributes to v_H. It is worth noting that the longitudinal degrees of freedom of the electromagnetic field are removed from the problem, by means of equations (63)-(66); they are replaced by the degrees of freedom of the charges. Similar interaction contributions arise from an external (transverse, purely radiation field).

By a similar procedure we may eliminate also the transverse field from the interaction problem, being left only with an interaction between charges and currents (particle-particle interaction). However, this is not a convenient procedure, because we measure photons (transverse field) in the experimental situations; noteworthy, such measurements are peformed far away from the charged particles, where the field is indeed a radiation field.

Perturbation Theory. We give below a few examples of perturbation-theoretical calculations by means of the boson theory of the Dirac equation. Since the parameter $\alpha = e^2/\hbar c = 1/137$ (fine-structure constant) of the perturbation series is much smaller than unity, we limit ourselves to the first order and some processes in the second order of the perturbation theory. These processes are chosen for their specific relevance for the Quantum Electrodynamics. Scattering amplitudes are not included, because their calculation is long and does not raise any interesting problem.

1. Spontaneous emission. Let us assume an initial state $\mid v_i >= c^*_{\mathbf{k}\sigma} \mid 0 >$, corresponding to an electron. The first-order perturbation state produced by the interaction V_p (equations (25), (60) and (67)) is

$$\mid v >_p^{(1)} = \frac{2e}{c\hbar^2}\sum_{\mathbf{k}'}\alpha_{k'}\beta_k\beta_{\mathbf{k}'+\mathbf{k}}\mathbf{k}\mathbf{e}_{-\mathbf{k}'}S_{\Delta\varepsilon}a^*_{-\mathbf{k}'}c^*_{\mathbf{k}'+\mathbf{k}\sigma}\mid 0 > \ , \tag{69}$$

where

$$S_{\Delta\varepsilon} = \frac{e^{i(\varepsilon_{\mathbf{k}'+\mathbf{k}}+\omega_{k'}-\varepsilon_k)t}-1}{\varepsilon_{\mathbf{k}'+\mathbf{k}}+\omega_{k'}-\varepsilon_k} \ , \ \Delta\varepsilon = \varepsilon_{\mathbf{k}'+\mathbf{k}}+\omega_{k'}-\varepsilon_k \tag{70}$$

and $\alpha_k = c\sqrt{2\pi\hbar/\omega_k}$, $\beta_k = c\sqrt{\hbar/2\varepsilon_k}$. Equation (69) gives the amplitude of spontaneous emission of a photon,

$$f = ec\sqrt{\frac{2\pi E_{ph}}{E_iE_f}}\sin\theta\cdot S_{\Delta\varepsilon} \ , \tag{71}$$

where $E_{ph} = \hbar\omega$ is the energy of the photon, $E_{i,f}$ are the initial and final energies of the electron and θ is the angle between the direction of propagation of the electron and the direction of propagation of the photon. For the probability of emission (polarized photon, per unit volume),

$$|S_{\Delta\varepsilon}|^2 = 2\pi t\delta(\Delta\varepsilon) \ . \tag{72}$$

If there is a width of the energy levels, the factor $S_{\Delta\varepsilon}$ becomes

$$S_{\Delta\varepsilon} = \frac{e^{i\Delta\varepsilon t}e^{-\gamma t}-1}{\Delta\varepsilon + i\gamma} \tag{73}$$

and $|S_{\Delta\varepsilon}|^2 \longrightarrow 1/\left(\Delta\varepsilon^2 + \gamma^2/4\right)$ for $t \longrightarrow \infty$ (a natural line breadth is caused by the field emitted by the charge).[31] On the other hand, $\pi\delta(\Delta\varepsilon) \longleftarrow (\gamma/2)/\left(\Delta\varepsilon^2 + \gamma^2/4\right)$ for $\gamma \ll \Delta\varepsilon$; it follows that the relevant time is of the order $t \simeq 1/\gamma$, as expected. This is the typical result for photon emission or absorption, dipole (multipole) radiation (from bound states), Zeeman and Stark and photoelectric effects. The results are similar with those of the radiation theory.[32]

2. Diamagnetic self-energy: a mass renormalization. The first non-vanishing contribution of the interaction v_d is of the order e^2:

$$\begin{gathered} \mid v >_d^{(2)} = -\tfrac{ie^2}{c^2\hbar^3} t\beta_k^2 \textstyle\sum_{\mathbf{k}'} \alpha_{\mathbf{k}'}^2 c_{\mathbf{k}\sigma}^* \mid 0 > - \\ -\tfrac{ie^2}{c^2\hbar^3} \textstyle\sum_{\mathbf{k}'\mathbf{q}} \alpha_{k'} \alpha_{\mathbf{k}'-\mathbf{q}} \beta_k \beta_{\mathbf{k}'+\mathbf{q}} \cdot \\ \cdot \left(\mathbf{e}_{-\mathbf{k}'} \mathbf{e}_{\mathbf{k}'-\mathbf{q}}\right) S_{\Delta\varepsilon} a_{-\mathbf{k}'}^* a_{\mathbf{k}'-\mathbf{q}}^* c_{\mathbf{k}+\mathbf{q}\sigma}^* \mid 0 > \ , \end{gathered} \tag{74}$$

where

$$\Delta\varepsilon = \omega_{k'} + \omega_{\mathbf{k}'-\mathbf{q}} - \varepsilon_k \ . \tag{75}$$

The first term in equation (74) gives an interacting state

$$\mid v > = \mid v_i > - \frac{ie^2}{c^2\hbar^3} t\beta_k^2 \sum_{\mathbf{k}'} \alpha_{\mathbf{k}'}^2 \mid v_i > \ , \tag{76}$$

which indicates a change

$$\Delta E_d = \frac{e^2}{c^2\hbar^2} \beta_k^2 \sum_{\mathbf{k}'} \alpha_{\mathbf{k}'}^2 \tag{77}$$

in the energy of the electron; or

$$\Delta E_d = \frac{e^2 c\hbar}{2\pi E} \int dk' \cdot k' \ , \tag{78}$$

where $E = \sqrt{p^2c^2 + m^2c^4}$ is the original energy of the electron. This self-energy is infinite. It is associated with photon fluctuations of the vacuum: the electron generates and absorbs a photon, according to the diamagnetic interaction $\sim (\overline{\psi}\psi)A^2$ (a factor 2 should be included for the two photon polarizations).

Such divergences are typical for higher-order terms in the perturbation series, including the second order. Their origin is twofold. On one hand, they arise from the point-like nature of the electron, which generates divergent quantities even in the classical Electrodynamics. On the other hand, they arise as a consequence of the deficient formulation of the interaction problem, which includes unphysical self-interaction. Both these points are associated with the Zitterbewegung. The renormalization technique of extracting finite results in the boson theory means the use of the Compton wavelength cutoff $k_0 = mc/\hbar$. By using this procedure in equation (78), we get

$$\Delta E_d = \frac{e^2 m^2 c^3}{4\pi\hbar E} \ , \tag{79}$$

[31] V. Weisskopf and E. Wigner, "Berechnung der natuerlichen Linienbreite auf Grund der Diracschen Lichttheorie", Z. Phys. **63** 54 (1930); "Uber die naturliche Linienbreite in der Strahlung des harmonischen Oszillators", Z. Phys. **65** 18 (1930). See also W. Heitler, *The Quantum Theory of Radiation*, Dover (1984).

[32] P. A. M. Dirac, "The quantum theory of the emission and absorption of radiation", Proc. Roy. Soc. **A114** 243 (1927); "The quantum theory of dispersion", Proc. Roy. Soc. **A114** 710 (1927). See also E. Fermi, "Quantum theory of radiation", Revs. Mod. Phys. **4** 87 (1932); W. Heitler, *The Quantum Theory of Radiation*, Dover (1984).

which, in the non-relativistic limit, becomes

$$\Delta E_d \simeq e^2/4\pi\lambda_c = \frac{e^2}{4\pi\hbar c}mc^2 \ . \tag{80}$$

This is a mass renormalization, due to photon fluctuations; $\alpha = e^2/\hbar c$ is the fine-structure constant. Similar contributions arise from second-order contributions.

3. Lamb shift. Let us consider a set of electron bound states denoted by n, with orthonormalized wavefunctions φ_n (instead of plane waves $e^{i\mathbf{kr}}$). The first-order interacting state generated by the interaction V_p is

$$\mid v >_p^{(1)} = -\frac{ie}{c\hbar^2}\sum_{n_1\mathbf{k}} \alpha_k\beta_n\beta_{n_1}\left[\mathbf{G}_{n_1 n}(\mathbf{k})\mathbf{e}_{-\mathbf{k}}\right] S_{\Delta\varepsilon} a^*_{-\mathbf{k}} c^*_{n\sigma} \mid 0> \ , \tag{81}$$

where $\Delta\varepsilon = \omega_k - \varepsilon_n$ and

$$\mathbf{G}_{nn'}(\mathbf{k}) = \int d\mathbf{r}\left(\varphi_n^* grad\varphi_{n'} - grad\varphi_n^*\cdot\varphi_{n'}\right)e^{i\mathbf{kr}} \tag{82}$$

(without the b-electrons). The second-order interacting state includes, apart from two photons, a contribution arising from the photon fluctuations, given by

$$\begin{gathered}\mid v >_p^{(2)} = \frac{e^2}{c^2\hbar^4}\sum_{n_1 n_2\mathbf{k}} \alpha_k^2\beta_n\beta_{n_1}\beta_{n_2}^2 \cdot \\ \cdot\left[\mathbf{G}_{n_1 n_2}(\mathbf{k})\mathbf{e}_{\mathbf{k}}\right]\left[\mathbf{G}_{n_2 n}(-\mathbf{k})\mathbf{e}_{\mathbf{k}}\right]\cdot \\ \cdot\int_0^t dt_1 e^{-i(\omega_k+\varepsilon_{n_2})t_1}\int_0^{t_1} dt_2 e^{i(\omega_k-\varepsilon_n+\varepsilon_{n_2})t_2} c^*_{n_1\sigma}\mid 0> \ .\end{gathered} \tag{83}$$

We can see that the electron emits and absorbs a photon and changes its state. Noteworthy, for free electrons this contribution is zero. A self-interacting contribution corresponds to $n_1 = n$. In addition, the main contribution arises from a degenerate state $n_2 = n'$, if it exists. In these circumstances, equation (83) becomes

$$\begin{gathered}\mid v >_p^{(2)} = \frac{e^2}{c^2\hbar^4}\sum_{\mathbf{k}} \alpha_k^2\beta_n^4\left[\mathbf{G}_{nn'}(\mathbf{k})\mathbf{e}_{\mathbf{k}}\right]\left[\mathbf{G}_{n'n}(-\mathbf{k})\mathbf{e}_{\mathbf{k}}\right]\cdot \\ \cdot\int_0^t dt_1 e^{-i(\omega_k+\varepsilon_n)t_1}\int_0^{t_1} dt_2 e^{i\omega_k t_2} c^*_{n\sigma}\mid 0> \ ,\end{gathered} \tag{84}$$

or

$$\begin{gathered}\mid v >_p^{(2)} = \frac{e^2}{c^2\hbar^4}\sum_{\mathbf{k}} \alpha_k^2\beta_n^4 \cdot \\ \cdot\left[\mathbf{G}_{nn'}(\mathbf{k})\mathbf{e}_{\mathbf{k}}\right]\left[\mathbf{G}_{n'n}(-\mathbf{k})\mathbf{e}_{\mathbf{k}}\right] S c^*_{n\sigma}\mid 0> \ ,\end{gathered} \tag{85}$$

where

$$S = \frac{1}{\omega_k}\left[\frac{e^{-i\varepsilon_n t}-1}{\varepsilon_n} - \frac{e^{-i(\varepsilon_n+\omega_k)t}-1}{\varepsilon_n+\omega_k}\right] \ . \tag{86}$$

For atomic bound states $\mathbf{G}_{nn'}(\mathbf{k})\mathbf{e}_{\mathbf{k}}$ and the range of k are of the order $1/a$, where a is the dimension of the atomic state. In the limit $k \longrightarrow 0$ equation (86) becomes

$$S = -\frac{\partial}{\partial\varepsilon_n}\frac{e^{-i\varepsilon_n t}-1}{\varepsilon_n} \ ; \tag{87}$$

in the limit of large t we get $S \simeq it/\varepsilon_n$, such that equation (85) becomes

$$\mid v >_p^{(2)} = it\frac{e^2\lambda_c^3}{8\pi\hbar a^4}\mid v_i > \ , \tag{88}$$

where $\varepsilon_n \simeq c/\lambda_c$. Therefore, we get an energy shift

$$\Delta E_p \simeq -E_n \left(\frac{e^2}{8\pi a E_n}\right)\left(\frac{\lambda_c}{a}\right)^3 ; \tag{89}$$

since E_n is of the order e^2/a, this shift is $\Delta E_p/E_n \simeq \frac{1}{8\pi}(\lambda_c/a)^3$. It is the splitting of the two degenerate states. The result is one-two orders of magnitude smaller than the standard result.[33]

4. Anomalous magnetic moment. Let us assume an external, uniform and constant, magnetic field $\mathbf{H}_0$. The first contribution of V_H (equation (59)) corresponding to $\mathbf{H}_0$ to the interacting state (apart from the zeroth-order contribution) arises in the third-order of the perturbation theory ($\mathbf{H}_0$ included). The structure of this contribution is

$$-\left(\frac{e}{c\hbar}\right)^3 \left[\overline{\psi}\left(\boldsymbol{\Sigma}\psi\right)\mathbf{H}\right]_1 \left[\overline{\psi}\left(\boldsymbol{\Sigma}\psi\right)\mathbf{H}\right]_2 \left[\overline{\psi}\left(\boldsymbol{\Sigma}\psi\right)\right]_3 \mathbf{H}_0 c^*_{\mathbf{k}\sigma} \mid 0> \ , \tag{90}$$

where the magnetic field is

$$\mathbf{H} = \sum_{\mathbf{q}} i\alpha_q \left(\mathbf{q}\times\mathbf{e}_{\mathbf{q}}\right)\left(a_{\mathbf{q}} - a^*_{-\mathbf{q}}\right) e^{i\mathbf{qr}} \tag{91}$$

and the suffixes $1, 2, 3$ denote the times; in equation (90) the space integration is included. Spin wavefunctions should be included, which amounts to products of spin operators. The full contribution to the interacting state is obtained by inserting the time integrations

$$\left(-\frac{i}{\hbar}\right)^3 \int_0^t dt_1 \cdot \int_0^{t_1} dt_2 \cdot \int_0^{t_2} dt_3 \ ; \tag{92}$$

a factor 3 is included for the three positions of $\mathbf{H}_0$ in equation (90). Obviously, only the c-operators contribute.

We use a parametrization

$$\begin{gathered}\mathbf{q} = q(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \ , \\ \mathbf{e}_{\mathbf{q}} = \mathbf{e}_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)\end{gathered} \tag{93}$$

and

$$\mathbf{q}\times\mathbf{e}_{\mathbf{q}} = \mathbf{e}_\varphi = (-\sin\varphi, \cos\varphi, 0) \tag{94}$$

for the photon vectors.

We can see that, in fact, the time t_3 does not appear in equation (90), as expected. The external field plays the role of a probe, which lasts a short time Δt; consequently, we replace the t_3-integration by

$$-\frac{i}{\hbar}\int_0^{\Delta t} dt_3 = -\frac{i\Delta t}{\hbar} = \frac{\delta t}{\hbar} = \frac{1}{mc^2} \ . \tag{95}$$

[33] W. E. Lamb and R. C. Retherford, "Fine structure of the hydrogen atom by a microwave method", Phys. Rev. **72** 241 (1947); H. Bethe, "The electromagnetic shift of the energy levels", Phys. Rev. **72** 339 (1947); J. B. French and V. F. Weisskopf, "The electromagnetic shift of the energy levels", Phys. Rev. **75** 1240 (1949); R. Feynman, "Relativistic cut-off for Quantum Electrodynamics", Phys. Rev. **74** 1430 (1948) (correction in "Space-time approach to Quantum Electrodynamics", Phys. Rev. **76** 769 (1949)); J. Schwinger, "Quantum Electrodynamics. III. The electromagnetic properties of the electron-Radiative corrections to scattering", Phys. Rev. **76** 790 (1949).

The remaining time integrations are

$$S=\int_0^t dt_1 e^{-i\varepsilon_{\mathbf{k}+\mathbf{q}}t_1}\cdot\int_0^{t_1} dt_2 e^{i(\varepsilon_{\mathbf{k}+\mathbf{q}}-\varepsilon_k)t_2}\ . \tag{96}$$

In this equation the wavevector $\mathbf{q}$ of the (emitted and absorbed) photon brings a comparatively small contribution, such that

$$S=-\frac{\partial}{\partial\varepsilon_k}\frac{e^{-i\varepsilon_k t}-1}{\varepsilon_k}\simeq\frac{it}{\varepsilon_k}\ . \tag{97}$$

Taking into account the spin contribution

$$[\boldsymbol{\sigma}_y(\mathbf{q}\times\mathbf{e_q})_y]^2=q^2\cos^2\varphi \tag{98}$$

(we recall the relationship $\sigma_i\sigma_j=\delta_{ij}+i\varepsilon_{ijk}\sigma_k$ for the Pauli matrices), we get the energy change

$$\Delta E_H=-\frac{3}{\hbar}\left(\frac{e}{c\hbar}\right)^3\sum_{\mathbf{q}}\frac{\cos^2\varphi(\mathbf{q})}{\varepsilon_k mc^2}q^2\alpha_q^2\beta_k^4\beta_{\mathbf{k}+\mathbf{q}}^2\left(\mathbf{H}_0\boldsymbol{\sigma}\right)\ . \tag{99}$$

In this expression we may approximate ε_k by ck_0 ($k_0=mc/\hbar$) and $\beta_{\mathbf{k}+\mathbf{q}}\simeq\beta_k$ (and average over directions); we get the relative change in the magnetic moment (the Bohr magneton $\mu=\mid e\mid\hbar/2mc$)

$$\Delta\mu/\mu\simeq\frac{3}{32\pi}\cdot\frac{e^2}{c\hbar} \tag{100}$$

(a factor 2 should be included for the two polarizations). The result is close to Schwinger's standard result.[34]

5. Pair creation. The interaction $v_E=\frac{ie}{2c\hbar}\overline{\psi}\left(\boldsymbol{\alpha}-\boldsymbol{\alpha}^*\right)\psi\mathbf{E}$ (equation (59)) is responsible for pair creation (destruction). The electric field is

$$\mathbf{E}=\sum_{\mathbf{q}}iq\alpha_q\left(\mathbf{e_q}a_{\mathbf{q}}-\mathbf{e}_{-\mathbf{q}}a^*_{-\mathbf{q}}\right)e^{i\mathbf{qr}}\ , \tag{101}$$

after the spatial integration the interaction becomes

$$\begin{aligned}V_E=-\tfrac{e}{c\hbar}\textstyle\sum_{\mathbf{kq}}q\alpha_q\beta_k\beta_{\mathbf{k}-\mathbf{q}}\left(c^*_{\mathbf{k}}\boldsymbol{\sigma}_y b^*_{-\mathbf{k}+\mathbf{q}}+b_{-\mathbf{k}}\boldsymbol{\sigma}_y c_{\mathbf{k}-\mathbf{q}}\right)\cdot\\ \cdot\left(\mathbf{e_q}a_{\mathbf{q}}-\mathbf{e}_{-\mathbf{q}}a^*_{-\mathbf{q}}\right)\ ,\end{aligned} \tag{102}$$

where the spin suffixes are included. We can see that an external photon is absorbed and generates an electron pair, with opposite spins. The interacting state is given by

$$-\frac{e}{c\hbar}\sum_{\mathbf{k}}q\alpha_q\beta_k\beta_{\mathbf{k}-\mathbf{q}}c^*_{\mathbf{k}}\sigma_y e_{\mathbf{q}y}b^*_{-\mathbf{k}+\mathbf{q}}e^{-i\omega_q t}\mid 0>\ , \tag{103}$$

where the time integration should be included. We get the amplitude

$$f=-\frac{ie}{c\hbar^2}q\alpha_q\beta_k\beta_{\mathbf{k}-\mathbf{q}}\cos\varphi\sin\varphi S_{\Delta\varepsilon} \tag{104}$$

[34] J. Schwinger, "On Quantum-Electrodynamics and the magnetic moment of the electron", Phys. Rev. **73** 416 (1948); "Quantum Electrodynamics. III. The electromagnetic properties of the electron-Radiative corrections to scattering", Phys. Rev. **76** 790 (1949).

for the pair $c^*_{\mathbf{k}+}b^*_{-\mathbf{k}+\mathbf{q},-}$, where $\Delta\varepsilon=\varepsilon_k+\varepsilon_{\mathbf{k}-\mathbf{q}}-\omega_q$ ($|S_{\Delta\varepsilon}|^2=2\pi t\delta(\Delta\varepsilon)$) and $\varphi=\varphi(\mathbf{q})$ is given by equation (94). We can see that the momentum is conserved, but the energy cannot be conserved, since the equation

$$\sqrt{k^2+k_0^2}+\sqrt{(\mathbf{k}-\mathbf{q})^2+k_0^2}=q \tag{105}$$

has not solutions (as it is well known).

The coupling of V_E with V_p or V_H leads to pair creation; V_H can be generated by an external magnetic field. Similarly, we can consider an interaction V_E generated by an external (static) electric field, like the field $grad(Ze/r)$ of a nucleus with charge Ze. This electric field is

$$\mathbf{E}=-\sum_{\mathbf{k}}\frac{4\pi iZe\mathbf{k}}{k^2}e^{-i\mathbf{kr}}\ ; \tag{106}$$

it generates an interaction

$$\begin{gathered}V_0=\frac{4\pi e^2Z}{c\hbar}\sum_{\mathbf{kk}_1}\frac{1}{k^2}\beta_{k_1}\beta_{\mathbf{k}_1+\mathbf{k}}\cdot\\ \cdot\left(c^*_{\mathbf{k}_1}\boldsymbol{\sigma}_yk_yb^*_{-\mathbf{k}_1-\mathbf{k}}+b_{-\mathbf{k}_1}\boldsymbol{\sigma}_yk_yc_{\mathbf{k}_1+\mathbf{k}}\right)\end{gathered} \tag{107}$$

(of the V_E-type). We couple this interaction with

$$\begin{gathered}V_p=-\frac{e}{c\hbar}\sum_{\mathbf{kq}}\alpha_q\beta_k\beta_{\mathbf{k}-\mathbf{q}}(2\mathbf{k}-\mathbf{q})\cdot\\ \cdot\left(c^*_{\mathbf{k}}c_{\mathbf{k}-\mathbf{q}}+b^*_{-\mathbf{k}+\mathbf{q}}b_{-\mathbf{k}}\right)\left(\mathbf{e}_{\mathbf{q}}a_{\mathbf{q}}-\mathbf{e}_{-\mathbf{q}}a^*_{-\mathbf{q}}\right)\ .\end{gathered} \tag{108}$$

The interaction V_0 does not depend on time; we use for it the integral given by equation (95). We apply

$$\frac{-2i}{\hbar mc^2}\int_0^t dt_1V_p(t_1)V_0 \tag{109}$$

to the state $a^*_{\mathbf{q}}\mid 0>$ (the factor 2 in equation (109) arises from the product $(V_p+V_0)(V_p+V_0)$). The temporal factor $S_{\Delta\varepsilon}$ implies

$$\varepsilon_{\mathbf{k}+\mathbf{q}}-\varepsilon_q=\omega_q\ \ , \tag{110}$$

an equation which is only approximately satisfied for small $\mathbf{k}$ and q of the order k_0. The momentum is not conserved, since the (large) nucleus is static. There are two amplitudes for creation of pairs $(\mathbf{k},-\mathbf{k}+\mathbf{q}-\mathbf{k}')$ and $(-\mathbf{k}+\mathbf{q},\mathbf{k}-\mathbf{k}')$, with opposite spins and undetermined $\mathbf{k}'$. The order of magnitude of these amplitudes is

$$f\simeq\pm\frac{4\pi ie^3Z}{\hbar mc^2k_0^2}\sqrt{\frac{2\pi c\hbar}{k_0}}\frac{k}{k'}\sin\theta\sin\theta'\cos\varphi'S_{\Delta\varepsilon}\ \ , \tag{111}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{q}$ and θ', φ' are the angles of $\mathbf{k}'$. This amplitude has the dimension $\sqrt{\alpha}\cdot vol^{3/2}$, where $\alpha=e^2/c\hbar$ is the fine-structure constant. The factor $vol^{3/2}$ is reduced by the density of states. Also, the amplitude given by equation (111) can be written as $f\simeq\sqrt{\alpha}Zr_e\lambda_c^{7/2}$, where $r_e=e^2/mc^2$ is the classical radius of the electron. Electron-positron pairs can also be created by polarizing the vacuum by an external electromagnetic field.[35]

The uncertainty implied by the second-order perturbation theory requires a density of states $\sim 1/\lambda_c^3$ for each electron; and a factor $\sim 1/\lambda_c$ for the photon; it follows a cross-section $\sigma\simeq\alpha Zr_e^2$,

[35] M. Apostol, "Dynamics of electron-positron pairs in a vacuum polarized by an external electromagnetic field", J. Mod. Opt. **58** 611 (2011).

which has the order of magnitude of the standard result.[36] The coupling of V_E with V_0 leads to Bremsstrahlung of an electron ($c^*_{\mathbf{k}\sigma} \mid 0>$) in an external field (and synchroton radiation).

The coupling of V_E (equation (102)) with V_p (equation (108)) leads to pair creation by the annihilation of two photons. We apply the second-order perturbation operator ($V_p V_E$) to the intial state $a^*_{\mathbf{q}} a^*_{-\mathbf{q}} \mid 0>$. The time integration leads to $S_{\Delta\varepsilon}$, where $\Delta\varepsilon = 2\varepsilon_k - 2\omega_q$, for the pair $(\mathbf{k}, -\mathbf{k})$ with opposite spins, and to $S_{\Delta\varepsilon}$, $\Delta\varepsilon = \varepsilon_k - \varepsilon_{\mathbf{k}-\mathbf{q}} - \omega_q$, an approximate energy conservation. The energy conservation $2\varepsilon_k - 2\omega_q$ is satisfied for $\mathbf{k} \longrightarrow 0$ and $q \simeq k_0$. The amplitude of formation of the pair $(\mathbf{k}, -\mathbf{k})$ is

$$f \simeq \mp \frac{4i}{\hbar^2} \left(\frac{e}{c\hbar} \right)^2 q^2 \alpha_q^2 \beta_0^2 \frac{\beta_q^2}{\varepsilon_q} \sin\Theta \cos\theta \sin\varphi \cdot S_{\Delta\varepsilon} \ , \tag{112}$$

where Θ is the angle between $\mathbf{k}$ and $\mathbf{q}$ and θ, φ are the angles of the wavevector $\mathbf{q}$ (with respect to the electron spin). For $q \simeq k_0$ the order of magnitude of the amplitude f is $f \simeq \alpha\lambda_c^3$ (where $S \simeq \tau \simeq \lambda_c/c$), in agreement with the standard result (we note that $r_e = \alpha\lambda_c$).[37] The process of pair formation by annihilation of two photons is related to the process of pair annihilation with the formation of two photons.[38]

6. Charge renormalization. Let us consider the interaction of a charge e with a static charge Q. The latter generates an interaction V_0 given by equation (107). The first non-vanishing correction to the initial state $c^*_{\mathbf{k}\sigma} \mid 0>$ appears in the second order of the perturbation theory; it is due to vacuum polarization. Before the time integration this contribution reads

$$\begin{gathered} \left(\tfrac{4\pi e Q}{c\hbar}\right)^2 \textstyle\sum_{\mathbf{q},\mathbf{q}'} \frac{1}{q^2 q'^2} \beta_k \beta^2_{\mathbf{k}-\mathbf{q}'} \beta_{\mathbf{k}-\mathbf{q}-\mathbf{q}'} (\sigma_y q_y)(\sigma_y q'_y) \cdot \\ \cdot e^{i(\varepsilon_{\mathbf{k}-\mathbf{q}'} - \varepsilon_k)t_1} e^{i\varepsilon_{\mathbf{k}-\mathbf{q}'} t_2} c^*_{\mathbf{k}-\mathbf{q}-\mathbf{q}'} \mid 0> \ . \end{gathered} \tag{113}$$

The main contribution to the state $\mathbf{k}\sigma$, with small $\mathbf{k}$, comes from small $\mathbf{q} = -\mathbf{q}'$. This is a quasi-static interaction; instead of two V_0 generated by Q, we may use only one generated by Q and another generated by e; then $(eQ)^2$ is replaced by $e^3 Q$. The integration over $\mathbf{q}'$ gives $q^2\Delta q \simeq 1/\lambda_c^3$. The main contribution of the time integration is

$$S_{\Delta\varepsilon} = \frac{e^{i(\varepsilon_{\mathbf{k}+\mathbf{q}} - \varepsilon_k)t} - 1}{\varepsilon_{\mathbf{k}+\mathbf{q}} - \varepsilon_k} \simeq it \ . \tag{114}$$

After averaging over angles we get a change

$$\Delta E \simeq \frac{1}{3\hbar} \left(\frac{4\pi e Q}{c\hbar} \right)^2 \frac{\beta_0^4}{\varepsilon_0} \lambda_c^3 \sum_{\mathbf{k}} \frac{1}{k^2} \tag{115}$$

in energy, which should be compared with the Coulomb interaction $4\pi e Q \sum_{\mathbf{k}} \frac{1}{k^2}$. It follows a charge renormalization of the order $\delta e/e \simeq \frac{\pi}{6}\alpha$, a result comparable with the standard results.[39]

7. Photon mass. The vacuum polarization accompanied by photon fluctuations in the perturbation series does not change the energy of the photon. However, the interaction with a "real"

[36] H. Bethe and W. Heitler, "On the stopping of fast particles and on the creation of positive electrons", Proc. Roy. Soc. **A146** 83 (1934).

[37] G. Breit and J. A. Wheeler, "Collision of two light quanta", Phys. Rev. **46** 1087 (1934).

[38] P. A. M. Dirac, "On the annihilation of electrons and protons", Proc. Cambr. Phil. Soc. **26** 361 (1930).

[39] J. Schwinger, "Quantum Electrodynamics. II. Vacuum polarization and self-energy", Phys. Rev. **75** 651 (1949); R. P. Feynman, "Space-time approach to Quantum Elctrodynamics", Phys. Rev. **76** 769 (1949); R. Serber, "Linear modifications in the Maxwell field equations", Phys. Rev. **48** 49 (!935); E. A. Uehling, "Polarization effects in the positron theory", Phys. Rev. **48** 55 (1935).

electron gives mass to the photon. Indeed, the interaction leads to

$$\frac{1}{c^2}\frac{\partial^2 A^\mu}{\partial t^2} - \Delta A^\mu + \frac{8\pi e^2}{c^2\hbar^2}\left(\overline{\psi}\psi\right)A^\mu = \frac{4\pi}{c}j^\mu - \frac{4\pi ie\hbar}{c}\partial_\nu\overline{\psi}\left(\sigma^{\mu\nu} - \sigma^{\mu\nu*}\right)\psi \ , \tag{116}$$

which is a wave equation with sources and mass. For an electron (both spin orientations) we get a photon mass given by

$$m_{ph}^2 = \frac{e^2\hbar^2}{Ec^2}\cdot\frac{1}{\lambda_c^3} = \alpha\left(mc^2/E\right)m^2 \ , \tag{117}$$

where E is the energy of the electron. The photons interacting with the electrons are similar with radiation propagating in matter (polaritons). A photon mass gives a screened Coulomb inteaction $\frac{1}{r}e^{-\sqrt{\alpha mc^2/E}\frac{r}{\lambda_c}}$.

Renormalization and Quantum Electrodynamics. The difficulties associated with the self-interaction are specific to a classical point charge with a structureless motion. The Dirac equation provides a structure to the motion of the charge, which makes meaningful part of the self-interaction. Indeed, the undetermined Zitterbewegung has a spatial extension, of the order of the Compton wavelength λ_c. This motion is undetermined and leads to infinities, because it implies a self-interaction for a (delocalized) point charge (and similarly for the electromagnetic field), but it may have physical, determined modifications. These modifications indicate an additional, superposed motion of the Zitterbewegung, either as a whole, or as its parts affecting other parts. The latter is a self-interaction which is meaningful and has finite contributions. The superposed motion is the boson motion. It is limited to wavevectors smaller than the Compton-wavelength cutoff. This cutoff effects, in fact, the renormalization and the regularization. The use of this procedure avoids the occurrence of the infinities. The self-interaction, which results finite from calculations, is the interaction of some parts of the Zitterbewegung with other parts of this motion. It assumes a spatial extension of the Zitterbewegung. In this respect, the self-interaction problem, as described herein within the boson theory of the Dirac equation, resembles the old conception of a spatially extended charge, without its difficulties (see Lorentz, Poincare, Abraham in Ref. [40]), or Heisenberg's "universelle Laenge theory".[41] In fact, the idea that a cutoff avoids divergences is obvious. The probem is to make convincing the existence of such a cutoff.

The boson theory of the Dirac field (equation) assigns the Compton wavelength λ_c to the spatial extension of the Zitterbewegung. This assignment is an order-of-magnitude procedure. Consequently, the results of this theory are only order-of-magnitude estimates.

Quantum Electrodynamics works with Dirac equation. Consequently, the infinities associated with the Zitterbewegung are unavoidable. However, the infinities of the interaction have the same structure as the infinities associated with a change in mass and charge, up to some finite contributions. Consequently, these infinities may simply be discarded, as arising from a renormalization of the mass and the charge, the finite contributions being thus the only relevant quantities. The renormalization can be viewed as a precise technique of eliminating the Zitterbewegung. Consequently, it is no surprise that the results of the Quantum Elctrodynamics are in excellent agreement with the experimental results; for instance, the Lamb shift[42] and the anomalous magnetic moment of the electron.[43]

[40]See, for instance, F. Rohrlich, *Classical Charged Particles*, World Scientific (2007).

[41]W. Heisenberg, "Ueber die in der Theorie der Elementarteilchen auftretende universelle Laenge", Ann. Physik **32** 20 (1938).

[42]W. E. Lamb and R. C. Retherford, "Fine structure of the hydrogen atom by a microwave method", Phys. Rev. **72** 241 (1947).

[43]J. E. Nafe, E. B. Nelson and I. I. Rabi, "The hyperfine structure of atomic hydrogen and deuterium", Phys.

However, the renormalization and regularization techniques hold in finite orders of the perturbation theory. For the infinite series of the perturbation theory these techiques fail, and the results become inconsistent.[44] This shows the artificial and unsafe (risky) character of these techniques. The boson theory of the Dirac field is free from such inconsistencies, though with less accurate numerical results.

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