Journal of Theoretical Physics

Founded and Edited by M. Apostol **319** (2020)

ISSN 1453-4428

On the structure of the Quantum Electrodynamics

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Introduction. The Maxwell equations in vacuum can be written as

$$\begin{aligned} div\boldsymbol{E} = 4\pi\rho \ , \ div\boldsymbol{H} = 0 \ , \\ curl\boldsymbol{E} = -\tfrac{1}{c}\tfrac{\partial \boldsymbol{H}}{\partial t} \ , \ curl\boldsymbol{H} = \tfrac{1}{c}\tfrac{\partial \boldsymbol{E}}{\partial t} + \tfrac{4\pi}{c}\boldsymbol{j} \ , \end{aligned} \tag{1}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ is the magnetic field, ρ is the charge density, $\boldsymbol{j}$ is the current density and c is the speed of light in vacuum; the charge is conserved, so we have the continuity equation

$$\frac{\partial \rho}{\partial t} + div\boldsymbol{j} = 0 \ . \tag{2}$$

If we take the *div* in the second-row equations, we get the first-row equations, so we have to solve only the former. These are two coupled sets each of three equations, with six unknown ($\boldsymbol{E}$ and $\boldsymbol{H}$); it follows that we have only three unknowns.

In equations (1) the charges and the currents determine the field. We can formulate the problem of the electromagnetic interaction of two charges, *i.e.* the interaction of a charge with the field generated by another charge; we may even talk about the interaction of two fields, each generated by distinct charges. But it would be quite improper to talk about self-interaction, *i.e.* the interaction of a charge with its own field, *i.e.* with itself, or the interaction of a field with itself; we know that the estimation of such effects leads to the Lorentz damping, which indicates the limits of such an improper formulation of the problem. In particular, such restrictions lead to distances larger than the classical charge radius e^2/mc^2, or Compton's wavelength $\hbar/mc = (e^2/mc^2)\frac{\hbar c}{e^2}$, where e is the charge, m is the particle mass, $\hbar$ is Planck's constant and $e^2/\hbar c = 1/137$ is the fine structure constant. Also, the fields should be restricted to Schwinger limit $e/(e^2/mc^2)^2 = m^2c^4/e^3$, or $mc^2/e(\hbar/mc) = (m^2c^4/e^3)\frac{e^2}{\hbar c}$, which is a very high field. Beyond these limits the Electromagnetism, either classical or quantum-mechanical, becomes meaningless.

Quantum Electrodynamics aims at describing the electromagnetic interaction of relativistic quantum-mechanical charges. The proper part of this interaction, as described above, which is an external-field type interaction, is given by Dirac's theorie of radiation. The Lamb shift and the anomalous magnetic moment of the electron forced the Quantum Electrodynamics to approach the improper self-interaction part. As it is well-known, this attempt leads to divergences.

It is claimed that Quantum Electrodynamics succeeds in removing the divergences by the so-called renormalization technique and the regularization of the divergent integrals; also, it is claimed that the finite results obtained this way are in very good agreement with the experimental data for the Lamb shift and the anomalous magnetic moment. It is shown herein that the Quantum

Electrodynamics is based on incorrect assumptions and incorrect manipulation of the mathematics. Consequently, its claims are doubtful, at least.

Photons. It is essential for the renormalization technique to work with an explicitly (manifestly) covariant formulation. To this end we introduce the field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix} \ ; \tag{3}$$

(in $F_{\mu\nu}$ the electric field changes sign). With this notation the Maxwell equations read

$$\begin{aligned} \partial_\nu F^{\mu\nu} &= -\tfrac{4\pi}{c} j^\mu \ , \\ \partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\nu} &= 0 \ , \end{aligned} \tag{4}$$

where the current density is

$$j^\mu = (c\rho, \boldsymbol{j}) \ , \ \partial_\mu j^\mu = 0 \ . \tag{5}$$

It is customary to introduce the potentials $A^\mu = (\Phi, \boldsymbol{A})$, where Φ is the scalar potential and $\boldsymbol{A}$ is the vector potential; then

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6}$$

and the Maxwell equations become

$$\partial_\nu \partial^\nu A^\mu - \partial^\mu (\partial_\nu A^\nu) = \frac{4\pi}{c} j^\mu \ . \tag{7}$$

It is claimed that the Maxwell equations given by equations (4) (or (7)) could be derived as Euler-Lagrange equations from the lagrangian density

$$L = -\frac{1}{16\pi c} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c^2} j_\mu A^\mu \ , \tag{8}$$

where A^μ are viewed as independent coordinates (the field part of equation (8) is proportional to $E^2 - H^2$). The quantization must be compatible with the relativistic form of the lagrangian; it involves commutation relations between the coordinates A^μ and the canonical conjugate momenta

$$\pi^\mu = \frac{\partial L}{\partial_t A_\mu} = \frac{1}{4\pi c^2} \left(\partial^\mu A^0 - \partial^0 A^\mu \right) = \frac{1}{4\pi c^2} F^{\mu 0} \ . \tag{9}$$

We can see that $\pi^0 = 0$, such that the canonical quantization cannot be done.

Actually, by the definition given by equation (6) of the potentials we have four potentials and need to express by them only three fields. Consequently, the four potentials are not independent. Therefore, we cannot treat them as independent coordinates in the variation of the lagrangian given by equation (8). On the other hand, in order to get the interaction contribution to the equations of motion, it is necessary to view the four potentials as independent variables. It follows that we cannot have a lagrangian theory of the electromagnetic field and charges in the covariant form.

In order to reduce the number of potentials from four to three we impose usually the Lorenz gauge

$$\partial_\mu A^\mu = 0 \ ; \tag{10}$$

this condition connects the scalar potential to the longitudinal component of the vector potential. We might think of using a Lagrange multiplier for this condition in the variation of the action, which would allow the use of the four potentials as independent variables. Unfortunately, the divergence given by equation (10) is ineffective in the action. The reason for not having a lagrangian theory for the electromagnetic field and charges in the covariant form resides in the fact that the scalar potential includes a static part (an instantaneous interaction) which does not describe a motion; while the lagrangian theory is specfically designed to describe a motion. It remains that we need to explicitly eliminate the redundant degree of freedom of the longitudinal component of the vector potential, or the scalar potential, and quantize the remaining part of transverse photons; we shall see that the scalar and the longitudinal "photons" disappear from the interaction problem.

The Quantum Electrodynamics adopts a different route. The lagrangian

$$L = -\frac{1}{8\pi}(\partial_\mu A_\nu)(\partial^\mu A^\nu) - \frac{1}{c} j_\mu A^\mu \tag{11}$$

is used to get the equations of motion

$$\partial_\nu \partial^\nu A^\mu = \frac{4\pi}{c} j^\mu \ . \tag{12}$$

and the conjugate momenta

$$\pi^\mu = \frac{\partial L}{\partial_t A_\mu} = -\frac{1}{4\pi c}\partial^0 A^\mu \ . \tag{13}$$

Although the Lorenz contribution $\partial_\mu A^\mu$ disappears from the equation of motion (12), the potentials are still treated as four independent coordinates. This procedure modifies the Maxwell equations (equations (12) are different from equations (7)) and leads to virtual photons and the infrared catastrophe associated with the soft photons. In addition, the canonical quantization of the four potentials involves unphysical quantum-mechanical conditions, which are circumvented only formally by various technical procedures (like the Gupta-Bleuler procedure).

It is difficult to assess the consequences of these modifications. For the S matrix, the photon propagator can be derived without resorting to the scalar potential and the longitudinal component of the vector potential; it is sufficient to use the transverse photons and the covariant form of the propagator; however, the latter continues to assume the existence of the redundant potentials.[1] The infrared divergences are eliminated by comparing the results with the classical results (where the Lorenz condition is used explictly); this technique is known as the Bloch-Nordsieck theorem. However, the elimination of the divergent contributions does not warrant the inexistence of other illegitimate, though finite, contributions.

Elimination of the superfluous potentials. We expand in Fourier series all the quantities, like the scalar potential

$$\Phi = \sum_{\boldsymbol{k}} \Phi_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{r}} \ , \ \Phi^*_{-\boldsymbol{k}} = \Phi_{\boldsymbol{k}} \ . \tag{14}$$

According to Maxwell equations the vector potential is a polar vector which changes sign under time reversal; therefore, additional conditions should be imposed on the Fourier expansion (as in Dirac theory of radiation), which, however, are immaterial for what follows. From the Lorenz gauge we get the longitudinal part of the vector potential

$$\boldsymbol{A}^l_{\boldsymbol{k}} = \frac{i\boldsymbol{k}}{k^2}\dot{\Phi}_{\boldsymbol{k}} \ . \tag{15}$$

[1] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 4, *Quantum Electrodynamics* (V. B. Berestetskii, E. M. Lifshitz and L. P. Pitaevskii), Pergamon (1982).

Making use of this relationship and the wave equation $\frac{1}{c^2}\ddot{\Phi}_{\boldsymbol{k}} + k^2\Phi_{\boldsymbol{k}} = 4\pi\rho_{\boldsymbol{k}}$, we get the electric field

$$\boldsymbol{E}_{\boldsymbol{k}} = -\frac{4\pi i\boldsymbol{k}}{k^2}\rho_{\boldsymbol{k}} - \frac{1}{c}\dot{\boldsymbol{A}}^t_{\boldsymbol{k}} \ , \tag{16}$$

where $\boldsymbol{A}^t_{\boldsymbol{k}}$ is the transverse part of the vector potential. The magnetic field is given by $\boldsymbol{H}_{\boldsymbol{k}} = i\boldsymbol{k}\times\boldsymbol{A}^t_{\boldsymbol{k}}$. The electromagnetic energy is

$$\mathcal{E} = \frac{1}{8\pi}\int\left(E^2+H^2\right)dV = \sum_{\boldsymbol{k}}\frac{2\pi}{k^2}\rho_{\boldsymbol{k}}\rho_{-\boldsymbol{k}} + \frac{1}{8\pi}\sum_{\boldsymbol{k}}\left(\frac{1}{c^2}\dot{\boldsymbol{A}}^t_{\boldsymbol{k}}\dot{\boldsymbol{A}}^t_{-\boldsymbol{k}} + k^2\boldsymbol{A}^t_{\boldsymbol{k}}\boldsymbol{A}^t_{-\boldsymbol{k}}\right) \ . \tag{17}$$

We can see that this energy includes the Coulomb energy $V_c = \sum_{\boldsymbol{k}}\frac{2\pi}{k^2}\rho_{\boldsymbol{k}}\rho_{-\boldsymbol{k}}$.

The interaction energy is computed from the mechanical work done per unit time

$$\begin{gathered} V = \int dtdV\boldsymbol{j}\boldsymbol{E} = \int dtdV\left(-\tfrac{1}{c}\boldsymbol{j}\tfrac{\partial\boldsymbol{A}}{\partial t} - \boldsymbol{j}grad\Phi\right) = \\ = -\tfrac{1}{c}\int dV\boldsymbol{j}\boldsymbol{A} + \int dtdVdiv\boldsymbol{j}\cdot\Phi = -\tfrac{1}{c}\int dV\boldsymbol{j}\boldsymbol{A} - \int dtdV\tfrac{\partial\rho}{\partial t}\Phi \ . \end{gathered} \tag{18}$$

In the mechanical action the last term in this equation becomes

$$\int dtdV\rho\frac{\partial\Phi}{\partial t} = \int dV\rho\Phi \ , \tag{19}$$

such that the interaction energy acquires the well-known form

$$V = \int dV\left(\rho\Phi - \frac{1}{c}\boldsymbol{j}\boldsymbol{A}\right) \ . \tag{20}$$

This energy can be decomposed into longitudinal and transverse contributions

$$\begin{gathered} V = \int dtdV\left[(\boldsymbol{j}\boldsymbol{E})_l + (\boldsymbol{j}\boldsymbol{E})_t\right] = \\ = \int dV\left(\rho\Phi - \tfrac{1}{c}\boldsymbol{j}_l\boldsymbol{A}_l\right) - \tfrac{1}{c}\int dV\boldsymbol{j}_t\boldsymbol{A}_t \ . \end{gathered} \tag{21}$$

Let us compute the longitudial part of this interaction by using the continuity equation $\boldsymbol{j}^l_{\boldsymbol{k}} = \frac{i\boldsymbol{k}}{k^2}\dot{\rho}_{\boldsymbol{k}}$ and the longitudinal part of the electric field given by equation (16); we get

$$V_l = \int dtdV(\boldsymbol{j}\boldsymbol{E})_l = -\int dt\sum_{\boldsymbol{k}}\frac{4\pi}{k^2}\dot{\rho}_{\boldsymbol{k}}\rho_{-\boldsymbol{k}} = -\sum_{\boldsymbol{k}}\frac{2\pi}{k^2}\rho_{\boldsymbol{k}}\rho_{-\boldsymbol{k}} = -V_c \ . \tag{22}$$

We can see that the longitudinal part of the interaction cancels out the Coulomb contribution to the electromagnetic energy, such that we are left with a total energy

$$\mathcal{E} + V = \frac{1}{8\pi}\sum_{\boldsymbol{k}}\left(\frac{1}{c^2}\dot{\boldsymbol{A}}^t_{\boldsymbol{k}}\dot{\boldsymbol{A}}^t_{-\boldsymbol{k}} + k^2\boldsymbol{A}^t_{\boldsymbol{k}}\boldsymbol{A}^t_{-\boldsymbol{k}}\right) - \frac{1}{c}\sum_{\boldsymbol{k}}\boldsymbol{j}^t_{\boldsymbol{k}}\boldsymbol{A}^t_{-\boldsymbol{k}} \tag{23}$$

arising only from the transverse fields. The work done by charges upon the field, which is the longitudinal part of the interaction, is stored in the Coulomb part of the field, and viceversa. We note that this is so only if we use explicitly the Lorenz gauge, which is not the procedure of the Quantum Electrodynamics.

The transverse part of the electromagnetic energy

$$\mathcal{E}_t = \frac{1}{8\pi}\sum_{\boldsymbol{k}}\left(\frac{1}{c^2}\dot{\boldsymbol{A}}^t_{\boldsymbol{k}}\dot{\boldsymbol{A}}^t_{-\boldsymbol{k}} + k^2\boldsymbol{A}^t_{\boldsymbol{k}}\boldsymbol{A}^t_{-\boldsymbol{k}}\right) \tag{24}$$

can be quantized by

$$\boldsymbol{A}_{\boldsymbol{k}}^{t} = \alpha_k \left(\boldsymbol{e}_{\boldsymbol{k}} a_{\boldsymbol{k}} + \boldsymbol{e}_{-\boldsymbol{k}} a_{-\boldsymbol{k}}^{*} \right) \ , \ \dot{\boldsymbol{A}}_{\boldsymbol{k}}^{t} = i\alpha_k \omega_k \left(\boldsymbol{e}_{\boldsymbol{k}} a_{\boldsymbol{k}} - \boldsymbol{e}_{-\boldsymbol{k}} a_{-\boldsymbol{k}}^{*} \right) \ , \tag{25}$$

where $\boldsymbol{e}_{\boldsymbol{k}}$ are real unit vectors perpendicular to $\boldsymbol{k}$ (polarization vectors), $\omega_k = ck$ and $\alpha_k = c\sqrt{2\pi\hbar/\omega_k}$; we get

$$\mathcal{E}_t = \sum_{\boldsymbol{k}} \frac{1}{2} \hbar\omega_k (a_{\boldsymbol{k}} a_{\boldsymbol{k}}^{*} + a_{\boldsymbol{k}}^{*} a_{\boldsymbol{k}}) \tag{26}$$

and equations (25) from the quantum-mecahnical equations of motion with the commutation relations $[a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}^{*}] = \delta_{\boldsymbol{k}\boldsymbol{k}'}$, $[a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}] = 0$. Two contributions of the type given by equation (26) exist, corresponding to the two transverse polarizations.

The S-matrix expansion. By means of the Dirac equation the Quantum Electrodynamics builds up the Dirac field for electrons. This field describes the electrons (and their antiparticles - the positrons) as quanta of energy, which can be created or destroyed in interaction processes. The electron field assumes, at one point, one electron or no electrons; there exists no more than one electron (positron) at one point (they are fermions). The free hamiltonian H_0 includes the electromagnetic energy, corresponding to the lagrangian given by equation (11), and the Dirac field energy. The interaction hamiltonian is

$$V = \frac{1}{c} \int dV j^{\mu} A_{\mu} \ , \tag{27}$$

where j^{μ} is the current density of the Dirac field. The wavefunction, which depends on the occupation numbers of photons and fermions, satisfies the Schroedinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = (H_0 + V)\psi \ ; \tag{28}$$

by $\psi = e^{-\frac{i}{\hbar} H_0 t} \varphi$ (the interaction representation), this equation becomes

$$i\hbar \frac{\partial \varphi}{\partial t} = V(t)\varphi \ , \ V(t) = e^{\frac{i}{\hbar} H_0 t} V e^{-\frac{i}{\hbar} H_0 t} \ . \tag{29}$$

The solution of this equation can be written as

$$\begin{gathered} \varphi = \varphi_0 - \tfrac{i}{\hbar} \textstyle\int_{t_0}^{t} dt_1 V(t_1)\varphi(t_1) = \\ = \varphi_0 - \tfrac{i}{\hbar} \textstyle\int_{t_0}^{t} dt_1 V(t_1)\varphi_0 + \left(-\tfrac{i}{\hbar}\right)^2 \textstyle\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 V(t_1)V(t_2)\varphi_0 + \ldots \ . \end{gathered} \tag{30}$$

This expansion describes scattering (interaction) processes. The wavefunction φ_0 is the incoming wavefunction corresponding to $t_0 = -\infty$; the wavefunction φ corresponding to $t = +\infty$ is the outgoing wavefunction; the operator S which connects φ to φ_0 by $\varphi = S\varphi_0$ is the S-matrix. The expansion given by equation (30) is the S-matrix expansion. The scalar product (φ, φ_0) gives the transition amplitudes. They are computed from equation (30) by using the Wick's theorem and the Feynman propagators.

The S-matrix expansion is written by Feynman and Dyson in Quantum Electrodynamics as

$$\begin{gathered} \varphi = \varphi_0 - \tfrac{1}{1!}\tfrac{i}{\hbar} \textstyle\int_{t_0}^{t} dt_1 T\left[V(t_1)\right]\varphi_0 + \tfrac{1}{2!}\left(-\tfrac{i}{\hbar}\right)^2 \textstyle\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T\left[V(t_1)V(t_2)\right]\varphi_0 + \ldots = \\ = T e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt_1 V(t_1)} \varphi_0 \ , \end{gathered} \tag{31}$$

where T denotes the chronological ordering: the interaction terms should be written from left to right in the order of the descending times. The Feynman-Dyson S-matrix expansion is incorrect. Indeed, let us compute

$$I = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 f(t_1) f(t_2) \tag{32}$$

for any function $f(t)$. We get

$$I = \int_{t_0}^{t} dt_1 f(t_1) F(t_1) - \left[F(t) - F(t_0)\right] F(t_0) \ , \tag{33}$$

where F is the primitive of the function f. Similarly, we get

$$\begin{gathered} I_T = \tfrac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T \left[f(t_1) f(t_2)\right] = \\ = \tfrac{1}{2} \int_{t_0}^{t} dt_1 \left[f(t_1) F(t_1) - F(t_1) f(t_1)\right] - \tfrac{1}{2} \left[F(t) - F(t_0)\right] F(t_0) + \tfrac{1}{2} F(t) \left[F(t) - F(t_0)\right] \ . \end{gathered} \tag{34}$$

We can see that $I \neq I_T$.

Dirac equation. The Dirac equation was devised to describe the motion of a relativistic quantum-mechanical electron. It reads

$$\gamma^{\mu} p_{\mu} \psi = mc\psi \ , \tag{35}$$

where γ^{μ} are the Dirac matrices, $p_{\mu} = i\hbar\partial_{\mu}$ are the electron momenta, $\partial_{\mu} = (\partial/c\partial t, grad)$, m is the electron mass and ψ is a four-component spinor. The equation can also be written as

$$i\hbar \frac{\partial \psi}{c\partial t} = \left(-i\hbar \boldsymbol{\alpha} grad + mc\beta\right) \psi \ , \tag{36}$$

where $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$ and $\beta = \gamma^0$. However, a relativistic electron has an energy pc of the order mc^2, which leads to a mechanical action of the order $mc^2\tau$; if $mc^2\tau <$a few $\hbar$, which means τ of the order of the Compton time $\tau_c = \hbar/mc^2$ and distances of the order of the Compton wavelength $\lambda_c = \hbar/mc$, then the Dirac equation is valid. On the contrary, if $\tau \gg \tau_c$ and distances d are such that $d \gg \lambda_c$, then we are in the quasi-classical limit and the Dirac equation becomes

$$E/c = \boldsymbol{\alpha} \boldsymbol{p} + mc\beta \ , \tag{37}$$

which is not a valid equation. The Dirac equation has not a classical limit. It follows that the Dirac equation cannot be used to describe the motion of a relativistic quantum-mechanical electron in this range of parameters. Moreover, for distances shorter than the Compton wavelength, or for shorter times than the Compton time, the electromagnetic field becomes meaningless. This can be seen, for instance, for an electron in the Coulomb field of a heavy nucleus with atomic number Z, when the field $Ze/(\hbar/mc)^2$ is comparable with Schwinger's field given by $eE(\hbar/mc) = mc^2$; we get in this case $Z = \hbar c/e^2 = 137$, which shows that such atoms cannot exist. For distances and times shorter than the Compton values, athough the Dirac equation is valid, it is not appropriate to descrbe the electromagnetic interaction. It remains that the great merit of the Dirac equation is the fact that it puts the electron equation in a relativistically invariant form, which leads to predicting the fermionic nature of the electrons, the one-half spin of the electron and the anti-electron (the positron), all consequences of the Relativity. The quantum-mechanical motion of the relativistic electron over distances and times of the order of the Compton values is the Zitterbewegung. The Zitterbewegung is not a physical motion. It is associated with the destruction and creation of the electron quantum. In the S-matrix processes, when absorbing or emitting a photon, the electron is destroyed and created, such that the uncertainty in energy is of the order mc^2. These unphysical

processes cannot be described by any equation. If it could, it would amount to describing by equations the quantum-mechanical uncertainty, which is non-sensical. Therefore, we must content ourselves with a basic cutoff in working with electron equations, of the order of the Compton wavelength for distances, and, correspondingly, the Compton time for durations. The effects associated with shorter distances and times than a cutoff of the order of the Compton cutoff are uncertainties.

Quantum-mechanical Klein-Gordon equation. The motion of the relativistic quantum-mechanical electron proceeds by a change δx^μ in the coordinates x^μ, denoted u^μ, according to the scheme $x^\mu \longrightarrow x^\mu + \delta x^\mu$, $\delta x^\mu = u^\mu$, $x^\mu \longrightarrow x^\mu + u^\mu$. We will take the first-order variations with respect to u^μ of the Dirac equation

$$\gamma^\mu \partial_\mu \psi = \frac{mc}{i\hbar}\psi \ . \tag{38}$$

According to the Dirac equation the Zitterbewegung implies that the coordinates should be viewed as matrices (like γ^μ). We write $x^\mu = s^\mu \cdot 1$, where $s^0 = ct$, $\mathbf{s} = \mathbf{r}$ and 1 denotes the unit matrix; we have $x_\mu x^\mu = s_\mu s^\mu \cdot 1 = s^2 \cdot 1$, where $s^2 = c^2t^2 - \mathbf{r}^2$. For δx^μ we need $\delta x_\mu \delta x^\mu = u_\mu u^\mu = u^2 = ds^2$; the (non-trivial) solution of this equation is

$$\delta x^\mu = u^\mu = \frac{1}{2} u \gamma^\mu \tag{39}$$

(since $\gamma_\mu \gamma^\mu = 4$). The first-order expansion of the Dirac equation is

$$\gamma^\mu \left(\partial_\mu \psi + u^\nu \partial_\nu \partial_\mu \psi \right) = \frac{mc}{i\hbar} \left(\psi + u^\nu \partial_\nu \psi \right) \ , \tag{40}$$

or

$$\partial^\mu \partial_\mu \psi = -\frac{m^2 c^2}{\hbar^2} \psi \ , \tag{41}$$

which corresponds to the Klein-Gordon equation $p_\mu p^\mu \psi = m^2 c^2 \psi$. We note that u is absorbed into ψ, hence the bosonic character of the latter. We write

$$\psi = \sum_{\mathbf{k}} c \sqrt{\frac{\hbar}{2\varepsilon_k}} \left(\psi_{\mathbf{k}\alpha} \right) e^{i\mathbf{kr}} \ , \tag{42}$$

where $\varepsilon_k = \varepsilon_{\mathbf{k}} = c\sqrt{k^2 + m^2 c^2/\hbar^2}$,

$$\left(\psi_{\mathbf{k}\alpha} \right) = \begin{pmatrix} c_{\mathbf{k},+1} \\ c_{\mathbf{k},-1} \\ b^*_{-\mathbf{k},-1} \\ b^*_{-\mathbf{k},+1} \end{pmatrix} \ , \tag{43}$$

and the c's and the b's satisfy usual bosonic commutation relations for four distinct types of bosons, corresponding to $c_{\mathbf{k}\sigma}$ and $b_{\mathbf{k}\sigma}$; $\sigma = \pm 1$ is the spin label and the c's and the b's correspond to positive and negative energies (frequencies), respectively. We attribute this field to the motion of the electrons. This bosonic field has an energy

$$H_e = \sum_{\boldsymbol{k}\sigma} \hbar \varepsilon_{\boldsymbol{k}} \left(c^*_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + b^*_{\boldsymbol{k}\sigma} b_{\boldsymbol{k}\sigma} \right) \ , \tag{44}$$

a momentum and a charge, similar to the Dirac field, and a current density given by

$$j^\mu = (c\rho, \boldsymbol{j}) = \frac{ie}{\hbar} \left(\overline{\psi} \partial^\mu \psi - \left(\partial^\mu \overline{\psi} \right) \psi \right) \ , \tag{45}$$

or

$$\rho = \frac{ie}{\hbar c^2}\left(\overline{\psi}\dot{\psi} - \dot{\overline{\psi}}\psi\right) \ , \ j_i = -\frac{ie}{\hbar}\left(\overline{\psi}\partial_i\psi - \partial_i\overline{\psi}\psi\right) \ , \tag{46}$$

where the adjoint $\overline{\psi}$ is the transposed and conjugate to ψ. This current density can be written immediately in terms of the c and b operators. In addition, the Klein-Gordon equation can be obtained from a lagrangian theory. The electron motion is the motion of the Zitterbewegung.

The linear approximation used in the expansion above spoils the effects of the electromagnetic field, such that, in the presence of the electromagnetic field, we need to use the covariant derivative $D_\mu = \partial_\mu - \frac{e}{ic\hbar}A_\mu$, according to $\psi \longrightarrow \psi + u^\mu D_\mu\psi$; we get the Klein-Gordon equation

$$\left(p_\mu - \frac{e}{c}A_\mu\right)\left(p^\mu - \frac{e}{c}A^\mu\right)\psi - \frac{ie\hbar}{2c}\sigma^{\mu\nu}F_{\mu\nu}\psi = m^2c^2\psi \tag{47}$$

with electromagnetic field. In addition, the limits of the Electromagnetism imply a lower bound upon the interaction (photons) wavelength, of the order of the Compton wavelength (as well as upon the electron wavelength). We can see that it amounts to applying twice the Dirac equation. We can see that the interaction terms are

$$v_p = \frac{1}{c}j_\mu A^\mu = \frac{ie}{c\hbar}\left[\overline{\psi}\left(\partial_\mu\psi\right) - \left(\partial_\mu\overline{\psi}\right)\psi\right]A^\mu \tag{48}$$

(where the gauge condition $\partial_\mu A^\mu = 0$ is used),

$$v_d = -\frac{e^2}{c^2\hbar^2}\left(\overline{\psi}\psi\right)A_\mu A^\mu \tag{49}$$

and

$$\begin{aligned} v_H &= -\tfrac{e}{2c\hbar}\overline{\psi}\left(\textstyle\sum + \sum^*\right)\psi\mathbf{H} \ , \\ v_E &= \tfrac{ie}{2c\hbar}\overline{\psi}\left(\boldsymbol{\alpha} - \boldsymbol{\alpha}^*\right)\psi\mathbf{E} \ , \end{aligned} \tag{50}$$

where

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \ , \ \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \tag{51}$$

and $\boldsymbol{\sigma}$ are the Pauli matrices. The interactions v_H and v_E arise from the $\sigma^{\mu\nu}F_{\mu\nu}$-term in equation (47). This is known as the "Pauli term". The symmetrization and antisymmetrization in equations (50) ensure the consistency of the equations of motion (for ψ and $\overline{\psi}$ and real energy).

Now, we can proceed to do perturbation theory and S-matrix expansion (using equation (30)) making use of the free electron hamiltonian H_e given by equation (44), the free photon hamiltonian $\mathcal{E}_t$ given by equation (26) (only transverse photons) and the interactions terms given above. Obviously, we must use a cutof wavelength of the order of the Compton wavelength $\lambda_c = \hbar/mc$. Similarly, we retain in v_p only the transverse part.

Experimental situation. The calculations performed by using the Dirac field imply using twice the Dirac equation, which amounts to using the Klein-Gordon equation. In the lowest order of the perturbation theory such calculations do not imply the electron self-energy, nor the vacuum polarization. The results are the same as those obtained by the Dirac theory of radiation, or by using the quantum-mechanical theory of perturbation, as described in Heitler's book.[2] To this approximation, the interaction does not occur over regions with dimension smaller than the Compton wavelength. These results are in good agreement with experimental measurements. Higher-order

[2]W. Heitler, *The Quatum Theory of Radiation*, Dover (1984).

calculations in the perturbation theory, involving electron self-energy and vacuum polarization, imply interactions over distances of the order of the Compton wavelength (self-interaction), and lead to divergences (ultraviolet divergences). Apart from their theoretical interest, such calculations were prompted by two experimentally measured effects, namely the Lamb shift and the anomalous magnetic momemt of the electron.

In order to eliminate these divergences the Quantum Electrodynamics devised a renormalization and regularization scheme. The resulting finite quantities are extremely small corrections. It is claimed that these results are in very good agreement with the experimental measurements.

The renormalization and regularization procedure is based on arbitrary modifications of the divergent integrals and on an obscure claim that the interaction effects should act upon bare particles with an infinite mass and an infinite charge, such that, cancelling out these two infinities, the one arising from interaction and the other arising from the bare particles, we would get finite results. This procedure is almost impossible to be applied in practice, and many errors were discovered in calculations of the Lamb shift and the anomalous magnetic moment of the electron.[3] Moreover, it seems that the experimental measurements are affected by unknown, unexplainable errors. Indeed, according to the boson theory described above, the results must depend on the cutoff used, which is of the order of the Compton wavelength but it has no definite value. These calculations, as well as the corresponding experimental results only tell that we are dealing with uncertainties; we can assess their order of magnitude, but, of course, we cannot give a definite value.

[3] O. Consa, "Something is rotten in the state of QED" (2020).