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Topological conductors

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Abstract

Quasi-free electrons sliding over the free surface of a solid are identified. We may say that this circumstance gives rise to topological conductors. Superficial plasmonic oscillations and polaritonic waves are highlighted for these electrons. It is shown that the surface electrons behave as a degenerate two-dimensional Fermi gas (thermodynamically equivalent with a two-dimensional Bose gas). A characterization as complete as possible is made for the surface electrons, including single-particle properties, transport properties, the solenoidal effect and the skin effect. The transport properties are governed by much shorter mean freepaths and lifetimes of the quasi-particles, due to te lower density of the surface electrons in comparison with the bulk electrons. Absorption rate of uniform quasi-static magnetic field is computed.

Introduction. Let us consider an ensemble of many fermions, leaving aside their interaction and the question of their stability. This is a very idealized picture, where we may expect surprising things. One of such things is related to the dimensionality of the space the fermions move in. Indeed, in three dimensions the density dn of fermionic states is proportional to $\sqrt{\varepsilon}d\varepsilon$, where ε is the energy of a fermionic state, while in two dimensions $dn \sim d\varepsilon$ and in one dimension $dn \sim d\varepsilon/\sqrt{\varepsilon}$. We can see that at the Fermi level there exists a large state density in three dimensions, which indicates the possibility of a normal Fermi liquid picture, when interaction is present; in two dimensions the state density does not depend on the Fermi level, so we may expect no difference between the thermodynamics of a two-dimensional gas of fermions and the thermodynamics of a two-dimensional gas of bosons. Indeed, this indication is verified by a direct calculation.[1] In one dimension the fermionic state density is very small at the Fermi level, such that we expect a dynamics governed entirely by bosonic density degrees of freedom, and no Fermi surface (points). We say that the one-dimensional models of fermions are bosonized.[2]

More surprises are claimed to appear in two-dimensional ensembles of many fermions. For instance, since in two dimensions there exist rotations only about one axis, it is suggested that the spin does not exist anymore for fermions in two dimesions. Moreover, an electron state around a localized magnetic flux may have any phase; in particular, for an integral number of magnetic flux quanta the electric charge of the electron would be fractional, because the electric charge multiplies the magnetic flux in the phase. So, it appeared the idea of composite electrons, *i.e.* electrons with an associated number of quanta of magnetic flux, which would have a fractional occupation number and a fractional statistics (any fraction, so their denomination of anyons).[3]

The fermion ensembles in two dimensions received a new impetus with the discovery of the graphene.[4] Since graphene sheets occur with a small size, the edge electronic states have been

brought into discussion.[5] The electric field preserves a Kramers degeneracy of the electronic spin states, related to the time reversal symmetry; but a magnetic field removes this degeneracy, and trasforms the graphene bulk in an insulator. This is not so for the edge states, which have a completely different spectrum than the bulk ones. The graphene edges remain conducting the electrical current. It is said that the edge states are protected by the time reversal symmetry, and they define a conductor surrounding an insulator; the latter is called a topological one, because its surface is a conductor.[6]

However, the surface is the deed of the devil. It seems that Pauli said that "God made the bulk; the surface was invented by the devil". We show in this paper that quasi-free surface electrons are ubiquituous

Topological conductors. Let us consider a solid consisting of N atoms (ions), placed at positions $\boldsymbol{r}_i$, $i = 1, 2, ...N$, and N electrons. The wavefunction of a free electron is $\frac{1}{\sqrt{V}}e^{i\boldsymbol{kr}}$,where $\boldsymbol{k}$ is the wavevector and V is the volume. An electron can be trapped in an atom (ion), where its wavefunction is $\frac{1}{\sqrt{v}}\chi(\boldsymbol{r}-\boldsymbol{r}_i)$; χ is a function localized over the volume v, which is of the order of the volume assigned in the solid to an atom placed at $\boldsymbol{r}_i$, such that $\frac{1}{v}\int d\boldsymbol{r} \mid \chi \mid^2 = 1$. The wavefunction of an electron trapped in any atom in solid is

$$\frac{1}{\sqrt{N}}\sum_i \frac{1}{\sqrt{v}}\chi(\boldsymbol{r}-\boldsymbol{r}_i) \ . \tag{1}$$

The amplitude of probability for a electron to be trapped in solid is

$$\frac{1}{\sqrt{N}}\sum_i \frac{1}{\sqrt{vV}}\int d\boldsymbol{r}\chi^*(\boldsymbol{r}-\boldsymbol{r}_i)e^{i\boldsymbol{kr}} = \frac{1}{\sqrt{N}}\sum_i \sqrt{\frac{v}{V}}e^{i\boldsymbol{kr}_i} \ , \tag{2}$$

such that the probability for the electron to be trapped is

$$P = \frac{v}{NV}\sum_{ij} e^{i\boldsymbol{k}(\boldsymbol{r}_i-\boldsymbol{r}_j)} \ . \tag{3}$$

If the atoms are distributed randomly, the summation in equation (3) gives $N^2\delta_{\boldsymbol{k}0}$, such that we get $P = Nv/V$. Since $Nv = V$, this probability is equal to unity. It follows that in an amorphous solid all the electrons can be trapped, and the solid is an insulator, as it is well known. If the solid is crytalline, the summation in equation (3) gives $N^2\delta_{\boldsymbol{kg}}$, where $\boldsymbol{g}$ is a vector of the reciprocal lattice. It follows that the electron states lying on the walls of the Brillouin zone are trapped. The corresponding energy bands are blocked by the exclusion principle. Therefore, again the solid is an insulator (actually, there are $2N$ electrons in each band, as a consequence of their spin states). If the band is not completed, the crystalline solid is a conductor, as it is well known.

The situation is different in a two-dimensional solid, *i.e.* a solid with the atoms arranged on a plane surface. In this case the electrons are delocalized over a transverse distance d, where $d > a$, $v = a^3$. The above probability reads $P = (N\sigma/S)\frac{a}{d}$, where S is the area of the solid, $V = Sd$, and $\sigma = a^2$ is the area of the cross-section of the volume assigned to an atom. Now, for an amorphous two-dimensional solid $N\sigma/S = 1$, such that $P = a/d < 1$. It follows that an amorphous two-dimensioanal solid is always a conductor. A similar conclusion holds for a crystalline two-dimensional solid, and for a one-dimensional solid.

An estimation of the distance d can be obtained from simple geometrical arguments. An electron moving along a single plane layer of atoms has a transverse distance $d = 2a$ at its disposition, while it has only the distance $d = a$ if the layer of atoms is sandwitched between two other atomic

layers. It the layer of atoms is deposited on a solid surface, the distance d becomes equal to a, but the extent of the atomic volume in the transverse direction is reduced to $a/2$. In both cases the ratio a/d is equal to $1/2$.

It is well known that the mean inter-atomic separation distance is modified at the surface of a solid, in comparison with the bulk, albeit to a very small extent. For instance, in an anharmonic solid the lattice parameter is slightly increased at the surface, or has an oscillatory behaviour.[13] This makes the electrons at the surface to have a small additional room, measured by the transverse distance $d > a$. On the other hand, the electrons and the ions at the surface form a charge inversion layer, where the electrons spill over the surface, leaving behind a positive ionic charge; the characteristic dimension of this layer is of the order of the mean inter-atomic distance.[8]-[10] Therefore, we may expect that a small amount of electrons at the surface of an insulator are quasi-free. Their motion is completely decoupled from the bulk electrons. We may say that this circumstance defines a topological conductor. A special discussion requires metals consisting of micro-crystallites, or metallic glasses, alloys, etc, where the electron band structure varies slightly locally (in a quasi-classical description). This variation is abrupt at the surface, where the energy bands are split and the surface quasi-free electrons occupy an incomplete band.

We give here an estimation of the thickness of the distorted superficial layer of a solid. As it is well known, at the atomic level the physical quantities vary abruptly and indeterminately in space and time. Usually, we average this motion over small spatial regions and small durations. Thereby, we get homogeneous domains, which microscopically may be sufficiently large, but macroscopically they are very small. At the surface this homogeneity is lost, such that we may estimate the thickness of the superficial layer as the dimension of a domain. The variation of the volume due to the atomic extent of the particles is $\delta\Omega/\Omega = a_0^3/a^3$, where a_0 is the atomic dimension and a is the mean inter-atomic separation distance; it follows $\delta\Omega = (a_0/a)^3 a^3 N_d$, where N_d is the number of atoms in a domain. On the other hand, the volume variation of a domain is $\delta\Omega = Aa = N_d^{2/3} a^3$, where A is the area of the surface which encloses the domain. These two relations lead to $N_d^{1/3} = (a/a_0)^3$, which indicates a dimension $D = N_d^{1/3} a = (a/a_0)^3 a$ for a domain. For instance, making use of typical values $a_0 = 1$Å and $a = 4$Å, we get $D = 64a = 256$Å and $N_d = (R/a)^3 \simeq 10^5$. This is an estimation of the size of the domains and the thickness of the superficial layer. An order of magnitude 10^2Å is estimated, by other arguments in Ref. [11]. Averaging over fractions of domains at the surface leads to surface electromagnetic fields distinct from the bulk fields, which sustain surface plasmon-polaritons.[12] In addition, the change in the electronic structure at the surface highlights surface electrons decoupled from the bulk.

it is expected that the density n of the surface electrons is much reduced in comparison with their bulk density n_b; we write $n = \beta n_b$ (the typical electron density in a solid is $n_b = 10^{22} cm^{-3}$). Indeed, the maximum deviation from homogeneity of a single domain describe above is one atom in 64 atoms, which indicates a factor $\beta = 1/64$ of the order $\beta = 10^{-2}$. On the other hand, the relative variation of the lattice parameter at the surface of an anharmonic solid is of the order 10^{-6}.[7] We may take an average order of magnitude 10^{-4} for the parameter β.

Collective motion. As it is well known, the existence of a (quasi-) plane surface leads to the existence of a surface plasmonic mode with frequency $\omega_p/\sqrt{2}$, where $\omega_p = (4\pi n_b e^2/m)^{1/2}$ is the bulk plasmon frequency; $-e$ and m are the electron charge and the electron mass.[13] This mode corresponds to a motion perpendicular to the surface and exists only in conjunction with the bulk plasmon. On the other hand, polaritonic modes may exist in the superficial layer, due to electromagnetic waves propagating along the surface. These modes have been identified recently in a wire with a circular cross-section;[12] they may guide electromagnetic waves along the surface, propagating dispersionless with the speed of light in vacuum.[14]-[17] These modes are called

surface plasmon-polariton modes, since they have a resonant behaviour at the surface plasmon frequency.

A different plasmon mode may appear along the finite dimension of a superficial layer. In general, the bulk plasmon is generated by the local internal (polarization) field. Indeed, if the charges q ($q=-e$) with concentration n_b suffer a local displacement $\boldsymbol{u}$, then a local charge density imbalance $-n_b q div\boldsymbol{u}$ occurs, which generates an internal field given by $div\boldsymbol{E}=-4\pi n_b q div\boldsymbol{u}$. This equation holds locally, with a spatial variation in all the three directions, in general. Consequently, the internal field is given by $\boldsymbol{E}=-4\pi n_b q\boldsymbol{u}$. In the equation of motion for a charge

$$m\ddot{\boldsymbol{u}}+m\omega_c^2\boldsymbol{u}+m\gamma\dot{\boldsymbol{u}}=q\boldsymbol{E}_0+q\boldsymbol{E}\ , \tag{4}$$

where ω_c is a characteristic (internal) frequency of the charge, γ is a small dissipation coefficient and $\boldsymbol{E}_0$ is the external electric field, the internal field generates the term $-4\pi n_b q^2\boldsymbol{u}$, which leads to the plasma frequency $\omega_p=(4\pi n_b q^2/m)^{1/2}$. For the motion of the superficial charges along a finite dimension L the div does not hold locally anymore; the smallest charge density imbalance is given approximately by $\delta n\simeq nu/L$, where $n=\beta n_b$ ($n_b=1/a^3$) is the surface electron density, u is the displacement and L is of the order of the linear dimension of the surface, for a body with a relatively smooth surface. The electric potential of the surface charges is of the order is $\varphi\simeq\beta^{1/3}q/a$. The corresponding interaction energy for a charge is $q\varphi(u/L)^2=(\beta^{1/3}q^2/a)(u/L)^2$, which indicates that a charge has an eigenfrequency ω_0 given by $(\beta^{1/3}q^2/a)(u/L)^2=m\omega_0^2u^2$, *i.e.*

$$\omega_0=\frac{1}{L}\sqrt{\beta^{1/3}\frac{q^2}{ma}}=\beta^{1/6}\frac{a}{L}\sqrt{\frac{q^2}{ma^3}}=\beta^{1/6}\omega_p\frac{a}{L}\ . \tag{5}$$

This eigenfrequency of the motion of the surface electrons is much lower than the bulk plasma frequency; indeed, for $L=10cm$ we get $\omega_0\simeq 10^{-8}\omega_p$ (for typical distances $a=4$Å. we may neglect the factor $\beta^{1/6}$). The equation of motion for these superficial charges $m\ddot{\boldsymbol{u}}+m\omega_c^2\boldsymbol{u}+m\omega_0^2\boldsymbol{u}+m\gamma\dot{\boldsymbol{u}}=q\boldsymbol{E}_0$ gives an internal field $\boldsymbol{E}_i=-m\omega_0^2\boldsymbol{u}/q=\boldsymbol{E}_0\omega_0^2/(\omega^2-\omega_c^2-\omega_0^2+i\omega\gamma)$, a total field $\boldsymbol{E}_t=\boldsymbol{E}_0+\boldsymbol{E}_i=\boldsymbol{E}_0(\omega^2-\omega_c^2+i\omega\gamma)/(\omega^2-\omega_c^2-\omega_0^2+i\omega\gamma)$, a surface dielectric function $\varepsilon=1-\omega_0^2/(\omega^2-\omega_c^2+i\omega\gamma)$ and a conductivity $\sigma=(nq^2/m)i\omega/((\omega^2-\omega_c^2+i\omega\gamma)$ ($\boldsymbol{j}=nq\dot{\boldsymbol{u}}=\sigma\boldsymbol{E}_t$). The resonance exhibited by the polarizability $\alpha=\omega_0^2/((\omega^2-\omega_c^2-\omega_0^2+i\omega\gamma)$ at $\omega^2=\omega_c^2+\omega_0^2$ may be tested experimentally ($\boldsymbol{E}_i=\alpha\boldsymbol{E}_0$). For the surface quasi-free electrons $\omega_c=0$. We note that the surface conductivity is smaller by the factor β than the bulk conductivity. $\omega_{ps}=(4\pi\beta n_b q^2/m)^{1/2}$ may be viewed as the plasma frequency of the surface electrons.

We note that the plasma frequency given above can be written as $\omega_0=v/R$, where $v=\sqrt{q^2/ma}$ is a characteristic velocity (we leave aside the factor $\beta^{1/6}$); for electrons and typical values of a this velocity is of the order $10^8cm/s$; for $L=10cm$ we get $\omega_0=10Mhz$ (and $\omega_p=10^{15}s^{-1}$). A similar set of frequencies cl/R, where c is the speed of light in vacuum and $l=1,2,,,,$ was obtained for polaritons moving along the circumference of a wire with radius R.[12]

Single-particle properties. Thermodynamics. We consider here the surface electrons of a long and thin wire, with length L and radius R of the circular circumference, $L\gg R$. We consider both L and R much larger than the mean inter-atomic separation distance a and omit the motion across the small thickness D of the surface layer. As it is well known, the thermodynamic properties of a two-dimensional gas of electrons are equivalent with the thermodynamic properties of a gas of bosons.[1] (In two dimensions there is no superfluid transition). The single-particle wavefunctions of the surface electrons are $\frac{1}{\sqrt{2\pi}}e^{il\varphi}\frac{1}{\sqrt{L}}e^{ikx}$, where φ is the angle on the circumference, l is any integer and k is the continuous wavevector along the length of the wire (direction x). The corresponding single-electron energies are given by

$$\varepsilon=\frac{\hbar^2}{2mR^2}l^2+\frac{\hbar^2}{2m}k^2\ . \tag{6}$$

These energies define a continuous two-dimensional Fermi sea, which is a circular disk in the variables $\sqrt{\hbar^2/2mR^2}l$ and $\sqrt{\hbar^2/2m}k$. The superficial density of electrons is given by

$$n_s = \frac{m}{\pi\hbar^2}\int_0^\infty d\varepsilon \frac{1}{e^{(\varepsilon-\mu)/T}+1} \ , \tag{7}$$

an equation wich defines the chemical potential μ (spin included). We can see that the thermal wavelength $\lambda = (\pi\hbar^2/2mT)^{1/2}$, where T is the temperature, is much smaller than the (macroscopic) dimensions R, L and the thickness of the layer of the superficial electrons, over a wide range of temperatures, such that the thermodynamcs is well defined and the gas is degenerate. From equation (7) we get $\mu = \pi\hbar^2 n_s/m$. Therefore, the chemical potential of the surface electrons is smaller by the factor $\beta^{2/3}$ than the chemical potential of the bulk electrons. The grand-partition potential is given by

$$\Omega = -\frac{mA}{\pi\hbar^2}\int_0^\infty d\varepsilon \frac{\varepsilon}{e^{(\varepsilon-\mu)/T}+1} = -E = -pA \ , \tag{8}$$

where A is the area of the cylindrical surface of the wire, E is the energy and p is the "pressure", *i.e.* the force per unit length. We get

$$E = \frac{mA}{2\pi\hbar^2}\mu^2\left(1+\frac{\pi^2}{3}T^2/\mu^2+...\right) \ , \tag{9}$$

whence, by means of $E = -T^2\partial(F/T)/\partial T$, we may get the free energy and all the other thermodynamic properties. The contribution of the surface electrons to the speific heat is $C_s = \pi mAT/3\hbar^2$.

Transport properties. The two-dimensional Fermi sea defined by the chemical potential μ computed above has a Fermi wavevector of the order $\beta^{1/3}1/a$, where a denotes the mean separation distance between the (bulk) electrons, in both directions, *i.e.* along the wire and along the circumference of the wire. In typical metals the Fermi velocity $v_F = \hbar/ma$ is of the order $10^7 - 10^8 cm/s$ ($n = 10^{22}cm^{-3}$). For the surface electrons it is diminished by the factor $\beta^{1/3}$. The single-particle elementary excitations are quasi-particles, with a lifetime governed by the thermal uncertainty, phonons, impurities, etc. The static electrical conductivity computed above reads $\sigma = nq^2/m\gamma = \beta\omega_p^2/4\pi\gamma$. This equation may be used to get the dissipation coefficient γ. In typical metals at room temperature $\gamma = 10^{13} - 10^{14}s^{-1}$ (and $\omega_p = 10^{15}s^{-1}$) for bulk electrons; we may assume that the same order of magnitude is preserved for the surface electrons. The parameter γ defines a lifetime $\tau_{lf} = 1/\gamma$ and a mean freepath $\Lambda = v_F\tau_{lf}$ of the order $0.1 - 1\mu m$ for bulk electrons at room temperature. For the surface electrons the mean freepath is $\Lambda_s = \beta^{1/3}\Lambda$. Similarly, the quasi-particles lifetime is of the order $\tau_{lf} = \hbar\mu/T^2$, which gives $\gamma = 10^{12}s^{-1}$ for bulk electrons ($\mu = 1eV$) at room temperature. For the surface electrons we get $\tau_{lf} = \beta^{2/3}\tau_{lfbulk}$, $\gamma_s = \beta^{-2/3}\gamma$ and $\Lambda_s = \beta\Lambda$. We can see that the mean freepath of the surface electrons may be appreciably smaller than the mean freepath of the bulk electrons. The transport properties of the surface electrons are much poorer than those of the bulk electrons, as a consequence of their much lower density. The dissipation coefficient is diminished in oscillations ($\omega \neq 0$). We note that the surface current along the wire is smaller by a factor D/R than the bulk current, where D is the thickness of the surface layer (estimated above as 256Å).

Solenoidal effect. Let us assume a uniform harmonic displacement

$$u_0 e^{-i\omega t} D\delta(r-b) \tag{10}$$

along the circumference of the wire with radius b; D is the tickness of the superficial region where this displacement is confined. It gives rise to a current density

$$\boldsymbol{j} = -i\omega nqu_0(0, -\sin\Phi, \cos\Phi)e^{-i\omega t}D\delta(r-b) \ , \tag{11}$$

were Φ is the angle in the (y,z)-plane. The vector potential is given by

$$\boldsymbol{A} = \frac{1}{c}\int d\boldsymbol{R}' \frac{\boldsymbol{j}(\boldsymbol{R}', t- \mid \boldsymbol{R}-\boldsymbol{R}' \mid /c)}{\mid \boldsymbol{R}-\boldsymbol{R}' \mid} \ , \tag{12}$$

where $\boldsymbol{R} = (x, r\cos\varphi, r\sin\varphi)$ and $\boldsymbol{R}' = (x', r'\cos\Phi, r'\sin\Phi)$. We introduce the angle $\varphi' = \varphi - \Phi$ and notice that the contribution to the integral comes from even functions of $\cos\varphi'$. The integral over x' gives the Hankel function $H_0^{(1)}$, such that we get[18]

$$\boldsymbol{A} = \frac{2\pi nq\omega}{c} u_0 bD \left(0, -\frac{z}{r}, \frac{y}{r}\right) \int_0^\pi d\varphi' \cos\varphi' H_0^{(1)} \left(\frac{\omega}{c}\sqrt{r^2+b^2-2rb\cos\varphi'}\right) e^{-i\omega t} \ . \tag{13}$$

In this expression we can use the asymptotic behaviour of the Hankel functions. We limit ourselves to quasi-static fields, *i.e.* fields, which satisfy the condition $\omega r/c$, $\omega b/c \ll 1$ (wavelengths much larger than the dimensions of the wire). The Hankel function goes like $H_0^{(1)}(z) \simeq \frac{2i}{\pi}\ln z$ in this limit, such that we get

$$\boldsymbol{A} = -\frac{2\pi inq\omega}{c} u_0 b^2 D \left(0, -\frac{z}{r^2+b^2}, \frac{y}{r^2+b^2}\right) e^{-i\omega t} \tag{14}$$

and the magnetic field

$$\boldsymbol{H} = curl A = -\frac{4\pi inq\omega}{c} u_0 b^2 D e^{-i\omega t} \left(\frac{b^2}{(r^2+b^2)^2}, 0, 0\right) = \frac{4\pi}{c} Dj \left(\frac{b^4}{(r^2+b^2)^2}, 0, 0\right) \ ; \tag{15}$$

this formula coincides with the well-known magnetic field along the axis of an infinite solenoid, $H_x = 4\pi Dj/c$ $(r \longrightarrow 0)$. The electric field is smaller by a factor $\omega b/c$ than the magnetic field.

Let us consider a uniform, quasi-static longitudinal magnetic field H applied along the axis of the wire. According to the Faraday equation $curl\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t}$ it generates an electric field E along the circumference, given by $E = i\omega rH/2c$. If electrical charges are present along this circumference, they get a displacement $u = -iqrH/2mc(\omega + i\gamma)$ and a current density $j = -nq^2\omega rH/2mc(\omega + i\gamma) = \sigma E$. For reasonable values of these parameters the displacement is very large. We can estimate the induced magnetic field H_i from the Maxwell-Ampere equation $curl\boldsymbol{H}_i = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}$, which leads to

$$H_i = \left(\frac{\omega^2}{2c^2} - \frac{\omega_p^2}{2c^2}\frac{\omega}{\omega + i\gamma}\right)\frac{r^2}{3}H = \frac{1}{2c^2}\left(\omega^2 + 4\pi i\sigma\omega\right)\frac{r^2}{3}H \ . \tag{16}$$

This field is uniform and directed along the axis of the wire. For reasonable values of the parameters (even if the factor β is included for the surface electrons), this induced magnetic field is much larger than the applied field and directed in the opposite direction. Therefore, we may expect that the applied magnetic field is expeled from the wire. This result is valid for metals. For insulators with surface electrons the conductivity is small and real, and the external magnetic field may penetrate.

The above estimation does not include the mutual dependence of the external and induced fields. This dependence is included by $curl\boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\sigma\boldsymbol{E}$, or $\Delta\boldsymbol{H} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} - \frac{4\pi\sigma}{c^2}\frac{\partial \boldsymbol{H}}{\partial t} = 0$; the second-order time derivative may be neglected, such that, the solution of this equation in cylindrical coordinates is a Bessel function of imaginary argument, which is damped over distances of the order $c/\sqrt{\mid \sigma \mid \omega} \simeq c/\omega_p$, *i.e.* the plasma wavelength. This is the well-known skin effect. For metals this wavelength is very small, for insulators it is large, irrespective of the presence of the surface electrons.

Quantum-mechanical effects. Let us focus now on the surface electrons in a wire with radius R; their single-particle wavefunctions are $\frac{1}{\sqrt{2\pi}}e^{il\varphi}\frac{1}{\sqrt{L}}e^{ikx}$ and their energy is given by equation (6). We assume an external uniform quasi-static electromagnetic field with the magnetic component $\boldsymbol{H}$ directed along the axis of the wire, $\boldsymbol{H}=(H,0,0)$, where $H=H_0\cos\omega t$. The vector potential is $\boldsymbol{A}=\frac{1}{2}H(0,-z,y)$; in cylindrical coordinates it has only the angular component $A_\varphi=\frac{1}{2}Hr$, where r is the radius of the cross-section. For the moment we allow also a motion along the small thickness D of the surface layer (coordinate r) and write the hamiltonian of these free electrons as

$$\mathcal{H}=\frac{1}{2m}\left(\boldsymbol{p_r}-\frac{q}{c}\boldsymbol{A}\right)^2+\frac{1}{2m}p_x^2\ . \tag{17}$$

The canonical transformation $\psi=e^{\frac{iq}{c\hbar}\int^{\boldsymbol{r}}\boldsymbol{A}d\boldsymbol{l}}\chi$ leads to the Schroedinger equation

$$i\hbar\frac{\partial\chi}{\partial t}=\frac{1}{2m}\left(p_{\boldsymbol{r}}^2+p_x^2\right)\chi-q\int^{\boldsymbol{r}}\boldsymbol{E}d\boldsymbol{l}\chi\ ; \tag{18}$$

we can see that this canonical transformation removes the vector potential and introduces the work done by the electric field $\boldsymbol{E}$ upon the electrons. This electric field is given by $\boldsymbol{E}=-\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}$ and has only the φ-component (acts along the circumference); it is proportional to the vector potential, dephased in time. Now, we can neglect the transverse motion and writes the Schroedinger equation as

$$i\hbar\frac{\partial\chi}{\partial t}=-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{R^2\partial\varphi^2}+\frac{\partial^2}{\partial x^2}\right)\chi-\frac{q\omega H_0R^2}{2c}\varphi\sin\omega t\ . \tag{19}$$

In this equation we may view the H_0-term as a perturbation, denoted by $V\sin\omega t$. The amplitude of transition from the initial state i to the final state f is given by

$$a_{fi}=-\frac{i}{\hbar}\int_0^t dt'V_{fi}\sin\omega t'e^{\frac{i}{\hbar}(\varepsilon_f-\varepsilon_i)t'}\ . \tag{20}$$

For absorption we retain only the term $e^{-i\omega t}$ in $\sin\omega t$. The transition does not imply the motion along the x-coordinate; it is a transition from the state l to the state l'. For a long time we get the number of transitions per unit time

$$\mid a_{fi}\mid^2/t=\frac{\pi^2C^2}{\hbar}\frac{1}{(l'-l)^2}\delta\left(\frac{\hbar^2}{2mR^2}(l'^2-l^2)-\hbar\omega\right)\ , \tag{21}$$

where $C=-q\omega H_0R^2/2c$. We sum (integrate) over the final states l' and get

$$\mid a\mid^2/t=\frac{\pi^2C^2}{\hbar}\frac{2mR^2}{\hbar^2}F(l)\ , \tag{22}$$

where

$$F(l)=\frac{16l^3}{B^2}\frac{2l^2+B}{(B+4l^2)^2} \tag{23}$$

and $B=2mR^2\omega/\hbar$. In order to get the total rate of absorption we should sum (integrate) over all the states l up to l_{max} given by $\mu=\frac{\hbar^2}{2mR^2}l_{max}^2$, which amounts to $B=\frac{\hbar\omega}{\mu}l_{max}^2$. It is easy to see that in equation (23) $l/l_{max}<1$ is much larger than $\sqrt{B}/l_{max}$. Finally, we multiply by $\hbar\omega$ to get the absorbed power given by

$$P=\frac{\pi^2q^2\omega^2H_0^2R^4}{4c^2\hbar}\ . \tag{24}$$

The electrons near the Fermi surface are quasi-classical, except those with small k (and large l) and those with small l (and large k). The states with small values of l are quantum-mechanical. If we neglect the transverse motion their wavefunction can be writen as

$$\psi = e^{\frac{iq}{2c\hbar}HR^2\varphi}\chi \ , \tag{25}$$

where $\chi = \frac{1}{\sqrt{2\pi}}e^{il\varphi}\frac{1}{\sqrt{L}}e^{ikx}$. The current density associated with the hamiltonian given by equation (17) is

$$\boldsymbol{j} = \frac{iq\hbar}{2m}\left(grad\psi^* \cdot \psi - \psi^* grad\psi\right) - \frac{q^2}{mc}\boldsymbol{A}\psi^*\psi \tag{26}$$

(leaving aside the spin contribution). If we neglect the perturbation and use the wavefunctiions given by equation (25), the diamagnetic contribution to the current (the $\boldsymbol{A}$-term) cancels out exactly the paramagentic contribution, such that we get

$$\boldsymbol{j} = \frac{1}{2\pi L}\left(\frac{q\hbar k}{m}, \frac{q\hbar l}{mR}, 0\right) \tag{27}$$

(in cylindrical coordinates x, φ, r; a proper normalization introduces a factor $1/RD$). Since there exist two values $\pm k$ and two values $\pm l$, the total current is zero.

If $qHR^2/2c\hbar = qH\pi R^2/ch = integer$, *i.e.* the flux $\Phi = \frac{ch}{q} \times integer$, for a constant magnetic field, then the wavefunctions ψ are the same as the wavefunctions χ; ch/q is the quantum of magnetic flux.

Concluding remarks. Quasi-free electrons are identified on the surface of solids, irrespective of the structure and the state, which define topological conductors. The surface electrons are decoupled from the bulk electrons. Their occurrence is caused by the surface discontinuity, where the solid properties vary abruptly. The surface electrons from a quasi-two-dimensional electron gas, with poor transport properties. They exhibit specfic plasmonic modes, sustain surface plasmon-polariton modes and are active in absorption of quasi-static magnetic fields.

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