Journal of Theoretical Physics

Founded and Edited by M. Apostol **331** (2021)

ISSN 1453-4428

Relativistic uncertainties

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Abstract

It is shown that the classical electromagnetism requires the pointlike charges to move indefinitely within a region with a finite classical electromagnetic radius. It is shown that the quantum-mechanical motion of a relativistic particle requires the pointlike particles to move indefinitely within a region with a finite radius. For massive particles this radius is of the order of the Compton wavelength. For photons it is of the order of the wavelength. This cutoff length, which governs relativistic uncertainties, should be included in the interaction effects. The reason for its occurrence is the quantization of the relativistic energy-momentum relation; the quantum-mechanical motion implies specific uncertainties in position, duration and dynamical variables. Particularly obvious is this indefinite motion in the presence of the matrices in the Dirac equation, which mix up the coordinates in the Zitterbewegung. The quantum-mechanical motion of the electron field is indefinite. The definite quantum-mechanical motion of the electrons is the motion of the Zitterbewegung, with observance of the Compton cutoff. Thus, for electromagnetically interactiong electrons we are led to a theory of "bosons" with charge and spin, corresponding to the electron anti-particle and spin states. The perturbation theory of these electromagnetically interacting bosons is presented, including electromagnetic quantum effects like spontaneous emission, mass and charge renormalization, Lamb shift and anomalous magnetic moment, pair creation and photon mass.

1 Classical electromagnetic radius

The representation of the pointlike charges by the Dirac δ-function leads to singular classical electromagnetic potentials at the position of the charge. The partial differential equations of the classical electromagnetism require to view the pointlike charges as having a radius, at least infinitesimal. In order to avoid the self-interaction the same Maxwell equations requires a finite classical electromagnetic radius for "pointlike" charges. We recall here the classical electromagnetic radius of a charge.

A charge in motion generates a propagating electromagnetic field. The charge moves in this field, so it interacts with its own field, as a consequence of the retardation. This is a self-interaction. Since we are interested in the field near the charge, we may leave aside the scalar potential and, according to the Lorenz gauge, the longitudinal part of the vector potential. The transverse vector potential $\boldsymbol{A}$ is generated according to the equation

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2}-\Delta\boldsymbol{A}=\frac{4\pi}{c}\boldsymbol{j}\ \ , \tag{1}$$

where $\boldsymbol{j}$ is the transverse current density. Let us assume a pointlike charge q with trajectory $\boldsymbol{r}_0(t)$, acted by an external force; its charge density is $\rho = q\delta(\boldsymbol{r}-\boldsymbol{r}_0)$ and its current density is $\boldsymbol{j} = q\boldsymbol{v}\delta(\boldsymbol{r}-\boldsymbol{r}_0)$, where $\boldsymbol{v} = \dot{\boldsymbol{r}}_0$. Equation (1) becomes

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \Delta\boldsymbol{A} = \frac{4\pi q}{c}\boldsymbol{v}\delta(\boldsymbol{r}-\boldsymbol{r}_0) \ . \tag{2}$$

The field $\boldsymbol{A}$ acts upon the charge. Consequently, the charge with mass m acquires an additional velocity $\boldsymbol{v}'$, whose (classical, non-relativistic) equation of motion is

$$m\frac{d\boldsymbol{v}'}{dt} = q\boldsymbol{E}_0 + \frac{1}{c}(\boldsymbol{v}+\boldsymbol{v}')\times\boldsymbol{H}_0 \ , \tag{3}$$

where $\boldsymbol{E}_0 = -\frac{1}{c}\frac{\partial\boldsymbol{A}_0}{\partial t}$ is the electric field and $\boldsymbol{H}_0 = curl\boldsymbol{A}_0$ is the magnetic field. The suffix 0 indicates that the fields are taken at position $\boldsymbol{r}_0$. This is the Lorentz force. Since we are interested in the region close to the charge, we may neglect the second term on the right in equation (3), and get

$$\frac{d\boldsymbol{v}'}{dt} = -\frac{q}{mc}\frac{d\boldsymbol{A}_0}{dt} \tag{4}$$

and $\boldsymbol{v}' = -\frac{q}{mc}\boldsymbol{A}_0$ (the relativistic equation of motion does not change these things). It follows that the self-interaction produces a reaction current density

$$\boldsymbol{j}' = q\boldsymbol{v}'\delta(\boldsymbol{r}-\boldsymbol{r}_0) = -\frac{q^2}{mc}\boldsymbol{A}\delta(\boldsymbol{r}-\boldsymbol{r}_0) \ , \tag{5}$$

which, in turn, determines a field given by

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}'}{\partial t^2} - \Delta\boldsymbol{A}' = -\frac{4\pi q^2}{mc^2}\boldsymbol{A}\delta(\boldsymbol{r}-\boldsymbol{r}_0) \ . \tag{6}$$

For $\boldsymbol{r}$ close to $\boldsymbol{r}_0$ we may limit ourselves to the static part of this field, which is $\boldsymbol{A}' = -\frac{q^2}{mc^2}\frac{\boldsymbol{A}_0}{|\boldsymbol{r}-\boldsymbol{r}_0|}$. The total field at the position of the charge is $\boldsymbol{A}_0 - \frac{q^2\boldsymbol{A}_0}{mc^2}\lim|_{\boldsymbol{r}\rightarrow\boldsymbol{r}_0}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}$. This shows that the charge must be viewed as having an undetermined position inside a region with dimension of order

$$a = \frac{q^2}{mc^2} \ ; \tag{7}$$

the field at the undetermined position of the charge is zero and the force acting on the charge is vanishing; this way, the self-interaction is avoided. Therefore, we must limit ourselves to a cutoff length of the order $a = q^2/mc^2$, a cutoff wavevector of the order $1/a$ and a cutoff frequency of the order c/a. The pointlike charge moves indefinitely inside a spherical region with radius of the order a; its self-energy is of the order $q^2/a = mc^2$. The parameter a is the classical electromagnetic radius of the charge q with mass m. It sets the limits of the classical electromagnetism. The fields are limited by $E_s = H_s = q/a^2 = m^2c^4/q^3$ (or $qE_sa = mc^2$, $m\omega^2 = q\omega H_s/c$, with $\omega = c/a$). For electrons $q = -e = -4.8\times10^{-10}esu$, $a \simeq 2.8\times10^{-13}cm^{-1}$ ($c/a \simeq 10^{23}s^{-1}$, $m \simeq 10^{-27}g$, $mc^2 \simeq 0.5MeV$, $1eV = 1.6\times10^{-12}erg$), $E_s = H_s \simeq 6\times10^{15}esu$ ($1esu = 3\times10^4V/m$). The Lorentz contraction along the direction of motion makes a length of the order $q^2/\mathcal{E}$ along this direction, where $\mathcal{E}$ is the charge energy. Being an unphyiscal, undetermined region, its dimension is not subject to the relativistic invariance.

The undefined motion of a pointlike charge inside of a region with dimensions of the order a has not very relevant consequences in the classical electromagnetism, except for its governing the Lorentz damping and the natural breath of the spectroscopic line. The radius a may be viewed as an uncertainty in the charge position, arising from its electromagnetic field.

2 Relativistic quantum uncertainty

The Schroedinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi \tag{8}$$

for a free particle with mass m has a temporal scale $\sim\frac{\hbar}{\tau}$ and a length scale $\sim\frac{\hbar^2}{2ml^2}$, where $\tau=\Delta t$ is a time variation and $l=\Delta x$ is a distance (x) variation. Since these two factors should be of the same order of magnitude,

$$l^2/\tau=l\Delta v\simeq\frac{\hbar}{2m}\ \ , \tag{9}$$

where v is the particle velocity, it follows

$$\Delta p\Delta x\simeq\frac{1}{2}\hbar\ \ , \tag{10}$$

where Δp is the mometum variation. This is the well-known quantum-mechanical uncertainty relation. Since the variations of the action cannot be smaller than $\hbar$, the uncertainty relationship is in fact $\Delta p\Delta x\geq\frac{1}{2}\hbar$. For $\hbar\rightarrow 0$ we recover the classical limit. Similarly, since the eenrgy $\mathcal{E}$ is given by $\mathcal{E}\psi=i\hbar\frac{\partial\psi}{\partial t}$, we get $\Delta\mathcal{E}\Delta t\geq\frac{1}{2}\hbar$. The above inequalities are not changed by the presence of the interaction.

The situation is completely changed for a relativistic particle. The quantization of the relativistic energy-momentum relation $\mathcal{E}^2=c^2p^2+m^2c^4$ leads to a Klein-Gordon-type equation

$$\hbar^2\frac{\partial^2\psi}{\partial t^2}-c^2\hbar^2\Delta\psi+m^2c^4\psi=0\ \ , \tag{11}$$

whatever ψ may mean. We can see that the time scale is $\tau=\frac{\hbar}{mc^2}$ and the distance scale is $l=\frac{\hbar}{mc}$. These are quantum-mechanical uncertainties, for a quantum-mechanical motion, where the durations cannot be shorter than τ and the distances cannot be shorter than l; if they were shorter, the quantum mechanical motion is undefined, because variations smaller than $\hbar$ of the mechanical action are meaningless. In addition, the motion proceeds with velocity c. This shows the inadequacy of the quantum-mechanical behaviour of a pointlike relativistic particle.[1] The reason is the square relativistic dependence of the energy $\mathcal{E}^2=c^2p^2+m^2c^4$. In dealing with relativistic quantum-mechanical effects we must be content with using a cutoff length of the order of the Compton wavelength $\lambda_c=\frac{\hbar}{mc}$ (and a cutoff time of the order $\hbar/mc^2$).[2, 3] The results depend on the choice of this cutoff, *i.e.* they are not definite.[4]

Moreover, the Klein-Gordon equation has not a classical limit. Indeed, if $\psi=Ae^{\frac{i}{\hbar}S}$ we get from equation (11)

$$\begin{gathered}\left(\frac{\partial S}{\partial t}\right)^2-c^2(gradS)^2-m^2c^4-\frac{\hbar^2}{A}\left(\frac{\partial^2A}{\partial t^2}-c^2\Delta A\right)=0\ , \\ \frac{\partial}{\partial t}\left(A^2\frac{\partial S}{\partial t}\right)-c^2div\left(A^2gradS\right)=0\ .\end{gathered} \tag{12}$$

In the classical limit $\hbar\rightarrow 0$ and a slowly-varying A the first equation (12) is the classical Hamilton -Jacobi equation, with S the classical action. The second equation is a continuity (conservation) equation. But $A^2\frac{\partial S}{\partial t}$ is not a positive quantity, and its motion proceeds with velocity c. It is not possible to give a meaning to this conservation equation. In the classical limit, when A varies slowly, this continuity equation leads to the wave equation $\frac{\partial^2S}{\partial t^2}-c^2\Delta S=0$ for the classical action, which is improper. This shows that it is impossible to establish a standard quantum-mechanical description for a relativistic particle. As it is well known, the proper picture is that of quantum fields, which, however, should be used with the cutoff described above. The Compton wavelength sets a quantum limit for the motion of the relativistic fields and for the fields themselves. The latter is the Schwinger limit $E_s = H_s = m^2c^3/\hbar \mid q \mid$, where q is the particle charge. For electrons $\lambda_c = \hbar/mc \simeq 3.8 \times 10^{-11} cm$ and $E_s = H_s \simeq 4 \times 10^{13} esu$. The ratio a/λ_c is $\alpha = a/\lambda_c = q^2/\hbar c$. For electrons $\alpha = e^2/\hbar c \simeq \frac{1}{137}$ is the fine-structure constant.

The same is true for photons ($m = 0$), where $\tau = \hbar/\mathcal{E}$ and $l = \hbar c/\mathcal{E}$ is the wavelength λ, as well as for the Dirac equation. In particular, this uncertainty is visible in the Dirac equation

$$\gamma^\mu p_\mu \psi = mc\psi \tag{13}$$

by the presence of the Dirac matrices γ^μ ($\mu = 0, 1, 2, 3$), which mix up te coordinates and the time over a length scale $\lambda_c = \hbar/mc$ and over durations $\tau = \hbar/mc^2$. This indefinite motion is the fermion Zitterbewegung.[5, 6] Also, in the classical limit the Dirac equation becomes the invalid equation $\mathcal{E} - c\boldsymbol{\alpha p} = mc\beta$, where $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$, $\beta = \gamma^0$. Moreover, the probability-conservation equation

$$\frac{\partial}{\partial t}(\psi^*\psi) + c \cdot grad(\psi^*\boldsymbol{\alpha}\psi) = 0 \tag{14}$$

shows that the probability current has an indefinite velocity $c\boldsymbol{\alpha}$, which indicates again the inadequacy of a standard quantum-mechanical treatment.[9, 10]

3 Charged "bosons" with spin

The quantum-mechanical motion of the electron field is undetermined. This indefiniteness leads not only to divergences, but also to ambiguities in the regularization and renormalization calculations. The determined quantum-mechanical motion of the electrons is the motion of the Zitterbewegung, with a cutoff length of the order of the Compton wavelength λ_c. The technical procedure of achieving a description of this motion is as follows.

We give a variation δx^μ to the coordinates x^μ, and denote it by u^μ, according to the scheme $x^\mu \longrightarrow x^\mu + \delta x^\mu$, $\delta x^\mu = u^\mu$, $x^\mu \longrightarrow x^\mu + u^\mu$. We will take the first-order variations with respect to u^μ of the Dirac equation

$$\gamma^\mu \partial_\mu \psi = \frac{mc}{i\hbar}\psi \ . \tag{15}$$

We write $x^\mu = s^\mu \cdot 1$, where $s^0 = ct$, $\mathbf{s} = \mathbf{r}$ and 1 denotes the unit matrix; we have $x_\mu x^\mu = s_\mu s^\mu \cdot 1 = s^2 \cdot 1$, where $s^2 = c^2t^2 - \mathbf{r}^2$. For δx^μ we need $\delta x_\mu \delta x^\mu = u_\mu u^\mu = u^2 = ds^2$; the (non-trivial) solution of this equation is

$$\delta x^\mu = u^\mu = \frac{1}{2}u\gamma^\mu \tag{16}$$

(since $\gamma_\mu\gamma^\mu = 4$). The first-order expansion of the Dirac equation is

$$\gamma^\mu \left(\partial_\mu\psi + u^\nu\partial_\nu\partial_\mu\psi\right) = \frac{mc}{i\hbar}\left(\psi + u^\nu\partial_\nu\psi\right) \ , \tag{17}$$

or

$$\partial^\mu\partial_\mu\psi = -\frac{m^2c^2}{\hbar^2}\psi \ , \tag{18}$$

which is the Klein-Gordon equation $p_\mu p^\mu \psi = m^2c^2\psi$. The displacement field u is absorbed into ψ, hence the bosonic character of the latter. The ψ in equation (18) originates in u multiplied by a bispinor. It is a boson field with four bispinor components. We write it as

$$\psi = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} \beta_{\boldsymbol{k}}(\psi_{k\alpha})e^{i\mathbf{kr}} \ , \tag{19}$$

where V is the volume,

$$(\psi_{\mathbf{k}\alpha}) = \begin{pmatrix} c_{\mathbf{k},+1} \\ c_{\mathbf{k},-1} \\ b^*_{-\mathbf{k},-1} \\ b^*_{-\mathbf{k},+1} \end{pmatrix} \ , \tag{20}$$

± 1 are spin labels and

$$\beta_{\boldsymbol{k}} = c\sqrt{\frac{\hbar}{2\varepsilon_k}} \ , \ \varepsilon_{\boldsymbol{k}} = c\sqrt{k^2 + k_0^2} \ , \ k_0 = mc/\hbar. \tag{21}$$

The c's and the b's operators satisfy usual bosonic commutation relations for four distinct types of bosons. The c-operators correspond to particles (electrons), while the b-operators correspond to antiparticles (positrons). The boson occupation number is the number of electrons, the presence of the cutoff λ_c ensures the exclusion principle.

In the presence of the electromagnetic field we need to use the covariant derivative $D_\mu = \partial_\mu - \frac{e}{ic\hbar} A_\mu$, according to $\psi \longrightarrow \psi + u^\mu D_\mu \psi$; we get the Klein-Gordon equation

$$\left(p_\mu - \frac{e}{c} A_\mu\right)\left(p^\mu - \frac{e}{c} A^\mu\right)\psi - \frac{ie\hbar}{2c}\sigma^{\mu\nu} F_{\mu\nu}\psi = m^2 c^2 \psi \tag{22}$$

with electromagnetic field, where A_μ are the electromagnetic potentials, $F_{\mu\nu}$ is the field tensor and $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ is the spin matrix.

The free equation of motion of the boson field (the Klein-Gordon equation (18)) reads

$$\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \Delta\psi + \frac{m^2 c^2}{\hbar^2}\psi = 0 \ . \tag{23}$$

If we multiply this equation by $\dot{\psi}^*$, the equation for ψ^* by $\dot{\psi}$, where ψ^* is the adjoint of ψ (transposed conjugate), and add the two equations, we get

$$\frac{\partial}{\partial t}\left(\frac{1}{c^2}\dot{\psi}^*\dot{\psi} + \partial_i \psi^* \partial_i \psi + \frac{m^2 c^2}{\hbar^2}\psi^*\psi\right) - \partial_i\left(\dot{\psi}^* \partial_i \psi + \partial_i \psi^* \dot{\psi}\right) = 0 \ , \tag{24}$$

where $i = 1, 2, 3$; hence, we can see that

$$w_e = \frac{1}{c^2}\dot{\psi}^*\dot{\psi} + \partial_i \psi^* \partial_i \psi + \frac{m^2 c^2}{\hbar^2}\psi^*\psi \tag{25}$$

is the energy density and

$$\mathbf{g}_e = -\frac{1}{c^2}\left(\dot{\psi}^* \partial_i \psi + \partial_i \psi^* \dot{\psi}\right) \tag{26}$$

is the momentum density. The total energy is

$$\begin{aligned} W_e = \int d\mathbf{r} w_e = \textstyle\sum_{\mathbf{k}\sigma} \beta_k^2 \frac{2\varepsilon_{\boldsymbol{k}}^2}{c^2} \left(c^*_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + b_{\mathbf{k}\sigma} b^*_{\mathbf{k}\sigma}\right) = \\ = \textstyle\sum_{\mathbf{k}\sigma} \hbar\varepsilon_{\boldsymbol{k}} \left(c^*_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + b_{\mathbf{k}\sigma} b^*_{\mathbf{k}\sigma}\right) = \\ = \textstyle\sum_{\mathbf{k}\sigma} \hbar\varepsilon_{\boldsymbol{k}} \left(c^*_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + b^*_{\mathbf{k}\sigma} b_{\mathbf{k}\sigma} + 1\right) \ . \end{aligned} \tag{27}$$

The time dependence $c_{\boldsymbol{k}\sigma}$, $b_{\boldsymbol{k}\sigma} \sim e^{-i\varepsilon_{\boldsymbol{k}} t}$ is established by the canonical equations of motion of these operators with the hamiltonian W_e. Similarly, the total momentum is

$$\begin{aligned} \mathbf{G}_e = \int d\mathbf{r} \mathbf{g}_e = \textstyle\sum_{\mathbf{k}\sigma} \hbar\mathbf{k} \left(c^*_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + b_{\mathbf{k}\sigma} b^*_{\mathbf{k}\sigma}\right) = \\ = \textstyle\sum_{\mathbf{k}\sigma} \hbar\mathbf{k} \left(c^*_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + b^*_{\mathbf{k}\sigma} b_{\mathbf{k}\sigma} + 1\right) \ . \end{aligned} \tag{28}$$

If we multiply the equations for ψ and ψ^* by ψ^* and ψ, respectively, and subtract the two equations from one another, we get another law of conservation, which reads

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left(\psi^*\dot{\psi}-\dot{\psi}^*\psi\right)-\partial_i\left(\psi^*\partial_i\psi-\partial_i\psi^*\psi\right)=0\ ; \tag{29}$$

hence,

$$\begin{aligned} Q=\tfrac{ie}{\hbar c^2}\int d\mathbf{r}\left(\psi^*\dot{\psi}-\dot{\psi}^*\psi\right)=e\sum\nolimits_{\mathbf{k}\sigma}\left(c^*_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}-b_{\mathbf{k}\sigma}b^*_{\mathbf{k}\sigma}\right)=\\ =e\sum\nolimits_{\mathbf{k}\sigma}\left(c^*_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}-b^*_{\mathbf{k}\sigma}b_{\mathbf{k}\sigma}-1\right) \end{aligned} \tag{30}$$

is the electric charge (where e is the particle charge) and

$$\begin{aligned} \mathbf{J}=-\tfrac{ie}{\hbar}\int d\mathbf{r}\left(\psi^* grad\psi-grad\psi^*\psi\right)=\\ =e\sum\nolimits_{\mathbf{k}\sigma}\tfrac{c^2\mathbf{k}}{\varepsilon_k}\left(c^*_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}-b_{\mathbf{k}\sigma}b^*_{\mathbf{k}\sigma}\right)=\\ =e\sum\nolimits_{\mathbf{k}\sigma}\tfrac{c^2\mathbf{k}}{\varepsilon_k}\left(c^*_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}-b^*_{\mathbf{k}\sigma}b_{\mathbf{k}\sigma}-1\right) \end{aligned} \tag{31}$$

is the electric current. With the notation

$$\rho=\tfrac{ie}{\hbar c^2}\left(\psi^*\dot{\psi}-\dot{\psi}^*\psi\right)\ ,\ j_i=-\tfrac{ie}{\hbar}\left(\psi^*\partial_i\psi-\partial_i\psi^*\psi\right)\ , \tag{32}$$

$$j^\mu=(c\rho,\mathbf{j})=\frac{ie}{\hbar}\left(\psi^*\partial^\mu\psi-(\partial^\mu\psi^*)\psi\right)\ , \tag{33}$$

equation (29) is the continuity equation $\partial_\mu j^\mu=0$.

4 Interaction

We adopt the energy W_e given by equation (27) as the free electron hamiltonian

$$H_e=\sum_{\mathbf{k}\sigma}\hbar\varepsilon_{\boldsymbol{k}}\left(c^*_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}+b^*_{\mathbf{k}\sigma}b_{\mathbf{k}\sigma}\right)\ , \tag{34}$$

where $\varepsilon_{\boldsymbol{k}}=c\sqrt{k^2+k_0^2}$, $k_0=mc/\hbar$ being the inverse of the Compton wavelength (Compton wavevector). The interaction of the electrons with the electromagnetic field is derived from equation (22); this equation can be written as

$$\begin{aligned} (p_\mu p^\mu-\tfrac{e}{c}p_\mu A^\mu-\tfrac{e}{c}A_\mu p^\mu+\tfrac{e^2}{c^2}A_\mu A^\mu+\\ +\tfrac{e\hbar}{c}\Sigma\mathbf{H}-\tfrac{ie\hbar}{c}\boldsymbol{\alpha}\mathbf{E}-m^2c^2)\psi=0\ , \end{aligned} \tag{35}$$

since

$$-\frac{ie\hbar}{2c}\sigma^{\mu\nu}F_{\mu\nu}=\frac{e\hbar}{c}\Sigma\mathbf{H}-\frac{ie\hbar}{c}\boldsymbol{\alpha}\mathbf{E}\ , \tag{36}$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field,

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}\ ,\ \boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix} \tag{37}$$

and $\boldsymbol{\sigma}$ are the Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; , \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; . \tag{38}$$

Hence, we get the equation of motion

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + \frac{m^2c^2}{\hbar^2}\psi + \frac{e}{c\hbar^2}p_\mu A^\mu\psi + \frac{e}{c\hbar^2}A_\mu p^\mu\psi - \frac{e^2}{c^2\hbar^2}A_\mu A^\mu\psi - \frac{e}{c\hbar}\sum\mathbf{H}\psi + \frac{ie}{c\hbar}\boldsymbol{\alpha}\mathbf{E}\psi = 0 \; . \tag{39}$$

The interaction energy densities are

$$v_p = \frac{1}{c}j_\mu A^\mu = \frac{ie}{c\hbar}\left[\psi^*\left(\partial_\mu\psi\right) - \left(\partial_\mu\psi^*\right)\psi\right]A^\mu \tag{40}$$

(where the gauge condition $\partial_\mu A^\mu = 0$ is used),

$$v_d = -\frac{e^2}{c^2\hbar^2}\left(\psi^*\psi\right)A_\mu A^\mu \tag{41}$$

and

$$v_H = -\frac{e}{c\hbar}\psi^*\sum\psi\mathbf{H} \; , \qquad v_E = \frac{ie}{c\hbar}\psi^*\boldsymbol{\alpha}\psi\mathbf{E} \; . \tag{42}$$

The interaction is obtained by integrating these densities over the whole space,

$$V_{p,d,H,E} = \int d\mathbf{r} v_{p,d,H,E} \; . \tag{43}$$

The labels p and d are chosen by analogy with the non-relativistic "paramagnetic" and "diamagnetic" contributions. The interactions v_H and v_E arise from the $\sigma^{\mu\nu}F_{\mu\nu}$-term in equation (22). This is known as the "Pauli term".[11]

The interaction v_p can be written as $\rho\Phi - \frac{1}{c}\boldsymbol{j}\boldsymbol{A}$, where the charge density ρ and the current density $\boldsymbol{j}$ are given by equation (32), Φ is the scalar potential and $\boldsymbol{A}$ is the vector potential. The vector potential has a longitudinal component $\boldsymbol{A}_l$ and a transverse component $\boldsymbol{A}_t$. The longitudinal component is related to the scalar potential through the Lorenz gauge,

$$\boldsymbol{A}_l = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}}\frac{i\boldsymbol{k}}{ck^2}\dot{\Phi}_{\boldsymbol{k}}e^{i\boldsymbol{k}\boldsymbol{r}} \tag{44}$$

($\frac{1}{c}\frac{\partial\Phi}{\partial t} + div\boldsymbol{A} = 0$). The scalar potential satisfies the Maxwell equation

$$\frac{1}{c^2}\ddot{\Phi} - \Delta\Phi = 4\pi\rho = \frac{4\pi ie}{\hbar c^2}\left(\psi^*\dot{\psi} - \dot{\psi}^*\psi\right) \; . \tag{45}$$

The longitudinal part $\rho\Phi - \frac{1}{c}\boldsymbol{j}\boldsymbol{A}_l$ of the interaction gives minus the Coulomb energy $V_c = \sum_{\mathbf{k}}\frac{2\pi}{k^2}\rho_{\mathbf{k}}\rho_{-\mathbf{k}}$, while the longitudinal part $\frac{1}{8\pi}\int d\mathbf{r}E_l^2$ of the electromagnetic field, where $\boldsymbol{E}_l$ is the longitudinal electric field, gives V_c. Therefore, we remain with the transverse interaction

$$v_p = -\frac{1}{c}\mathbf{j}\mathbf{A}_t \tag{46}$$

and the transverse part of the energy of the electromagnetic field

$$H_{em}=H_{ph}=\sum_{\mathbf{k}}\hbar\omega_{\boldsymbol{k}}a_{\mathbf{k}}^{*}a_{\mathbf{k}}\ \ , \tag{47}$$

which corresponds to photons with energy $\hbar\omega_{\boldsymbol{k}}=\hbar ck$ (for a fixed polarization, equation. The transverse vector potential reads

$$\boldsymbol{A}_t=\frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}}\alpha_{\boldsymbol{k}}\boldsymbol{e}_{\boldsymbol{k}}\left(a_{\boldsymbol{k}}e^{i\boldsymbol{kr}}+a_{\boldsymbol{k}}^{*}e^{-i\boldsymbol{kr}}\right)\ \ , \tag{48}$$

where $\alpha_{\boldsymbol{k}}=c\sqrt{2\pi\hbar/\omega_k}$ and $\boldsymbol{e}_{\boldsymbol{k}}$ is the polarization vector. Since the longitudinal field has a vanishing free hamiltonian ($V_c=\frac{1}{8\pi}\int d\mathbf{r}E_l^2=0$ for $\ddot{\Phi}_{\mathbf{k}}+\omega_{\boldsymbol{k}}^2\Phi_k=0$ in equation (45)), this field cannot be quantized as an independent kind of photons (there is no scalar or longitudinal "photon").

In all other interaction terms where the scalar potential and the longitudinal vector potential occur, we replace them by their expressions given by the equation of motion (45):

$$\begin{gathered}\rho_{\mathbf{k}}=\tfrac{e}{\hbar c^2\sqrt{V}}\textstyle\sum_{\mathbf{k}'\sigma}\beta_{\mathbf{k}'+\mathbf{k}}\beta_{\mathbf{k}'}\left(\varepsilon_{\mathbf{k}'+\mathbf{k}}+\varepsilon_{\mathbf{k}'}\right)\cdot\\ \cdot\left(c_{\mathbf{k}'+\mathbf{k}\sigma}^{*}c_{\mathbf{k}'\sigma}-b_{-\mathbf{k}'-\mathbf{k}-\sigma}b_{-\mathbf{k}'-\sigma}^{*}\right)\ \ ,\\ \Phi_{\mathbf{k}}=-\tfrac{4\pi e}{\hbar\sqrt{V}}\textstyle\sum_{\mathbf{k}'\sigma}\frac{\beta_{\mathbf{k}'+\mathbf{k}}\beta_{\mathbf{k}'}\left(\varepsilon_{\mathbf{k}'+\mathbf{k}}+\varepsilon_{\mathbf{k}'}\right)}{\left(\varepsilon_{\mathbf{k}'+\mathbf{k}}-\varepsilon_{\mathbf{k}'}\right)^2-\omega_k^2}\cdot\\ \cdot\left(c_{\mathbf{k}'+\mathbf{k}\sigma}^{*}c_{\mathbf{k}'\sigma}-b_{-\mathbf{k}'-\mathbf{k}-\sigma}b_{-\mathbf{k}'-\sigma}^{*}\right)\ \ ,\end{gathered} \tag{49}$$

$$\begin{gathered}\mathbf{A}_{t\mathbf{k}}=\alpha_{\boldsymbol{k}}\left(\mathbf{e}_{\mathbf{k}}a_{\mathbf{k}}+\mathbf{e}_{-\mathbf{k}}a_{-\mathbf{k}}^{*}\right)\ ,\ \alpha_{\boldsymbol{k}}=c\sqrt{\tfrac{2\pi\hbar}{\omega_k}}\ ,\\ \mathbf{A}_{l\mathbf{k}}=\tfrac{4\pi e}{\hbar\sqrt{V}}\textstyle\sum_{\mathbf{k}'\sigma}\frac{i\mathbf{k}}{k\omega_k}\frac{\beta_{\mathbf{k}'+\mathbf{k}}\beta_{\mathbf{k}'}\left(\varepsilon_{\mathbf{k}'+\mathbf{k}}^2-\varepsilon_{\mathbf{k}'}^2\right)}{\left(\varepsilon_{\mathbf{k}'+\mathbf{k}}-\varepsilon_{\mathbf{k}'}\right)^2-\omega_{\boldsymbol{k}}^2}\cdot\\ \cdot\left(c_{\mathbf{k}'+\mathbf{k}\sigma}^{*}c_{\mathbf{k}'\sigma}-b_{-\mathbf{k}'-\mathbf{k}-\sigma}b_{-\mathbf{k}'-\sigma}^{*}\right)\end{gathered} \tag{50}$$

and

$$\mathbf{E}_{l\mathbf{k}}=-\frac{4\pi i\mathbf{k}}{k^2}\rho_{\mathbf{k}}\ . \tag{51}$$

In addition, from the continuity equation we get the Fourier component of the longitudinal current

$$\begin{gathered}\mathbf{j}_{l\mathbf{k}}=\tfrac{e}{\hbar c^2\sqrt{V}}\textstyle\sum_{\mathbf{k}'\sigma}\frac{\mathbf{k}}{k^2}\beta_{\mathbf{k}'+\mathbf{k}}\beta_{\mathbf{k}'}\left(\varepsilon_{\mathbf{k}'+\mathbf{k}}^2-\varepsilon_{\mathbf{k}'}^2\right)\cdot\\ \cdot\left(c_{\mathbf{k}'+\mathbf{k}\sigma}^{*}c_{\mathbf{k}'\sigma}-b_{-\mathbf{k}'-\mathbf{k}-\sigma}b_{-\mathbf{k}'-\sigma}^{*}\right)\ .\end{gathered} \tag{52}$$

For convenience, also we give below a few bilinear forms:

$$\psi^*\psi=\frac{1}{V}\sum_{\mathbf{k}\mathbf{k}'\sigma}\beta_{\boldsymbol{k}}\beta_{\boldsymbol{k}'}\left(c_{\mathbf{k}\sigma}^{*}c_{\mathbf{k}'\sigma}+b_{-\mathbf{k},-\sigma}b_{-\mathbf{k}',-\sigma}^{*}\right)e^{-i(\mathbf{k}-\mathbf{k}')\mathbf{r}}\ \ , \tag{53}$$

$$\begin{gathered}j_0=\tfrac{e}{c\hbar V}\textstyle\sum_{\mathbf{k}\mathbf{k}'\sigma}\beta_{\boldsymbol{k}}\beta_{\boldsymbol{k}'}\left(\varepsilon_{\boldsymbol{k}}+\varepsilon_{\boldsymbol{k}'}\right)\cdot\\ \cdot\left(c_{\mathbf{k}\sigma}^{*}c_{\mathbf{k}'\sigma}-b_{-\mathbf{k},-\sigma}b_{-\mathbf{k}',-\sigma}^{*}\right)e^{-i(\mathbf{k}-\mathbf{k}')\mathbf{r}}\ \ ,\end{gathered} \tag{54}$$

$$\mathbf{j} = \frac{e}{\hbar V} \sum_{\mathbf{k}\mathbf{k}'\sigma} \beta_{\boldsymbol{k}} \beta_{\boldsymbol{k}'} \left(\mathbf{k} + \mathbf{k}'\right) \cdot \\ \cdot \left(c^*_{\mathbf{k}\sigma} c_{\mathbf{k}'\sigma} + b_{-\mathbf{k},-\sigma} b^*_{-\mathbf{k}',-\sigma}\right) e^{-i(\mathbf{k}-\mathbf{k}')\mathbf{r}} \ . \tag{55}$$

The longitudinal contribution $\Phi^2 - \frac{1}{c^2}(\mathbf{A}_l^2 + 2\mathbf{A}_l\mathbf{A}_t)$ in the interaction v_d (equation (41)) brings an interaction of the order e^3, at least; if we limit ourselves to e^2-orders at most, we may use

$$v_d \simeq \frac{e^2}{c^2\hbar^2} \left(\psi^*\psi\right) \mathbf{A}_t \mathbf{A}_t \ . \tag{56}$$

Similarly, only the transverse field contributes to v_H. It is worth noting that the longitudinal degrees of freedom of the electromagnetic field are removed from the problem, by means of equations (49)-(52); they are replaced by the degrees of freedom of the charges. Similar interaction contributions arise from an external (transverse, purely radiation field).

For convenience, the interactions $v_{H,E}$ (equations (42)) can be written as

$$v_H = -\frac{e}{2c\hbar} \psi^* \left(\sum + \sum^*\right) \psi \mathbf{H} \ , \\ v_E = \frac{ie}{2c\hbar} \psi^* \left(\boldsymbol{\alpha} - \boldsymbol{\alpha}^*\right) \psi \mathbf{E} \ . \tag{57}$$

In the first equation (57) only the matrices $\sigma_{x,z}$ remain, while in the second equation (57) only the matrix σ_y remains.

5 Perturbation theory

5.1 Interaction representation

The state vectors $\mid v >$ are obtained by the action of the electron and photon creation operators upon the vacuum ($c^*_{\mathbf{k}\sigma}$, $b^*_{\mathbf{k}\sigma}$, $a^*_{\mathbf{k}}$). The time evolution of the state vectors is given by Schroedinger equation

$$i\hbar \frac{\partial}{\partial t} \mid v >= (H_0 + V) \mid v > \ , \tag{58}$$

where $H_0 = H_e + H_{ph}$ and $V = V_p + V_d + V_H + V_E$, given above. The electron hamiltonian is

$$H_e = \sum_{\mathbf{k}\sigma} \hbar\varepsilon_{\boldsymbol{k}} \left(c^*_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + b^*_{\mathbf{k}\sigma} b_{\mathbf{k}\sigma}\right) \ , \tag{59}$$

where $\varepsilon_{\boldsymbol{k}} = c\sqrt{k^2 + k_0^2}$, $k_0 = mc/\hbar$ being the inverse of the Compton wavelength. The spin label takes two values $\sigma \pm 1$. The photon hamiltonian is

$$H_{ph} = \sum_{\mathbf{k}} \hbar\omega_{\boldsymbol{k}} a^*_{\mathbf{k}} a_{\mathbf{k}} \ , \tag{60}$$

where $\hbar\omega_{\boldsymbol{k}} = \hbar ck$ (for a fixed polarization). The interaction is given by

$$V_{p,d,E,H} = \int d\boldsymbol{r} v_{p,d,E,H} \ , \tag{61}$$

where the interaction densities $v_{p,d,E,H}$ are given above.

We adopt the interaction representation, where $\mid v >= e^{-\frac{i}{\hbar}H_0 t}\widetilde{\mid v >}$, which means that all the operators have the form

$$\widetilde{O} = e^{\frac{i}{\hbar}H_0 t} O e^{-\frac{i}{\hbar}H_0 t} \quad , \tag{62}$$

such that their equation of motion is

$$\dot{\widetilde{O}} = \frac{i}{\hbar}[H_0, \widetilde{O}] \; ; \tag{63}$$

we can see that $\widetilde{a}_{\mathbf{k}} = e^{-i\omega_k t} a_{\mathbf{k}}$, $\widetilde{c}_{\mathbf{k}\sigma} = e^{-i\varepsilon_k t} c_{\mathbf{k}\sigma}$ and $\widetilde{b}_{\mathbf{k}\sigma} = e^{-i\varepsilon_k t} b_{\mathbf{k}\sigma}$. Equation (58) becomes

$$i\hbar\frac{\partial}{\partial t}\widetilde{\mid v >} = e^{\frac{i}{\hbar}H_0 t} V e^{-\frac{i}{\hbar}H_0 t}\widetilde{\mid v >} \; ; \tag{64}$$

we denote all the $\widetilde{O}$-operators by $O(t)$ and all the states $\widetilde{\mid v >}$ by $\mid v(t) >$, and write

$$i\hbar\frac{\partial}{\partial t} \mid v(t) >= V(t) \mid v(t) > \quad . \tag{65}$$

The solution of this equation is

$$\mid v(t) >=\mid v(t_i) > -\frac{i}{\hbar}\int_{t_i}^{t} dt' V(t') \mid v(t') > \quad , \tag{66}$$

or

$$\begin{gathered} \mid v(t) >=\mid v(t_i) > -\frac{i}{\hbar}\int_{t_i}^{t} dt' V(t') \mid v(t_i) > + \\ + \left(-\frac{i}{\hbar}\right)^2 \int_{t_i}^{t} dt' V(t') \int_{t_i}^{t'} dt'' V(t'') \mid v(t_i) > + ... \quad , \end{gathered} \tag{67}$$

where t_i is the initial moment of time. The initial time is $t_i = 0$ and the final time is $t_f = t$. The product $< v_f \mid v_i >$, where $\mid v_{i,f} >=\mid v(t_{i,f}) >$, is the amplitude of transition from the initial to the final state (with normalized states). We note that the Schroedinger evolution equation is not relativistically invariant. However, in computing the transition probabilities we may let the time go to infinity (and the initial moment to minus infinity), such that the solution gets a constant coefficient (the scattering matrix), which is irrelevant for the relativistic invariance. The interaction includes a position integral over the whole space. The relativistic invariance is improper for an evolution equation which does not include a position or a moment of time.

In an elementary act of interaction there exists a photon with energy $\hbar\omega_{\boldsymbol{k}} = \hbar ck$ and momentum $\hbar\boldsymbol{k}$ and an electron with energy $e^2 k$, such that the ratio of these two energies gives the coupling constant $\alpha = e^2/\hbar c$. For electrons $\alpha = 1/137$ is the fine structure constant. It follows that calculations up to the second-order of the perturbation theory suffice. In addition, whenever divergent integrals occur we use a cutoff wavevector of the order $k_0 = mc/\hbar$ (the inverse of the Compton wavelength).

5.2 Interacting electron

The initial state of an electron is $\mid v_i >= c^*_{\mathbf{k}\sigma} \mid 0 >$. The first-order perturbation state produced by the interaction v_p is

$$\mid v >_p^{(1)}= \frac{2e}{c\hbar^2}\sum_{\mathbf{k}'} \alpha_{\boldsymbol{k}'}\beta_{\boldsymbol{k}}\beta_{\mathbf{k}'+\mathbf{k}}\mathbf{k}\mathbf{e}_{-\mathbf{k}'} S_{\Delta\varepsilon} a^*_{-\mathbf{k}'} c^*_{\mathbf{k}'+\mathbf{k}\sigma} \mid 0 > \quad , \tag{68}$$

where

$$S_{\Delta\varepsilon}=\frac{e^{i(\varepsilon_{\mathbf{k'+k}}+\omega_{\boldsymbol{k'}}-\varepsilon_{\boldsymbol{k}})t}-1}{\varepsilon_{\mathbf{k'+k}}+\omega_{\boldsymbol{k'}}-\varepsilon_{\boldsymbol{k}}}\ ,\ \Delta\varepsilon=\varepsilon_{\mathbf{k'+k}}+\omega_{\boldsymbol{k'}}-\varepsilon_{\boldsymbol{k}} \tag{69}$$

and $\alpha_{\boldsymbol{k}}=c\sqrt{2\pi\hbar/\omega_{\boldsymbol{k}}}$, $\beta_{\boldsymbol{k}}=c\sqrt{\hbar/2\varepsilon_{\boldsymbol{k}}}$. The electron changes its state and emits a photon. The second-order perturbation state produced by the same interaction v_p is

$$\begin{array}{c}\mid v >_p^{(2)}=\frac{e^2}{c^2\hbar^4}\sum_{\mathbf{k'}}\alpha_{\boldsymbol{k}}\alpha_{\mathbf{k'-k}}\beta_{\boldsymbol{k}}\beta_{\mathbf{k'+k}}\beta_{k'}^2\left[2\mathbf{k'}\mathbf{e}_{-\mathbf{k}}\right]\cdot\\ \cdot\left[(\mathbf{k'}+\mathbf{k})\mathbf{e}_{\mathbf{k}-\mathbf{k'}}\right]\frac{S_{\Delta\varepsilon}+S_{\Delta\varepsilon_2}}{\Delta\varepsilon_1}a^*_{-\mathbf{k}}a^*_{\mathbf{k}-\mathbf{k'}}c^*_{\mathbf{k'+k}\sigma}\mid 0>\ ,\end{array} \tag{70}$$

where

$$\begin{array}{c}\Delta\varepsilon_1=\varepsilon_{\boldsymbol{k'}}+\omega_{\mathbf{k'-k}}-\varepsilon_{\boldsymbol{k}}\ ,\ \Delta\varepsilon_2=\varepsilon_{\boldsymbol{k'}}-\omega_{\boldsymbol{k}}-\varepsilon_{\mathbf{k'+k}}\ ,\\ \Delta\varepsilon=\Delta\varepsilon_1-\Delta\varepsilon_2\ ;\end{array} \tag{71}$$

the electron emits two photons (an additional term is present for bound states). The first non-vanishing contribution of the interaction v_d is of the order e^2:

$$\begin{array}{c}\mid v >_d^{(2)}=-\frac{ie^2}{c^2\hbar^3}t\beta_{\boldsymbol{k}}^2\sum_{\mathbf{k'}}\alpha_{\mathbf{k'}}^2c^*_{\mathbf{k}\sigma}\mid 0>-\\ -\frac{ie^2}{c^2\hbar^3}\sum_{\mathbf{k'q}}\alpha_{\boldsymbol{k'}}\alpha_{\mathbf{k'-q}}\beta_{\boldsymbol{k}}\beta_{\mathbf{k'+q}}\cdot\\ \cdot\left(\mathbf{e}_{-\mathbf{k'}}\mathbf{e}_{\mathbf{k'-q}}\right)S_{\Delta\varepsilon}a^*_{-\mathbf{k'}}a^*_{\mathbf{k'-q}}c^*_{\mathbf{k+q}\sigma}\mid 0>\ ,\end{array} \tag{72}$$

where

$$\Delta\varepsilon=\omega_{\boldsymbol{k'}}+\omega_{\mathbf{k'-q}}-\varepsilon_{\boldsymbol{k}}\ . \tag{73}$$

The contributions of the other interactions (v_H, v_E) are estimated in the same way.

5.3 Spontaneous emission

Equation (68) gives the amplitude of spontaneous emission of a photon,

$$f=ec\sqrt{\frac{2\pi E_{ph}}{E_iE_f}}\sin\theta\cdot S_{\Delta\varepsilon}\ , \tag{74}$$

where $E_{ph}=\hbar\omega$ is the energy of the photon, $E_{i,f}$ are the initial and final energies of the electon and θ is the angle between the direction of propagation of the electron and the direction of propagation of the photon. For the probability of emission (polarized photon, per unit volume),

$$\left|S_{\Delta\varepsilon}\right|^2=2\pi t\delta(\Delta\varepsilon)\ . \tag{75}$$

If there is a width of the energy levels, the factor $S_{\Delta\varepsilon}$ becomes

$$S_{\Delta\varepsilon}=\frac{e^{i\Delta\varepsilon t}e^{-\gamma t}-1}{\Delta\varepsilon+i\gamma} \tag{76}$$

and $\left|S_{\Delta\varepsilon}\right|^2\longrightarrow 1/\left(\Delta\varepsilon^2+\gamma^2/4\right)$ for $t\longrightarrow\infty$ (a natural line breadth is caused by the field emitted by the charge).[1] On the other hand, $\pi\delta(\Delta\varepsilon)\longleftarrow(\gamma/2)/\left(\Delta\varepsilon^2+\gamma^2/4\right)$ for $\gamma\ll\Delta\varepsilon$; it follows that the relevant time is of the order $t\simeq 1/\gamma$, as expected. This is the typical result for photon emission or absorption, dipole (multipole) radiation (from bound states), Zeeman, Stark and photoelectric effects. The results are similar with those of the radiation theory.[12]-[15]

[1] V. Weisskopf and E. Wigner, "Berechnung der natuerlichen Linienbreite auf Grund der Diracschen Lichttheorie", Z. Phys. **63** 54 (1930); "Ueber die naturliche Linienbreite in der Strahlung des harmonischen Oszillators", Z. Phys. **65** 18 (1930). See also W. Heitler, *The Quantum Theory of Radiation*, Dover (1984).

5.4 Diamagnetic self-energy: a mass renormalization

The first term in equation (72) gives an interacting state

$$\mid v >=\mid v_i > -\frac{ie^2}{c^2\hbar^3}t\beta_{\boldsymbol{k}}^2\sum_{\mathbf{k}'}\alpha_{\mathbf{k}'}^2 \mid v_i > \ , \tag{77}$$

which indicates a change

$$\Delta E_d = \frac{e^2}{c^2\hbar^2}\beta_{\boldsymbol{k}}^2\sum_{\mathbf{k}'}\alpha_{\mathbf{k}'}^2 \tag{78}$$

in the energy of the electron; or

$$\Delta E_d = \frac{e^2c\hbar}{2\pi E}\int dk'\cdot k' \ , \tag{79}$$

where $E=\sqrt{p^2c^2+m^2c^4}$ is the original energy of the electron. This self-energy is infinite. It is associated with photon fluctuations of the vacuum: the electron generates and absorbs a photon, according to the diamagnetic interaction $\sim(\psi^*\psi)A^2$ (a factor 2 should be included for the two photon polarizations).

Such divergences are typical for higher-order terms in the perturbation series, including even some second-order contributions. Their origin is twofold. On one hand, they arise from the pointlike nature of the electron, which generates a Coulomb static potential (related to the integration extended to infinity in equation (79)). On the other hand, they arise in higher-orders of the perturbations series as a consequence of the deficient formulation of the interaction problem. Indeed, the interaction problem is based on two equations. First, the electromagnetic field satisfies the Maxwell equations

$$\frac{1}{c^2}\frac{\partial^2 A_\mu}{\partial t^2}-\Delta A_\mu=\frac{4\pi}{c}j_\mu \ , \tag{80}$$

where the current j_μ is a bilinear form of ψ and ψ^*. These later quantities are expressed with creation and destruction operators, which obey the Schroedinger equation of motion; for an operator O, this equation reads $\dot{O}=\frac{i}{\hbar}[H,O]$, where H is the hamiltonian which includes the interaction with the electromagnetic field. Starting with the second order of the perturbation theory, we have a contribution to j_μ from the field, by using Schroedinger equation; introducing this contribution in equation (80), it gives a field-field self-interaction, which is unphysical. Similarly, we may express the field by means of the current density j_μ from equation (80) and introduce it into the Schroedinger equation; then, starting with the second-order of the perturbation theory, we have an interaction of the particle (charge) with itself, which again is unphysical. In fact, the separation of the charges from their fields is impossible in higher orders of the perturbation theory, as it is known from the impossibility of eliminating the fields from the classical equation of motion of the charges in higher-orders of relativistic corrections. Ultimately, this entanglement of the charges and their fields arises from the formulation of the interaction problem; it manifests itself as an indeterminacy of the motion over short distances. The relativistic quantum-mechanical motion is not determined over distances of the order of the Compton wavelength.

It is worthwhile recalling in this context the relativistic equation of motion

$$\mu c\frac{du^\mu}{dt}=\frac{1}{c}F^{\mu\nu}j_\nu \tag{81}$$

for a mass density μ, where u^μ is the four-velocity ($u^\mu=dx^\mu/ds$) and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the electromagnetic field (the change of μ and t makes this a relativistic invariant equation); the current density is $j_\mu = \rho dx_\mu/dt = \rho u_\mu(ds/dt)$, where ρ is the charge density (which changes like μ). We may solve this equation for j^μ (u^μ) as a function of $F^{\mu\nu}$, introduce this j^μ in equation (80) and get fields determined by themselves. Similarly, we may solve equation (80) for fields as functions of u^μ (j^μ), introduce these fields in equation (81) and get four-velocities determined by four-velocities. In both cases we get field or particle (charge) self-interaction, which is unphysical. We cannot disentangle the fields from the charges. This situation is well known from the relativistic corrections to the classical lagrangian of the charges, where, in higher-orders of relativistic corrections, the charges cannot be separated from fields. Similarly, the Dirac equation

$$\gamma^\mu\left(p_\mu - \frac{e}{c}A_\mu\right)\psi = mc\psi \tag{82}$$

is entangled with Maxwell equations (80), the indeterminacy of the motion over short distances being embodied in the Dirac matrices.

It follows that in computing higher-order contributions to the perturbation series of the relativistic quantum-mechanical quantities one should be content with the approximation scheme provided by averaging the Dirac equation over distances of the order of the Compton wavelength, *i.e.* with the approximation scheme provided by the boson field; this scheme gives results with a "certain uncertainty". The fact that in Dirac equation the indeterminacy of the short-distance motion is associated with the Dirac matrices may support the attempt of disentangling the fields from the charges, by means, for instance of the renormalization technique, carried out in a covariant fashion. Though a reasonable idea, the renormalization implies regularization techniques which are improper. The standard renormalization scheme forces the separation of the particles from the fields; and it implies a subtraction of infinities in the regularization of the Feynman integrals which is arbitrary. The experimental measurements force, indeed, the particle-field separation, and we may admit that the two separations, the theoretical one and the experimental one, may coincide. However, the subtraction of the infinities remains arbitrary, undetermined. In addition, the existence of competing interaction with other charges (like mesons), makes any agreement problematic, and, ultimately, the agreement with the experiment, if it exists, is accidental. In fact, as it is well known, the consistent application of the separation procedure in all orders of the perturbation theory, makes the interaction infinite or zero.[3, 16, 17] The renormalization and the regularization techniques are inconsistent.

The cutoff in equation (79) is of the order of the inverse of the Compton wavelength $\lambda_c = \frac{\hbar}{mc}$; by using it, we get

$$\Delta E_d = \frac{e^2m^2c^3}{4\pi\hbar E} \ , \tag{83}$$

which, in the non-relativistic limit, becomes $\Delta E_d \simeq e^2/4\pi\lambda_c = (1/4\pi)\cdot$
$\cdot(e^2/\hbar c)mc^2$. This is a mass renormalization, due to photon fluctuations ($\alpha = e^2/\hbar c = 1/137$ is the fine structure constant). Similar results are obtained from second-order contributions.

5.5 Lamb shift

Let us consider a set of electron bound states denoted by n, with orthonormalized wavefunctions φ_n (instead of plane waves $e^{i\mathbf{kr}}$). The first-order interacting state generated by the interaction v_p is

$$\mid v >_p^{(1)} = -\frac{ie}{c\hbar^2}\sum_{n_1\mathbf{k}}\alpha_{\boldsymbol{k}}\beta_n\beta_{n_1}\left[\mathbf{G}_{n_1n}(\mathbf{k})\mathbf{e}_{-\mathbf{k}}\right]S_{\Delta\varepsilon}a^*_{-\mathbf{k}}c^*_{n\sigma}\mid 0> \ , \tag{84}$$

where $\Delta\varepsilon = \omega_{\boldsymbol{k}} - \varepsilon_n$ and

$$\mathbf{G}_{nn'}(\mathbf{k}) = \int d\mathbf{r}\left(\varphi_n^* grad\varphi_{n'} - grad\varphi_n^* \cdot \varphi_{n'}\right) e^{i\mathbf{kr}} \tag{85}$$

(without the b-electrons). The second-order interacting state includes, apart from two photons, a contribution arising from the photon fluctuations, given by

$$\begin{gathered}\mid v >_p^{(2)}= \tfrac{e^2}{c^2\hbar^4}\sum_{n_1n_2\mathbf{k}} \alpha_{\boldsymbol{k}}^2\beta_n\beta_{n_1}\beta_{n_2}^2\cdot \\ \cdot\left[\mathbf{G}_{n_1n_2}(\mathbf{k})\mathbf{e}_{\mathbf{k}}\right]\left[\mathbf{G}_{n_2n}(-\mathbf{k})\mathbf{e}_{\mathbf{k}}\right]\cdot \\ \cdot\int_0^t dt_1 e^{-i(\omega_{\boldsymbol{k}}+\varepsilon_{n_2})t_1}\int_0^{t_1} dt_2 e^{i(\omega_{\boldsymbol{k}}-\varepsilon_n+\varepsilon_{n_2})t_2} c_{n_1\sigma}^* \mid 0> \ .\end{gathered} \tag{86}$$

We can see that the electron emits and absorbs a photon and changes its state. Noteworthy, for free electrons this contribution is zero. A self-interacting contribution corresponds to $n_1 = n$. In addition, the main contribution arises from a degenerate state $n_2 = n'$, if it exists. In these circumstances, equation (86) becomes

$$\begin{gathered}\mid v >_p^{(2)}= \tfrac{e^2}{c^2\hbar^4}\sum_{\mathbf{k}} \alpha_{\boldsymbol{k}}^2\beta_n^4\left[\mathbf{G}_{nn'}(\mathbf{k})\mathbf{e}_{\mathbf{k}}\right]\left[\mathbf{G}_{n'n}(-\mathbf{k})\mathbf{e}_{\mathbf{k}}\right]\cdot \\ \cdot\int_0^t dt_1 e^{-i(\omega_{\boldsymbol{k}}+\varepsilon_n)t_1}\int_0^{t_1} dt_2 e^{i\omega_{\boldsymbol{k}}t_2} c_{n\sigma}^* \mid 0> \ ,\end{gathered} \tag{87}$$

or

$$\begin{gathered}\mid v >_p^{(2)}= \tfrac{e^2}{c^2\hbar^4}\sum_{\mathbf{k}} \alpha_{\boldsymbol{k}}^2\beta_n^4\cdot \\ \cdot\left[\mathbf{G}_{nn'}(\mathbf{k})\mathbf{e}_{\mathbf{k}}\right]\left[\mathbf{G}_{n'n}(-\mathbf{k})\mathbf{e}_{\mathbf{k}}\right] S c_{n\sigma}^* \mid 0> \ ,\end{gathered} \tag{88}$$

where

$$S = \frac{1}{\omega_k}\left[\frac{e^{-i\varepsilon_n t}-1}{\varepsilon_n} - \frac{e^{-i(\varepsilon_n+\omega_{\boldsymbol{k}})t}-1}{\varepsilon_n+\omega_{\boldsymbol{k}}}\right] \ . \tag{89}$$

For atomic bound states $\mathbf{G}_{nn'}(\mathbf{k})\mathbf{e}_{\mathbf{k}}$ and the range of k are of the order $1/a$, where a is the dimension of the atomic state. In the limit $k \longrightarrow 0$ equation (89) becomes

$$S = -\frac{\partial}{\partial\varepsilon_n}\frac{e^{-i\varepsilon_n t}-1}{\varepsilon_n} \ ; \tag{90}$$

in the limit of large t we get $S \simeq it/\varepsilon_n$, such that equation (88) becomes

$$\mid v >_p^{(2)}= it\frac{e^2\lambda_c^3}{8\pi\hbar a^4} \mid v_i > \ , \tag{91}$$

where $\varepsilon_n \simeq c/\lambda_c$. Therefore, we get an energy shift

$$\Delta E_p \simeq -E_n\left(\frac{e^2}{8\pi a E_n}\right)\left(\frac{\lambda_c}{a}\right)^3 \ ; \tag{92}$$

since E_n is of the order e^2/a, this shift is $\Delta E_p/E_n \simeq \frac{1}{8\pi}(\lambda_c/a)^3$. It is the splitting of the two degenerate states. The result is smaller by one order of magnitude than the standard result.[18]-[22] The standard result is $\Delta E \simeq 10^3 MHz$ ($1eV \simeq 2.4\times10^{14}Hz$), $\Delta E/E \simeq 3\times10^{-7}$ ($E = 13.6eV$), while $\Delta E_p/E_n \simeq 2\times10^{-8}$ (equation (92) with $\lambda_c = 3.8\times10^{-11}cm$ and $a = 0.53$Å). The main source of errors is the restriction to long-wavelength photons in equation (90) and the approximation $\varepsilon_n \simeq c/\lambda_c$. A slight variation in the cutoff λ_c may compensate the difference. The "mass renormalization" applied by Bethe[19] is improper both in using the non-relativistic approximation and the cutoff $242eV$.

5.6 Spin-flip photon emission

In the interaction $v_H = -\frac{e}{2c\hbar}\psi^*(\boldsymbol{\Sigma}+\Sigma^*)\psi\mathbf{H}$ it is convenient to use the spin wavefunctions φ_σ in writing the field ψ. The interacting-electron initial state $c^*_{\mathbf{k}\sigma} \mid 0>$ becomes

$$\begin{gathered} \mid v >^{(1)}_H = \frac{e}{c\hbar^2}\sum_{\mathbf{k}'}\sin\varphi(\mathbf{k}')k'\alpha_{\boldsymbol{k}'}\beta_{\boldsymbol{k}}\beta_{\mathbf{k}'+\mathbf{k}}\cdot \\ \cdot S_{\Delta\varepsilon}a^*_{-\mathbf{k}'}c^*_{\mathbf{k}'+\mathbf{k},-\sigma} \mid 0> \ , \end{gathered} \tag{93}$$

where $\varphi(\mathbf{k}')$ is the angle in the parametrization

$$\begin{gathered} \mathbf{k}' = k'(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \ , \\ \mathbf{e}_{\mathbf{k}'} = \mathbf{e}_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \end{gathered} \tag{94}$$

and

$$\mathbf{k}' \times \mathbf{e}_{\mathbf{k}'} = \mathbf{e}_\varphi = (-\sin\varphi, \cos\varphi, 0) \tag{95}$$

(actually, related to the angle between the spin and the propagation wavevector of the photon); the spin operator $\boldsymbol{\sigma}$ is directed along the vector $\mathbf{e}_\varphi$ (where only the σ_x-component remains), such that, on emitting a photon, the magnetic field changes and the spin is reversed. Equation (93) gives the amplitude of spin-flip photon emission ($\Delta\varepsilon = \varepsilon_{\mathbf{k}'+\mathbf{k}} + \omega_{k'} - \varepsilon_k$). The second-order correction, as well as the contributions of the interaction v_E can be estimated in the same way; the latter implies vacuum polarization and photon fluctuations.

5.7 Anomalous magnetic moment

Let us assume an external, uniform and constant, magnetic field $\mathbf{H}_0$. The first contribution of v_H with $H_\mathbf{0}$ to the interacting state (apart from the zeroth order contribution) arises in the third-order of the perturbation theory. The structure of this contribution is

$$-\left(\frac{e}{c\hbar}\right)^3 \left[\psi^*\left(\boldsymbol{\Sigma}\psi\right)\mathbf{H}\right]_1 \left[\psi^*\left(\boldsymbol{\Sigma}\psi\right)\mathbf{H}\right]_2 \left[\psi^8\left(\boldsymbol{\Sigma}\psi\right)\right]_3 \mathbf{H}_0 c^*_{\mathbf{k}\sigma} \mid 0> \ , \tag{96}$$

where the magnetic field is

$$\mathbf{H} = \sum_{\mathbf{q}} i\alpha_{\boldsymbol{q}}\left(\mathbf{q}\times\mathbf{e}_\mathbf{q}\right)\left(a_\mathbf{q} - a^*_{-\mathbf{q}}\right)e^{i\mathbf{qr}} \tag{97}$$

and the suffixes $1, 2, 3$ denote the times; in equation (96) the space integration is included. Spin wavefunctions should be included, which amounts to products of spin operators. The full contribution to the interacting state is obtained by inserting the time integrations

$$\left(-\frac{i}{\hbar}\right)^3 \int_0^t dt_1 \cdot \int_0^{t_1} dt_2 \cdot \int_0^{t_2} dt_3 \ ; \tag{98}$$

a factor 3 is included for the three positions of $\mathbf{H}_0$ in equation (96). Obviously, only the c-operators contribute.

We can see that, in fact, the time t_3 does not appear in equation (96), as expected. The external field plays the role of a probe, which lasts a short time Δt; consequently, we replace the t_3-integration by

$$-\frac{i}{\hbar}\int_0^{\Delta t} dt_3 = -\frac{i\Delta t}{\hbar} = \frac{\delta t}{\hbar} = \frac{1}{mc^2} \ . \tag{99}$$

The remaining time integrations are

$$S=\int_0^t dt_1 e^{-i\varepsilon_{\mathbf{k}+\mathbf{q}}t_1}\cdot\int_0^{t_1} dt_2 e^{i(\varepsilon_{\mathbf{k}+\mathbf{q}}-\varepsilon_{\boldsymbol{k}})t_2} \ . \tag{100}$$

In this equation the wavevector $\mathbf{q}$ of the (emitted and absorbed) photon brings a comparatively small contribution, such that

$$S=-\frac{\partial}{\partial\varepsilon_k}\frac{e^{-i\varepsilon_{\boldsymbol{k}}t}-1}{\varepsilon_{\boldsymbol{k}}}\simeq\frac{it}{\varepsilon_{\boldsymbol{k}}} \ . \tag{101}$$

Taking into account the spin contribution

$$[\boldsymbol{\sigma}_y(\mathbf{q}\times\mathbf{e_q})_y]^2=q^2\cos^2\varphi \tag{102}$$

(we recall the relationship $\sigma_i\sigma_j=\delta_{ij}+i\varepsilon_{ijk}\sigma_k$ for the Pauli matrices), we get the energy change

$$\Delta E_H=-\frac{3}{\hbar}\left(\frac{e}{c\hbar}\right)^3\sum_{\mathbf{q}}\frac{\cos^2\varphi(\mathbf{q})}{\varepsilon_k mc^2}q^2\alpha_{\boldsymbol{q}}^2\beta_{\boldsymbol{k}}^4\beta_{\mathbf{k}+\mathbf{q}}^2\left(\mathbf{H}_0\boldsymbol{\sigma}\right) \ . \tag{103}$$

In this expression we may approximate $\varepsilon_{\boldsymbol{k}}$ by ck_0 ($k_0=mc/\hbar$) and $\beta_{\mathbf{k}+\mathbf{q}}\simeq\beta_{\boldsymbol{k}}$ (and average over directions); we get the relative change in the magnetic moment (the Bohr magneton $\mu=\mid e\mid\hbar/2mc$)

$$\Delta\mu/\mu\simeq\frac{3}{32\pi}\cdot\frac{e^2}{\hbar c} \tag{104}$$

(a factor 2 should be included for the two polarizations). The result is comparable with Schwinger's standard result ($\Delta\mu/\mu=\alpha/2\pi$).[22, 23] The result should be multiplied by a factor $(k_c/k_0)^3$, where k_c is the cutoff. The standard result is obtained for $k_c\simeq k_0/\sqrt{2}$, characteristic for a gaussian distribution. The standard result implies an improper regularization of a divergent integral (see, for instance, Ref. [15]).

5.8 Pair creation: photon annihilation

The interaction $v_E=\frac{ie}{2c\hbar}\psi^*\left(\boldsymbol{\alpha}-\boldsymbol{\alpha}^*\right)\psi\mathbf{E}$ is responsible for pair creation (destruction). The electric field is

$$\mathbf{E}=\sum_{\mathbf{q}}iq\alpha_{\boldsymbol{q}}\left(\mathbf{e_q}a_{\mathbf{q}}-\mathbf{e}_{-\mathbf{q}}a_{-\mathbf{q}}^*\right)e^{i\mathbf{qr}} \ , \tag{105}$$

after the spatial integration the interaction becomes

$$V_E=-\tfrac{e}{c\hbar}\textstyle\sum_{\mathbf{kq}}q\alpha_{\boldsymbol{q}}\beta_{\boldsymbol{k}}\beta_{\mathbf{k}-\mathbf{q}}\left(c_{\mathbf{k}}^*\boldsymbol{\sigma}_y b_{-\mathbf{k}+\mathbf{q}}^*+b_{-\mathbf{k}}\boldsymbol{\sigma}_y c_{\mathbf{k}-\mathbf{q}}\right)\cdot \\ \cdot\left(\mathbf{e_q}a_{\mathbf{q}}-\mathbf{e}_{-\mathbf{q}}a_{-\mathbf{q}}^*\right) \ , \tag{106}$$

where the spin suffixes are included. We can see that an external photon is absorbed and generates an electron pair, with opposite spins. The interacting state is given by

$$-\frac{e}{c\hbar}\sum_{\mathbf{k}}q\alpha_{\boldsymbol{q}}\beta_{\boldsymbol{k}}\beta_{\mathbf{k}-\mathbf{q}}c_{\mathbf{k}}^*\sigma_y e_{\mathbf{q}y}b_{-\mathbf{k}+\mathbf{q}}^*e^{-i\omega_q t}\mid 0> \ , \tag{107}$$

where the time integration should be included. We get the amplitude

$$f=-\frac{ie}{c\hbar^2}q\alpha_{\boldsymbol{q}}\beta_{\boldsymbol{k}}\beta_{\mathbf{k}-\mathbf{q}}\cos\varphi\sin\varphi S_{\Delta\varepsilon} \tag{108}$$

for the pair $c^*_{\mathbf{k}+}b^*_{-\mathbf{k}+\mathbf{q},-}$, where $\Delta\varepsilon = \varepsilon_{\boldsymbol{k}} + \varepsilon_{\mathbf{k}-\mathbf{q}} - \omega_{\boldsymbol{q}}$ ($|S_{\Delta\varepsilon}|^2 = 2\pi t\delta(\Delta\varepsilon)$) and $\varphi = \varphi(\mathbf{q})$ is given by equation (94). We can see that the momentum is conserved, but the energy cannot be conserved, since the equation

$$\sqrt{k^2+k_0^2} + \sqrt{(\mathbf{k}-\mathbf{q})^2+k_0^2} = q \tag{109}$$

has no solutions (as it is well known).

5.9 Pair creation: external field

The coupling of v_E with v_p or v_H leads to pair creation; v_H can be generated by an external magnetic field. Similarly, we can consider an interaction v_E generated by an external (static) electric field, like the field $grad(Ze/r)$ of a nucleus with charge Ze. This electric field is

$$\mathbf{E} = -\sum_{\mathbf{k}} \frac{4\pi i Ze\mathbf{k}}{k^2} e^{-i\mathbf{kr}} \ ; \tag{110}$$

it generates an interaction

$$\begin{gathered} V_0 = \tfrac{4\pi e^2 Z}{c\hbar} \textstyle\sum_{\mathbf{k}\mathbf{k}_1} \tfrac{1}{k^2}\beta_{\boldsymbol{k}_1}\beta_{\mathbf{k}_1+\mathbf{k}} \cdot \\ \cdot \left(c^*_{\mathbf{k}_1}\boldsymbol{\sigma}_y k_y b^*_{-\mathbf{k}_1-\mathbf{k}} + b_{-\mathbf{k}_1}\boldsymbol{\sigma}_y k_y c_{\mathbf{k}_1+\mathbf{k}}\right) \end{gathered} \tag{111}$$

(of the v_E-type). We couple this interaction with

$$\begin{gathered} V_p = -\tfrac{e}{c\hbar}\textstyle\sum_{\mathbf{kq}} \alpha_{\boldsymbol{q}}\beta_{\boldsymbol{k}}\beta_{\mathbf{k}-\mathbf{q}}(2\mathbf{k}-\mathbf{q}) \cdot \\ \cdot \left(c^*_{\mathbf{k}}c_{\mathbf{k}-\mathbf{q}} + b^*_{-\mathbf{k}+\mathbf{q}}b_{-\mathbf{k}}\right)\left(\mathbf{e}_{\mathbf{q}}a_{\mathbf{q}} - \mathbf{e}_{-\mathbf{q}}a^*_{-\mathbf{q}}\right) \ . \end{gathered} \tag{112}$$

The interaction V_0 does not depend on time; we use for it the integral given by equation (99). We apply

$$\frac{-2i}{\hbar mc^2}\int_0^t dt_1 V_p(t_1)V_0 \tag{113}$$

to the state $a^*_{\mathbf{q}} \mid 0>$ (the factor 2 in equation (113) arises from the product $(V_p+V_0)(V_p+V_0)$). The temporal factor $S_{\Delta\varepsilon}$ implies

$$\varepsilon_{\mathbf{k}+\mathbf{q}} - \varepsilon_{\boldsymbol{q}} = \omega_{\boldsymbol{q}} \ , \tag{114}$$

an equation which is only approximately satisfied for small $\mathbf{k}$ and $\boldsymbol{q}$ of the order k_0. The momentum is not conserved, since the (large) nucleus is static. There are two amplitudes for creation of pairs $(\mathbf{k}, -\mathbf{k}+\mathbf{q}-\mathbf{k}')$ and $(-\mathbf{k}+\mathbf{q}, \mathbf{k}-\mathbf{k}')$, with opposite spins and undetermined $\mathbf{k}'$. The order of magnitude of these amplitudes is

$$f \simeq \pm\frac{4\pi i e^3 Z}{\hbar mc^2 k_0^2}\sqrt{\frac{2\pi c\hbar}{k_0}}\frac{k}{k'}\sin\theta\sin\theta'\cos\varphi' S_{\Delta\varepsilon} \ , \tag{115}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{q}$ and θ', φ' are the angles of $\mathbf{k}'$. This amplitude has the dimension $\sqrt{\alpha}\cdot vol^{3/2}$, where $\alpha = e^2/c\hbar$ is the fine-structure constant. The factor $vol^{3/2}$ is reduced by the density of states. Also, the amplitude given by equation (115) can be written as $f \simeq \sqrt{\alpha}Zr_e\lambda_c^{7/2}$, where $r_e = e^2/mc^2$ is the classical radius of the electron. Electron-positron pairs can also be created by polarizing the vacuum by an external electromagnetic field.[24]

The uncertainty implied by the second-order perturbation theory requires a density of states $\sim 1/\lambda_c^3$ for each electron; and a factor $\sim 1/\lambda_c$ for the photon; it follows a cross-section $\sigma \simeq \alpha Z r_e^2$, which has the order of magnitude of the standard result.[25] This is the Bethe-Heitler process. The coupling of V_E with V_0 leads to the Bremsstrahlung of an electron ($c^*_{\mathbf{k}\sigma} \mid 0>$) in an external field (and synchroton radiation).

5.10 Pair creation: two-photon annihilation

The coupling of V_E (equation (106)) with V_p (equation (112)) leads to pair creation by the annihilation of two photons. We apply the second-order perturbation operator (V_pV_E) to the intial state $a^*_{\mathbf{q}}a^*_{-\mathbf{q}} \mid 0>$. The time integration leads to $S_{\Delta\varepsilon}$, where $\Delta\varepsilon = 2\varepsilon_{\boldsymbol{k}} - 2\omega_{\boldsymbol{q}}$, for the pair $(\mathbf{k}, -\mathbf{k})$ with opposite spins, and to $S_{\Delta\varepsilon}$, $\Delta\varepsilon = \varepsilon_{\boldsymbol{k}} - \varepsilon_{\mathbf{k}-\mathbf{q}} - \omega_{\boldsymbol{q}}$, an approximate energy conservation. This is the Breit-Wheeler process. The energy conservation $2\varepsilon_k - 2\omega_q$ is satisfied for $\mathbf{k} \longrightarrow 0$ and $q \simeq k_0$. The amplitude of formation of the pair $(\mathbf{k}, -\mathbf{k})$ is

$$f \simeq \mp\frac{4i}{\hbar^2}\left(\frac{e}{c\hbar}\right)^2 q^2\alpha^2_{\boldsymbol{q}}\beta_0^2\frac{\beta_q^2}{\varepsilon_q}\sin\Theta\cos\theta\sin\varphi\cdot S_{\Delta\varepsilon} \ , \tag{116}$$

where Θ is the angle between $\mathbf{k}$ and $\mathbf{q}$ and θ, φ are the angles of the wavevector $\mathbf{q}$ (with respect to the electron spin). For $q \simeq k_0$ the order of magnitude of the amplitude f is $f \simeq \alpha\lambda_c^3$ (where $S \simeq \tau \simeq \lambda_c/c$), in agreement with the standard result (we note that $r_e = \alpha\lambda_c$).[26] The process of pair formation by annihilation of two photons is related to the process of pair annihilation with the formation of two photons.[27]

5.11 Charge renormalization

Let us consider the interaction of a charge e with a static charge Q. The latter generates an interaction V_0 given by equation (111). The first non-vanishing correction to the initial state $c^*_{\mathbf{k}\sigma} \mid 0>$ appears in the second order of the perturbation theory; it is due to vacuum polarization. Before effecting the time integration, this contribution reads

$$\begin{gathered}\left(\tfrac{4\pi eQ}{c\hbar}\right)^2\sum_{\mathbf{q},\mathbf{q}'}\tfrac{1}{q^2q'^2}\beta_{\boldsymbol{k}}\beta^2_{\mathbf{k}-\mathbf{q}'}\beta_{\mathbf{k}-\mathbf{q}-\mathbf{q}'}(\sigma_y q_y)(\sigma_y q'_y)\cdot \\ \cdot e^{i(\varepsilon_{\mathbf{k}-\mathbf{q}'}-\varepsilon_{\boldsymbol{k}})t_1}e^{i\varepsilon_{\mathbf{k}-\mathbf{q}'}t_2}c^*_{\mathbf{k}-\mathbf{q}-\mathbf{q}'} \mid 0> \ .\end{gathered} \tag{117}$$

The main contribution to the state $\mathbf{k}\sigma$, with small $\mathbf{k}$, comes from small $\mathbf{q} = -\mathbf{q}'$. This is a quasi-static interaction; instead of two V_0 generated by Q, we may use only one generated by Q and another generated by e; then $(eQ)^2$ is replaced by e^3Q. The integration over $\mathbf{q}'$ gives $q^2\Delta q \simeq 1/\lambda_c^3$. The main contribution of the time integration is $S_{\Delta\varepsilon}/\varepsilon_k$,

$$S_{\Delta\varepsilon} = \frac{e^{i(\varepsilon_{\mathbf{k}+\mathbf{q}}-\varepsilon_{\boldsymbol{k}})t}-1}{\varepsilon_{\mathbf{k}+\mathbf{q}}-\varepsilon_{\boldsymbol{k}}} \simeq it \ . \tag{118}$$

After averaging over angles we get a change

$$\Delta E \simeq \frac{1}{3\hbar}\left(\frac{4\pi eQ}{c\hbar}\right)^2\frac{\beta_0^4}{\varepsilon_0}\lambda_c^3\sum_{\mathbf{k}}\frac{1}{k^2} \tag{119}$$

in energy, which should be compared with the Coulomb interaction $4\pi eQ\sum_{\mathbf{k}}\frac{1}{k^2}$. It follows a charge renormalization of the order $\delta e/e \simeq \frac{\pi}{6}\alpha$, a result comparable with the standard results.[28]-[31]

5.12 Photon mass

The vacuum polarization accompanied by photon fluctuations in the perturbation series does not change the energy of the photon. However, the interaction with a "real" electron gives mass to the photon. Indeed, the interaction leads to

$$\begin{aligned} &\tfrac{1}{c^2}\tfrac{\partial^2 A^\mu}{\partial t^2} - \Delta A^\mu + \tfrac{8\pi e^2}{c^2\hbar^2}\left(\psi^*\psi\right) A^\mu = \\ &= \tfrac{4\pi}{c} j^\mu - \tfrac{4\pi i e\hbar}{c}\partial_\nu \psi^* \left(\sigma^{\mu\nu} - \sigma^{\mu\nu *}\right)\psi \ , \end{aligned} \tag{120}$$

which is a wave equation with sources and mass. For an electron (both spin orientations) we get a photon mass given by

$$m_{ph}^2 = \frac{e^2\hbar^2}{Ec^2} \cdot \frac{1}{\lambda_c^3} = \alpha\left(mc^2/E\right) m^2 \ , \tag{121}$$

where E is the energy of the electron. The photons interacting with the electrons are similar with radiation propagating in matter (polaritons). The photon mass leads to a screened Coulomb interaction $\frac{1}{r} e^{-\frac{m_{ph} c}{\hbar} r}$.

6 Conclusion

The quantization of the relativistic motion implies a cutoff length of the order of the Compton wavelength (and corresponding thresholds in energy, momentum, duration). Being associated to an undefined, unphysical motion, it is not subject to the relativistic invariance requirement.

Acknowledgments. The author is indebted to the members of the Laboratory of Theoretical Physics at Magurele-Bucharest for many fruitful discussions. This work has been supported by the Scientific Research Agency of the Romanian Government through Grant PN 19060101/2019.

References

[1] V. F. Weisskopf, "On the self-energy and the electromagnetic field of the electron", Phys. Rev. **56** 72-85 (1939).

[2] L. Landau and R. Peierls, "Erweiterung des Unbestimmtheitsprinzips fur die relativistische Quantentheorie", Z. Phys. **69** 56-69 (1931).

[3] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 4, *Quantum Electrodynamics* (V. Berestetskii, E. Lifshitz, L. Pitaevskii), Butterworth-Heinemann (1971).

[4] T. A. Welton, "Some observable effects of the quantum-mechanical fluctuations of the electromagnetic field", Phys. Rev. **74** 1157-1167 (1948).

[5] E. Schroedinger, "Ueber die kraeftefreie Bewegung in der relativistischen Quantenmechanik", Sitzungsberichte der Preussischen Akademie der Wissenschaften, Berlin, 418 (1930)

[6] E. Schroedinger, "Zur Quantendynamik des Elektrons", Sitzungsberichte der Preussischen Akademie der Wissenschaften, Berlin, 63 (1931).

[9] G. Breit, "An interpretation of Dirac's theory of the electron", Proc. Nat. Acad. Sci. **14** 553-559 (1928)

[10] W. H. Furry and J. R. Oppenheimer, "On the theory of the electron and positive", Phys. Rev. **45** 245-262 (1934).

[11] W. Pauli, "Relativistic field theories of elementary particles", Revs. Mod. Phys. **13** 203-232 (1941).

[12] P. A. M. Dirac, "The quantum theory of the emission and absorption of radiation", Proc. Roy. Soc. **A114** 243-265 (1927)

[13] P. A. M. Dirac "The quantum theory of dispersion", Proc. Roy. Soc. **A114** 710-718 (1927).

[14] E. Fermi, "Quantum theory of radiation", Revs. Mod. Phys. **4** 87-132 (1932).

[15] W. Heitler, *The Quantum Theory of Radiation*, Dover (1984).

[16] L. Landau, "On the Quantum Theory of Fields", in *Niels Bohr and the Development of Physics*, ed. W. Pauli, Pergamon Press (1955) and references therein.

[17] M. Gell-Mann and F. E. Low, "Quantum Electrodynamics at small distances", Phys. Rev. **95** 1300-1312 (1954).

[18] W. E. Lamb and R. C. Retherford, "Fine structure of the hydrogen atom by a microwave method", Phys. Rev. **72** 241-243 (1947).

[19] H. Bethe, "The electromagnetic shift of the energy levels", Phys. Rev. **72** 339-341 (1947).

[20] ; J. B. French and V. F. Weisskopf, "The electromagnetic shift of the energy levels", Phys. Rev. **75** 1240-1248 (1949).

[21] R. Feynman, "Relativistic cut-off for Quantum Electrodynamics", Phys. Rev. **74** 1430-1438 (1948) (correction in "Space-time approach to Quantum Electrodynamics", Phys. Rev. **76** 769-789 (1949)).

[22] J. Schwinger, "Quantum Electrodynamics. III. The electromagnetic properties of the electron-Radiative corrections to scattering", Phys. Rev. **76** 790-817 (1949).

[23] J. Schwinger, "On Quantum-Electrodynamics and the magnetic moment of the electron", Phys. Rev. **73** 416-417 (1948).

[24] M. Apostol, "Dynamics of electron-positron pairs in a vacuum polarized by an external electromagnetic field", J. Mod. Opt. **58** 611-618 (2011).

[25] H. Bethe and W. Heitler, "On the stopping of fast particles and on the creation of positive electrons", Proc. Roy. Soc. **A146** 83-112 (1934).

[26] G. Breit and J. A. Wheeler, "Collision of two light quanta", Phys. Rev. **46** 1087-1091 (1934).

[27] P. A. M. Dirac, "On the annihilation of electrons and protons", Proc. Cambr. Phil. Soc. **126** 361-365 (1930).

[28] J. Schwinger, "Quantum Electrodynamics. II. Vacuum polarization and self-energy", Phys. Rev. **75** 651-679 (1949).

[29] R. P. Feynman, "Space-time approach to Quantum Electrodynamics", Phys. Rev. **76** 769-789 (1949).

[30] R. Serber, "Linear modifications in the Maxwell field equations", Phys. Rev. **48** 49-54 (1935).

[31] E. A. Uehling, "Polarization effects in the positron theory", Phys. Rev. **48** 55-63 (1935).