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The Navier-Stokes equation and a fully developed turbulence

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Abstract

In fairly general conditions we give explicit (smooth) solutions for the potential flow. We show that, rigorously speaking, the equations of the fluid mechanics have not rotational solutions. However, within the usual approximations of an incompressible fluid and an isentropic flow, the remaining Navier-Stokes equation has approximate vorticial (rotational) solutions, generated by viscosity. In general, the vortices are unstable, and a discrete distribution of vorticial solutions is not in mechanical equilibrium; it forms an unstable vorticial liquid. On the other hand, these solutions may exhibit turbulent, fluctuating instabilities for large variations of the velocity over short distances. We represent a fully developed turbulence as a homogeneous, isotropic and highly-fluctuating distribution of singular centres of turbulence. A regular mean flow can be included. In these circumstances the Navier-Stokes equation exhibits three time scales. The equations of the mean flow can be disentangled from the equations of the fluctuating part, which is reduced to a vanishing inertial term. This latter equation is not satisfied after averaging out the temporal fluctuations. However, for a homogeneous and isotropic distribution of non-singular turbulence centres the equation for the inertial term is satisfied trivially, *i.e.* both the average fluctuating velocity and the average fluctuating inertial term are zero. If the velocity is singular at the turbulence centres, we are left with a quasi-ideal classical gas of singularities, or a solution of singularities in quasi thermal equilibrium in the background fluid. This is an example of an emergent dynamics. We give three examples of vorticial liquids.

1 Introduction

In fairly general conditions we give explicit (smooth) solutions for the potential flow. As it is well known, the fluids may develop turbulence. In its extreme manifestation the turbulent flow displays very irregular, disordered velocities, fluctuating in time at each point in space. This is known as a fully developed turbulence. By using such fluctuating velocities, besides a steady mean velocity, the Navier-Stokes equation becomes an infinite hierarchy of equations for velocity mean correlation functions, known as Reynolds's equations,[1] which need closure assumptions. According to the experimental observations, it was realized that such irregular movements of the fluid exhibit distributions of swirls (eddies, vortices), of various magnitude and vorticities; it is likely that the large eddies transfer energy to the small eddies, which dissipate it.[2]-[4] Statistical concepts like correlations, homogeneity and isotropy have been introduced in the theory of turbulence,[5, 6] and dimensional analysis and similarity arguments allowed the derivation of the energy spectrum of the turbulent eddies.[7]-[9]

Meanwhile, the relation of this statistical turbulence with the Navier-Stokes equation remained unclear.[10]-[13] Could the Navier-Stokes equation describe a turbulent motion? To what extent and in what sense? Has the Navier-Stokes equation smooth and stable solutions? What is the appropriate representation of a turbulent field of velocities?[14]

The dynamics of the vorticity has enjoyed much interest (see Refs. [15, 16] and References therein). The results depend on model assumptions. Dynamical-system concepts and statistical models have been invoked in studies of turbulence, with chaotic behaviour, intermittency and coherent structures (see, for instance, Refs. [17]-[21]). In particular, by analogy with the quantum turbulence, "Turbulent flows may be regarded as an intricate collection of mutually-interacting vortices", and "Vortex filaments may thus be seen as the fundamental structure of turbulence,... ".[17]

The difficulties exhibited by the Navier-Stokes equation are related to the viscosity, which governs the vorticity, and the inertial term, which is quadratic in velocity. We show in this paper that the viscosity term in the Navier-Stokes equation may produce vorticity, provided the fluid is incompressible and the flow is isentropic. Although such an approximate treatment may look reasonable, we can see that, rigorously speaking, the fluids cannot exhibit vorticity. Moreover, we give arguments that the vortices are unstable.

Further, we show in this paper that large variations of the velocity over short distances lead to highly fluctuating, swirling instabilities, controlled by viscosity. This is characteristic for the phenomenon of a fully developed turbulence. In this case, the inertial term acquires a major role in describing the flow. We represent a fully developed turbulence as a superposition of fluctuating velocities, associated to a discrete set of turbulence centres. A mean flow may be included. In general, the Navier-Stokes equation, averaged over fluctuations, is not satisfied. On the other hand, a homogeneous and isotropic distribution of (non-singular) turbulence centres leads to vanishing averages of velocity and inertial term, such that the Navier-Stokes equation is satisfied trivially. If the turbulence centres are singular, we are left with a gas of singularities (or a solution of singularities in the background fluid), which is in quasi thermal equilibrium. The corresponding Navier-Stokes equation for the fluid of singularities is reduced to Newton's equation of motion, with a small friction.

We illustrate the above descripton with three examples of vorticial liquids (filamentary liquid, coulombian and dipolar liquid).

2 Potential flow. Incompressible fluid

Let us consider a potential flow of an incompressible fluid. The velocity $\boldsymbol{v} = grad\Phi$ is given by the gradient of a potential Φ, which satisfies the Laplace equation

$$\Delta\Phi = 0 \tag{1}$$

(incompressiblity condition $div\boldsymbol{v} = 0$). The viscosity term $\sim \Delta\boldsymbol{v}$ is zero, such that we are left with Euler's equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}grad)\boldsymbol{v} = -\frac{1}{\rho}gradp \ , \tag{2}$$

where ρ is the density and p denotes the pressure. By using the well-known identity

$$(\boldsymbol{v}grad)\boldsymbol{v} = -\boldsymbol{v} \times curl\boldsymbol{v} + grad(v^2/2) \ , \tag{3}$$

equation (2) becomes

$$\frac{\partial \boldsymbol{v}}{\partial t} + grad(v^2/2 + p/\rho) = 0 \ , \tag{4}$$

where $curl\boldsymbol{v} = 0$. As it is well known, by using equation (1), this equation leads to

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(grad\Phi)^2 + \frac{p}{\rho} = 0 \ . \tag{5}$$

In this equation p should be viewed as the variation of the pressure with respect to equilibrium. We assume that p does not depend on Φ and the time.

In equations (1) and (5) the variables may be separated. Let $g(\boldsymbol{r})$ be a solution of equation (1) (satisfying the boundary conditions); the potential can be written as $\Phi = f(t)g(\boldsymbol{r})$, where the function $f(t)$ satisfies equation (5),

$$\frac{df}{dt} + \frac{1}{2g}f^2(gradg)^2 + \frac{p}{\rho g} = 0 \ . \tag{6}$$

The acceptable solution of this equation (for $f(0) = 0$) is

$$f(t) = \frac{\sqrt{2 \mid p \mid /\rho}}{\mid gradg \mid} \tanh \frac{\sqrt{\mid p \mid /2\rho} \mid gradg \mid}{g} t \tag{7}$$

for $p < 0$. It may happen that the boundary conditions for the equation $\Delta\Phi = 0$ depend on time, thus providing the time derivative $\dot{\Phi}$; in that case equation (5) gives the pressure.

3 Potential flow. Compressible fluid

Let us write down the equations of the fluid mechanics

$$\begin{gathered}
\tfrac{\partial \rho}{\partial t} + \rho div\boldsymbol{v} + \boldsymbol{v}grad\rho = 0 \ , \\
\rho\tfrac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}grad)\boldsymbol{v} = -gradp - \rho grad\varphi + \\
+\eta\Delta\boldsymbol{v} + \left(\tfrac{1}{3}\eta + \zeta\right) grad\, div\boldsymbol{v} \ , \\
\rho T\left(\tfrac{\partial s}{\partial t} + \boldsymbol{v}grads\right) = \kappa\Delta T + \sigma'_{ij}\partial_j v_i \ ,
\end{gathered} \tag{8}$$

where p is the internal pressure, φ is an external potential, η and ζ are the viscosity coefficients, T is the temperature, s is the entropy per unit masss, κ is the thermoconductivity and

$$\sigma'_{ij} = \eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}div\boldsymbol{v}\right) + \zeta\delta_{ij}div\boldsymbol{v} \ . \tag{9}$$

is the viscosity tensor. In the Navier-Stokes equation (the second equation (8)) the forces which determine the velocity are $gradp$ and $\rho grad\varphi$, where $p = p(\rho, T)$ is a function of density and temperature. For all the usual flows the relative variations $\delta\rho/\rho_0$ of the density, $\delta T/T_0$ of the temperature, $\delta p/p_0$ of the pressure, $\delta s/s_0$ of the entropy as well as the variation $\delta\varphi/\varphi_0$ of the external potential are small, in comparison with their equilibrium values, labelled by the suffix 0, when the fluid is at rest. Consequently, we may view the velocity $\boldsymbol{v}$ as a first-order quantity, and linearize the above equations as

$$\begin{gathered}\frac{\partial\rho}{\partial t}+\rho_0 div\boldsymbol{v}=0\ ,\\ \rho_0\frac{\partial\boldsymbol{v}}{\partial t}=-gradp-\rho_0 grad\varphi+\\ +\eta\Delta\boldsymbol{v}+\left(\frac{1}{3}\eta+\zeta\right)grad\,div\boldsymbol{v}\ ,\\ \rho_0 T_0\frac{\partial s}{\partial t}=\kappa\Delta T\ .\end{gathered} \tag{10}$$

We note that within this approximation there is no heat source, and the first-order equation of energy conservation is reduced to an identity.

The density and entropy variations can be written as

$$\begin{gathered}\delta\rho=\frac{\rho_0}{K}\delta p-\beta\rho_0\delta T\ ,\\ \delta s=-\frac{\beta}{\rho_0}\delta p+\frac{c_p}{T_0}\delta T\ ,\end{gathered} \tag{11}$$

where K is the isothermal modulus of compressibility ($1/K=-\frac{1}{V}(\partial V/\partial p)_T$, $V=1/\rho$), $\beta=\frac{1}{V}(\partial V/\partial T)_p$ is the dilatation coefficient and c_p is the specific heat per unit mass at constant pressure, all at equilibrium. In deriving equations (11) the Gibbs free energy $d\Phi=Vdp-sdT$ is used.

Part of the temperature variation in equation (11) is compensated by pressure variation, as in an adiabatic process; we denote this contribution by δT_1. The remaining part, denoted by δT, corresponds to the conducted heat. Therefore, we write

$$\begin{gathered}\delta\rho=\frac{\rho_0}{K}\delta p-\beta\rho_0\delta T_1-\beta\rho_0\delta T\ ,\\ \delta s=-\frac{\beta}{\rho_0}\delta p+\frac{c_p}{T_0}\delta T_1+\frac{c_p}{T_0}\delta T=\frac{c_p}{T_0}\delta T\ ,\end{gathered} \tag{12}$$

whence

$$\frac{\beta}{\rho_0}\delta p=\frac{c_p}{T_0}\delta T_1 \tag{13}$$

and

$$\delta\rho=\frac{\rho_0}{K}\left(1-\frac{\beta^2T_0K}{\rho_0c_p}\right)\delta p-\beta\rho_0\delta T\ . \tag{14}$$

In this equation we use the thermodynamic relation

$$\frac{\beta^2T_0K}{\rho_0}=c_p-c_v\ , \tag{15}$$

where c_v is the specific heat per unit mass at constant volume.[22] Therefore, equation (14) becomes

$$\delta\rho=\frac{\rho_0c_v}{Kc_p}\delta p-\beta\rho_0\delta T\ , \tag{16}$$

or

$$\delta p=\frac{Kc_p}{\rho_0c_v}\delta\rho+\frac{\beta Kc_p}{c_v}\delta T\ . \tag{17}$$

Now we use two other thermodynamic relations

$$\frac{c_p}{c_v}K = K_{ad} \ , \ \beta K = \alpha \ , \tag{18}$$

where K_{ad} is the adiabatic modulus of compressibility and $\alpha = (\partial p/\partial T)_V$ is the thermal pressure coefficient.[22] Finally, we get

$$\delta p = \frac{K_{ad}}{\rho_0}\delta\rho + \frac{c_p}{c_v}\alpha\delta T \ , \tag{19}$$

which is used in the Navier-Stokes equation. Equations (10) become

$$\begin{gathered}\frac{\partial\rho}{\partial t} + \rho_0 div\boldsymbol{v} = 0 \ , \\ \rho_0\frac{\partial\boldsymbol{v}}{\partial t} = -\frac{K_{ad}}{\rho_0}grad\rho - \rho_0 grad\varphi - \frac{c_p}{c_v}\alpha gradT + \\ +\eta\Delta\boldsymbol{v} + \left(\frac{1}{3}\eta + \zeta\right) grad\, div\boldsymbol{v} \ , \\ \rho_0 c_p\frac{\partial T}{\partial t} = \kappa\Delta T \ ,\end{gathered} \tag{20}$$

where the second equation (12) is used. We can see that the temperature equation (20) is independent; it describes the transport of an external temperature, which may provide a source for the velocity in the Navier-Stokes equation. We may leave aside this external temperature.

Let us seek a potential-flow solution of the above equations, where the velocity is derived from a potential Φ, by

$$\boldsymbol{v} = grad\Phi \ . \tag{21}$$

We notice that $curl\boldsymbol{v} = 0$ and $curl\, curl\boldsymbol{v} = 0$, *i.e.* $\Delta\boldsymbol{v} = grad\, div\boldsymbol{v}$. Therefore, the Navier-Stokes equation can be written as

$$\begin{gathered}\rho_0\frac{\partial\boldsymbol{v}}{\partial t} = -\frac{K_{ad}}{\rho_0}grad\rho - \rho_0 grad\varphi + \\ +\left(\frac{4}{3}\eta + \zeta\right) grad\, div\boldsymbol{v} \ .\end{gathered} \tag{22}$$

By using equation (21), we obtain

$$\begin{gathered}\frac{\partial\rho}{\partial t} + \rho_0\Delta\Phi = 0 \ , \\ \frac{\partial\Phi}{\partial t} + \frac{K_{ad}}{\rho_0^2}\rho + \varphi - \frac{1}{\rho_0}\left(\frac{4}{3}\eta + \zeta\right)\Delta\Phi = 0 \ ,\end{gathered} \tag{23}$$

up to a function of time, where ρ and φ should be viewed as their corresponding variations. By an additional time differentiation we obtain

$$\frac{\partial^2\Phi}{\partial t^2} - \frac{K_{ad}}{\rho_0}\Delta\Phi + \dot{\varphi} - \frac{1}{\rho_0}\left(\frac{4}{3}\eta + \zeta\right)\Delta\dot{\Phi} = 0 \ . \tag{24}$$

This equation provides the potential Φ, therefore the velocity $\boldsymbol{v}$ through equation (21) and the density ρ through the first equation (23).

Equation (24) is the wave equation with friction (the term $\sim \Delta\dot{\Phi}$) and sources ($-\dot{\varphi}$). The ratio K_{ad}/ρ_0 is the square of the sound velocity $c = \sqrt{K_{ad}/\rho_0}$. The elementary solutions $e^{-i\omega t}e^{i\boldsymbol{kr}}$ of this (homogeneous) equation are damped plane waves

$$e^{\mp ickt}e^{i\boldsymbol{kr}}e^{-\frac{\sigma k^2}{c}t} \ , \tag{25}$$

for $\sigma k \ll c$, where $\sigma = (4\eta/3 + \zeta)/2\rho_0$. The relaxation time is much longer than the wave period. A wave propagating along the x-direction is proportional to $\sim e^{-\gamma x}$, with the attenuation coefficient $\gamma = \sigma k^2/c = \sigma\omega^2/c^3$. This is the well-known absorption coefficient for sound (without the κ-contribution).

4 Vorticity

Euler's equation for an ideal fluid can be written as

$$\frac{d\boldsymbol{v}}{dt} = -gradw \ , \tag{26}$$

where w is the enthalpy ($dw = \frac{1}{\rho}dp$); the pressure p is a function of density ρ. By taking the *curl*, we get

$$curl\frac{d\boldsymbol{v}}{dt} = 0 \ . \tag{27}$$

On the other hand,

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}grad)\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial t} + \left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_f \ , \tag{28}$$

where the suffix f indicates that the derivative is taken along the flow. Equation (27) becomes

$$\frac{\partial}{\partial t}curl\boldsymbol{v} + \left(\frac{\partial}{\partial t}curl\boldsymbol{v}\right)_f = 0 \ ; \tag{29}$$

since the two variations of the $curl\boldsymbol{v}$ are independent, we get

$$curl\boldsymbol{v} = 0 \ , \tag{30}$$

i.e. the vorticity $curl\boldsymbol{v}$ is conserved along the flow. Therefore, we cannot create, or destroy, vorticity $curl\boldsymbol{v}$ in the flow of an ideal fluid. Equation (30) is valid in the absence of special external force, which do not derive from a gradient. This is Helmholtz's circulation law. As it is well known, an ideal fluid supports only an irrotational (potential) flow, where the velocity is derived from a scalar potential ($\boldsymbol{v} = grad\Phi$). By knowing the equation of state of the fluid, the Euler equation and the continuity equation are fully determined.

For a real, viscid, fluid the Navier-Stokes equation is

$$\rho\frac{d\boldsymbol{v}}{dt} = -gradp + \eta\Delta\boldsymbol{v} + \left(\frac{1}{3}\eta + \zeta\right) grad\, div\boldsymbol{v} \ , \tag{31}$$

where η, ζ are the viscosity coefficients. By taking the *curl*, we get

$$\begin{gathered} curl\left(\rho\tfrac{d\boldsymbol{v}}{dt}\right) = grad\rho \times \tfrac{d\boldsymbol{v}}{dt} + \rho curl\tfrac{d\boldsymbol{v}}{dt} = \\ = \eta\Delta curl\boldsymbol{v} \ ; \end{gathered} \tag{32}$$

we can see that the viscosity η can generate vorticity ($curl\boldsymbol{v} \neq 0$). In general, the velocity $\boldsymbol{v}$ rotates about the vorticity $curl\boldsymbol{v}$. This is a vortex.

Equation (32) is in conflict with the continuity equation

$$\frac{d\rho}{dt} + \rho div\boldsymbol{v} = 0 \ . \tag{33}$$

Indeed, if $curl\boldsymbol{v} \neq 0$, the velocity should be derived from a *curl* (not from a *grad*!), *i.e.* we should have $\boldsymbol{v} = curl\boldsymbol{A}$, where $\boldsymbol{A}$ is a vector potential. Consequently, the fluid should be incompressible ($div\boldsymbol{v} = div\, curl\boldsymbol{A} = 0$). The density should be constant, both in time and space (along the flow).

This indicates that in a compressible fluid we cannot have vortices. Usually, the variations of the density are small, such that they may be neglected for the present purpose.

Therefore, we may limit to an incompressible fluid ($div\boldsymbol{v} = 0$), for which equation (32) becomes

$$curl\frac{d\boldsymbol{v}}{dt} = \nu\Delta curl\boldsymbol{v} \ , \tag{34}$$

or

$$\frac{\partial}{\partial t}curl\boldsymbol{v} - curl\left(\boldsymbol{v} \times curl\boldsymbol{v}\right) = \nu\Delta curl\boldsymbol{v} \ , \tag{35}$$

where $\nu = \eta/\rho$ is the kinematical viscosity and we have used the identity $(\boldsymbol{v}grad)\boldsymbol{v} = -\boldsymbol{v} \times curl\boldsymbol{v} + grad(v^2/2)$. This is the equation of vorticity; it can also be written as

$$\frac{\partial}{\partial t}curl\boldsymbol{v} + curl\left[(\boldsymbol{v}grad)\boldsymbol{v}\right] = \nu\Delta curl\boldsymbol{v} \ . \tag{36}$$

This equation gives the velocity. The pressure is obtained from the Navier-Stokes equation. If we write the Navier-Stokes equation as

$$\frac{\partial}{\partial t}curl\boldsymbol{A} + (\boldsymbol{v}grad)\boldsymbol{v} = -\frac{1}{\rho}gradp + \nu\Delta curl\boldsymbol{A} \ , \tag{37}$$

we get

$$div\left[(\boldsymbol{v}grad)\boldsymbol{v} + \tfrac{1}{\rho}gradp\right] = 0 \tag{38}$$

and

$$div\left(\frac{\partial\boldsymbol{v}}{\partial t} - \nu\Delta\boldsymbol{v}\right) = 0 \ , \tag{39}$$

which is an identity ($div\boldsymbol{v} = 0$, $\boldsymbol{v} = curl\boldsymbol{A}$). In some cases a particular solution of these equations is provided by

$$\begin{gathered}(\boldsymbol{v}grad)\boldsymbol{v} = -grad(p/\rho) \ , \\ \tfrac{\partial\boldsymbol{v}}{\partial t} = \nu\Delta\boldsymbol{v} \ .\end{gathered} \tag{40}$$

We can see that the Navier-Stokes equation is split into (the derivatives of) a diffusion (heat) equation and an equilibrium equation; the first equation (40) indicates an equilibrium between the pressure force $-grad(p/\rho)$ and Euler's force $(\boldsymbol{v}grad)\boldsymbol{v}$. The diffusion equation (40) holds also for the vorticity, because the above equations are valid for a non-vanishing vorticity. It is easy to see that these equations generalize the equations for the Couette flow.

However, we have also the heat-transfer equation. For an incompressible fluid it reads

$$\rho c_p\frac{dT}{dt} = \kappa\Delta T + \frac{1}{2}\eta\left(\partial_i v_j + \partial_j v_i\right)^2 \ , \tag{41}$$

or

$$\frac{dT}{dt} = \chi\Delta T + \frac{1}{2}\frac{\nu}{c_p}\left(\partial_i v_j + \partial_j v_i\right)^2 \ , \tag{42}$$

where c_p is the specific heat per unit mass at constant pressure, κ is the thermoconductivity, and $\chi = \kappa/\rho c_p$ is the thermometric conductivity. For an incompressible fluid this equation can be transformed into an equation for the derivatives of the pressure,

$$\frac{dp}{dt} = \chi\Delta p + \frac{1}{2}\frac{\alpha\nu}{c_p}\left(\partial_i v_j + \partial_j v_i\right)^2 \ , \tag{43}$$

where $\alpha = (\partial p/\partial T)_v$ is the thermal pressure coefficient. In general, equation (43) is not compatible with the Navier-Stokes equation. Therefore, rigorously speaking, we cannot have vorticity in an incompressible fluid either. Usually, the coefficient ν/c_p is very small (of the order $10^{-24} - 10^{-25} g \cdot cm^2/s$), such that, for small gradients of velocity, we have a low rate of entropy production (though, a factor of the order $10^{23} K/erg$ should be taken into account). Under these conditions, we may assume that the flow is isentropic and the heat-transfer equation may be neglected.

In order to get an idea of how large the variations of the density and the temperature can be, we may estimate a change δp in pressure from $\delta p \simeq \rho v^2$. A velocity $v = 100km/h$, which is fairly large, produces a change $\delta p \simeq 10^4 dyn/cm^2$ in air ($\rho = 10^{-3} g/cm^3$), whose normal pressure is $10^6 dyn/cm^2$; therefore, $\delta p/p \simeq 10^{-2}$. Such a velocity ($\simeq 3 \times 10^3 cm/s$) is close to the mean thermal velocity $\simeq 10^4 cm/s$ (for normal air), and close to the sound velocity in normal air $c \simeq 3.5 \times 10^4 cm/s$. For this velocity we still expect local thermal equilibrium. The change in density is given by $\delta p = K(\delta\rho/\rho)$, where $K = -V(\partial p/\partial V)$ is the (say, isothermal) modulus of compression. For air $K \simeq 10^6 dyn/cm^2$, for water $K \simeq 10^{10} dyn/cm^2$, such that we get $\delta\rho/\rho \simeq 10^{-2}$, 10^{-6}. The change in temperature is obtained from $\delta p = \alpha T(\delta T/T)$, where $\alpha = (\partial p/\partial T)_V$ is the thermal pressure coefficient (at constant volume V). For water $\alpha \simeq 10^{22}/cm^3$, for gases it is much higher; for normal temperature $T = 300K$ we get $\delta T/T \simeq 10^{-4}$, or much lower. Consequently, we may expect an almost ideal, incompressible flow. We note that, although we neglect the viscosity in the heat-transfer equation, we keep it in the Navier-Stokes equation.

Therefore, within these approximations (incompressibility and constant entropy), we are left with equation (36) and the Navier-Stokes equation for a vorticial flow. In general, an external pressure which satisfies the Navier-Stokes equation (or equation (38)) is very special, such that, if they exist, the vortices might be, in fact, unstable. They develop an Euler's force, which is difficult to be compensated by an external force.

We note that, under the conditions stated above, the viscosity may generate (unstable) vortices. In the next section we show that the viscosity may generate another type of instabilities.

For small variations of the velocity we may neglect the inertial term in the vorticity equation (36), which becomes

$$\frac{\partial}{\partial t} curl\boldsymbol{v} = \nu\Delta curl\boldsymbol{v} \ . \tag{44}$$

By making use of $\boldsymbol{v} = curl\boldsymbol{A}$, we get

$$(\Delta - grad\, div)\left(\frac{\partial \boldsymbol{A}}{\partial t} - \nu\Delta\boldsymbol{A}\right) = 0 \ . \tag{45}$$

A solution of this equation is provided by

$$\frac{\partial \boldsymbol{A}}{\partial t} - \nu\Delta\boldsymbol{A} = 0 \ , \tag{46}$$

which leads to

$$\boldsymbol{A} = \boldsymbol{A}_0 e^{-\lambda\nu t} e^{\pm i\sqrt{\lambda}r}/r \ , \tag{47}$$

where $\boldsymbol{A_0}$ and λ are two constants. The velocity acquires the form

$$\boldsymbol{v} = -\boldsymbol{A}_0 \times grad\left(e^{-\lambda\nu t} e^{\pm i\sqrt{\lambda}r}/r\right) \ , \tag{48}$$

and the pressure is uniform within this approximation. We note that, although the spatial dependence of the solution does not depend on viscosity, it is generated by the viscosity term $\nu\Delta\boldsymbol{v}$.

5 Instabilities

The equation of energy conservation for an incompressible fluid is

$$\begin{gathered}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right)+div\left[\boldsymbol{v}\left(\frac{1}{2}\rho v^2+p\right)-\frac{1}{2}\eta grad(v^2)\right]+\\+\eta\left(\partial_j v_i\right)^2=0\ ;\end{gathered}\tag{49}$$

it is obtained by multiplying by $\boldsymbol{v}$ the Navier Stokes equation (31) for an incompressible fluid ($div\boldsymbol{v}=0$). The div-term represents a transport of energy and mechanical work of the pressure, and an energy flux associated with collisions (viscosity); the term $\eta\left(\partial_j v_i\right)^2$ represents the heat produced by viscosity. We integrate this equation over a volume V enclosed by a surface S,

$$\begin{gathered}\frac{\partial}{\partial t}\int dV\left(\frac{1}{2}\rho v^2\right)+\oint dS\left[v_n\left(\frac{1}{2}\rho v^2+p\right)-\frac{1}{2}\eta\partial_n(v^2)\right]+\\+\eta\int dV\left(\partial_j v_i\right)^2=0\ \ ,\end{gathered}\tag{50}$$

where v_n is the velocity component normal to the surface and ∂_n is the derivative along the normal to the surface.

We compare the orders of magnitude of the surface terms and the η-volume term, and get ratios of the form $\frac{Sl}{V}R$, $\frac{Sl}{V}(p/\rho v^2)R$, $\frac{Sl}{V}$, where l is the distance over which the velocity varies and $R=vl/\nu$ is the Reynolds number. For moderate Reynolds numbers and $Sl/V\ll 1$ we can neglect the surface contributions in comparison with the heat term. By writing

$$\boldsymbol{v}=f(t)\boldsymbol{u}(\boldsymbol{r})\ \ ,\tag{51}$$

the above equation becomes

$$\frac{\partial}{\partial t}f^2\cdot\int dV\left(\frac{1}{2}\rho u^2\right)+\eta f^2\int dV\left(\partial_j u_i\right)^2=0\ .\tag{52}$$

We can see that the time dependence of the velocity is a damped exponential. The flow is stable, as a consequence of the dissipated heat. The η-term in equation (52) gives, in fact, the increase of entropy.

Let us assume that the integration domains are sufficiently small, such that Sl/V is of the order of unity; the velocity varies over a distance l inside the domains, but we assume that it suffers a large discontinuity across the surface, over a small distance $\delta\ll l$. Then, it is easy to see that the dominant term in equation (50) is the collision term, such that equation (50) becomes

$$\frac{\partial}{\partial t}f^2\cdot\int dV\left(\frac{1}{2}\rho u^2\right)-\frac{1}{2}\eta f^2\oint dS\partial_n(u^2)=0\ .\tag{53}$$

We can see that for a positive normal derivative the flow is unstable. The viscosity is insufficient to disipate the energy as heat, and the energy is transferred by molecular collisions (viscosity) through surfaces of discontinuities. The process occurs in small domains, with large discontinuities of velocity across their surface, and the instabilities imply returning, swirling and fluctuating, velocities. This is the turbulence phenomenon. We note that the instabilities are governed by viscosity, which gives also vorticity (when the entropy production is neglected). Moreover, we note that the inertial term does not appear in instabilities, though it plays an important role in turbulence.

The above arguments can be extended to compressible fluids, including the variations of the temperature. Indeed, the energy conservation in this case reads

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho\varepsilon\right)=-\partial_j\left[\rho v_j\left(\frac{1}{2}v^2+w\right)-v_i\sigma_{ij}^{'}-\kappa\partial_j T\right] \ , \tag{54}$$

where ε is the internal energy per unit mass, w is the enthalpy per unit mass and

$$\sigma_{ij}^{'}=\eta\left(\partial_i v_j+\partial_j v_i-\frac{2}{3}\delta_{ij}div\boldsymbol{v}\right)+\zeta\delta_{ij}div\boldsymbol{v} \tag{55}$$

is the viscosity tensor. For a smooth flow and a sufficiently large volume the surface term in equation (54) can be neglected, and the energy is conserved, as it is well known. However, in the surface integral we have terms of the form

$$\oint dS\left[\eta v_i\left(\partial_i v_n+\partial_n v_i\right)-\left(\frac{2}{3}\eta-\zeta\right)v_n div\boldsymbol{v}+\kappa\partial_n T\right] \ , \tag{56}$$

which imply normal derivatives to the surface, both of velocity and temperature. By collecting these contributions, we get

$$\oint dS\left[\eta\partial_n(v^2/2)+\left(\frac{1}{3}\eta+\zeta\right)\partial_n(v_n^2/2)+\kappa\partial_n T\right] \ . \tag{57}$$

We can see that for large normal derivatives across the surface, both for velocity and temperature, these terms may lead to instabilities.

6 Turbulence

As it is well known, for a moderate turbulence, *i.e.* for slowly varying fluctuations, we may decompose the velocity field into a mean velocity and a fluctuating part, and limit ourselves to the time averaged Navier-Stokes equation. This way we get the Reynolds equations, for which the mean energy is coupled to the fluctuating energy, via model assumptions. A fully developed turbulence exhibits highly-varying fluctuations, such that we need to consider the time-dependent Navier-Stokes equation.

The turbulent instabilities ocurring in a fully developed turbulence exhibit large variations of the velocity over small distances. In this case we may assume that the velocity is split into a mean-flow velocity $\boldsymbol{v}_0$ and a fluctuating part $\boldsymbol{v}$, where the mean-flow velocity $\boldsymbol{v}_0$ may have a slight time variation, while the fluctuating velocity $\boldsymbol{v}$ is a rapidly varying velocity. By using this decomposition for an incompressible fluid, the Navier-Stokes equation reads

$$\begin{aligned}&\tfrac{\partial\boldsymbol{v}_0}{\partial t}+\tfrac{\partial\boldsymbol{v}}{\partial t}+\left(\boldsymbol{v}_0 grad\right)\boldsymbol{v}_0+\left(\boldsymbol{v}_0 grad\right)\boldsymbol{v}+\left(\boldsymbol{v}grad\right)\boldsymbol{v}_0+\\&+\left(\boldsymbol{v}grad\right)\boldsymbol{v}=-\tfrac{1}{\rho}gradp_0-\tfrac{1}{\rho}gradp+\nu\Delta\boldsymbol{v}_0+\nu\Delta\boldsymbol{v} \ ,\end{aligned} \tag{58}$$

where p_0 is the pressure corresponding to the main flow and p is the fluctuating part of the pressure. In this equation we have three distinct types of time variations, such that it should be viewed as three equations

$$\begin{gathered}\tfrac{\partial\boldsymbol{v}_0}{\partial t}+\left(\boldsymbol{v}_0 grad\right)\boldsymbol{v}_0=-\tfrac{1}{\rho}gradp_0+\nu\Delta\boldsymbol{v}_0 \ ,\\ \tfrac{\partial\boldsymbol{v}}{\partial t}+\left(\boldsymbol{v}_0 grad\right)\boldsymbol{v}+\left(\boldsymbol{v}grad\right)\boldsymbol{v}_0=-\tfrac{1}{\rho}gradp+\nu\Delta\boldsymbol{v} \ ,\\ \left(\boldsymbol{v}grad\right)\boldsymbol{v}=0 \ ;\end{gathered} \tag{59}$$

similarly, the continuity equation should be split into

$$div\boldsymbol{v}_0 = 0 \ , \ div\boldsymbol{v} = 0 \ . \tag{60}$$

The first equation (59) is an independent equation, which gives the main flow velocity $\boldsymbol{v}_0$. Since the main part of the velocity is taken by the fluctuating velocity, we may neglect the quadratic term in this equation. The energy conservation and the heat transfer for this equation are given by

$$\frac{\partial}{\partial t}(v_0^2/2) = -\partial_i \left[v_{0i}\left(p_0/\rho + v_0^2/2\right) - \nu\partial_i(v_0^2/2)\right] -$$

$$-\nu(\partial_i v_{0j})^2 \ , \tag{61}$$

$$T_0\frac{ds_0}{dt} = \chi\Delta T_0 + \nu(\partial_i v_{0j})^2 \ ,$$

where T_0 is the temperature of the main flow, s_0 is the entropy per unit mass of the main flow and χ is the thermometric conductivity.

Having solved the mean-flow equation we can pass to solve the second equation (59) for the fluctuating velocity $\boldsymbol{v}$, with $\boldsymbol{v}_0$ as a parameter. This equation has its own energy-conservation and heat-transfer equations. We note that the temperature and the entropy of the mean flow are different from the temperature and the entropy of the fluctuating part of the flow, which means that the two components of the flow (the mean flow and the fluctuating flow) are not in thermal equilbrium. Indeed, if we multiply the first equation (59) by $\boldsymbol{v}$ and the second equation (59) by $\boldsymbol{v}_0$, we get cross-terms of the form $T_0\frac{ds}{dt} + T\frac{ds_0}{dt}$ in the heat-transfer equation, where T and s are the temperature and the entropy of the fluctuating flow. This indicates a heat exchange between the two components of the flow.

We are left with the third equation (59), which, in general, is not satisfied. We conclude that the fully developed turbulence does not satisfy the Navier-Stokes equation. The quadratic term of the third equation (59) is equivalent to a rapidly varying internal force (Euler's force), which cannot be compensated by any physical external force. The fully developed turbulence is unstable.

Under these conditions it is reasonable to be interested in time averaged quantities. Then, the fluctuating part of the flow is reduced to

$$\overline{(\boldsymbol{v}grad)\,\boldsymbol{v}} = 0 \ . \tag{62}$$

Since $div\boldsymbol{v} = 0$, the components of the velocity $\boldsymbol{v}$ are not independent. In general, equation (62) is not satisfied, which means that the turbulent motion is unstable even on average. We note that a similar decomposition is valid for a compressible fluid, as long as the velocity, density and entropy fluctuations are independent of one another.

Since a fully developed turbulence originates in large variations of the velocity across small distances, it is reasonable to associate these variations with a discrete distribution of positions $\boldsymbol{r}_i$,which we call centres of turbulence. Further on, we assume that this is a homogeneous and isotropic distribution, such that we may write the velocity field as

$$\boldsymbol{v} = \sum_i \boldsymbol{v}_i(t, R_i) \ , \tag{63}$$

where $\boldsymbol{R}_i = \boldsymbol{r} - \boldsymbol{r}_i$. If $curl\boldsymbol{v}_i \neq 0$, this velocity field represents a vorticial liquid, which is unstable. We assume that the fluctuating velocities are independent at distinct positions, *i.e.* $\overline{\boldsymbol{v}_i} = 0$ and $\overline{\boldsymbol{v}_i\boldsymbol{v}_j} \sim \delta_{ij}$. Equation (62) becomes

$$\overline{(\boldsymbol{v}grad)\,\boldsymbol{v}} = \sum_i \overline{(\boldsymbol{v}_i\boldsymbol{R}_i/R_i)\,(d\boldsymbol{v}_i/dR_i)} \ . \tag{64}$$

The conditions of homogeneity and isotropy imply that $\boldsymbol{v}_i$ in the above equation may be replaced by the same velociy $\boldsymbol{u}$. For a sufficiently dense set of positions $\boldsymbol{r}_i$ we can define a density ρ_v of such points, which is a constant. Then, equation (64) can be transformed into the integral

$$\overline{(\boldsymbol{v}grad)\,\boldsymbol{v}} = \rho_v \int dR \cdot R^2 \int do\overline{(\boldsymbol{u}\boldsymbol{R}/R)\,(d\boldsymbol{u}/dR)} \ . \tag{65}$$

If the radial integral is finite, the result of integration in the above equation is zero, due to the integration over the solid angle o, such that the term $\overline{(\boldsymbol{v}grad)\,\boldsymbol{v}}$ is zero. In this case we can say that the Navier-Stokes equation is trivially satisfied on average, being reduced to the mean-flow equation (first equation (59)).

If the radial integral in equation (65) is singular for $\boldsymbol{R} = 0$, as it may often happen for vortices, we are left with a discrete set of singularities, extending over a small characteristic distance a, where the singularity is

$$\boldsymbol{u} \sim (a/R)^n \ , \ n > 1 \ . \tag{66}$$

In each of these regions there exists a mass M of fluid, which can be carried by the background fluid and, at the same time, they may have their own motion.

The averaged energy-conservation equation derived from the second equation (59),

$$\frac{1}{2}\boldsymbol{v}_0 grad\overline{v^2} + \partial_j\left(\overline{v_i v_j} v_{0i}\right) - \frac{1}{2}\nu\Delta\overline{v^2} = -\nu\overline{(\partial_j v_i)^2} \ , \tag{67}$$

shows that the fluctuating motion produces heat which is partly transported by the mean flow (the *div*-terms integrated over a volume are irrelevant). This dissipated heat should be compensated from the outside. Therefore, we are left with a quasi ideal classical gas of singular vortices, or a solution of vortices in the background fluid, in thermal quasi equilibrium. This is an example of emergent dynamics.[23]

7 Gas of singularities

Let us assume a homogeneous, isotropic, fluctuating distribution of singular centres of turbulence localized at $\boldsymbol{r}_i$ with mass M, as described above. Their density is

$$\rho = M\sum_i \delta(\boldsymbol{r} - \boldsymbol{r}_i) \tag{68}$$

and

$$\frac{\partial\rho}{\partial t} = -M\sum_i \boldsymbol{u}_i grad\delta(\boldsymbol{r} - \boldsymbol{r}_i) \ , \tag{69}$$

where $\boldsymbol{u}_i = d\boldsymbol{r}_i/dt$ is their velocity. The velocity field of the singularities is

$$\boldsymbol{u} = v\sum_i \boldsymbol{u}_i\delta(\boldsymbol{r} - \boldsymbol{r}_i) \ , \tag{70}$$

where v is the small volume over which the δ-function is localized, such that $v = a^3$ and $M = \rho v$. Let us compute

$$\begin{aligned} div(\rho\boldsymbol{u}) &= vMdiv\textstyle\sum_{ij}\delta(\boldsymbol{r} - \boldsymbol{r}_i)\boldsymbol{u}_j\delta(\boldsymbol{r} - \boldsymbol{r}_j) = \\ &= Mdiv\textstyle\sum_i \boldsymbol{u}_i\delta(\boldsymbol{r} - \boldsymbol{r}_i) = \\ &= M\textstyle\sum_i \boldsymbol{u}_i grad\delta(\boldsymbol{r} - \boldsymbol{r}_i) = -\frac{\partial\rho}{\partial t} \ ; \end{aligned} \tag{71}$$

we can see that the continuity equation is satisfied.

Now, let us focus on the Navier-Stokes equation

$$\rho\frac{\partial \boldsymbol{u}}{\partial t}+\rho(\boldsymbol{u}grad)\boldsymbol{u}=-gradp+\eta\Delta\boldsymbol{u}\ , \tag{72}$$

and let us compute each term in this equation for our fluid of singularities. We have

$$\tfrac{\partial \boldsymbol{u}}{\partial t}=v\textstyle\sum_i \dot{\boldsymbol{u}}_i\delta(\boldsymbol{r}-\boldsymbol{r}_i)-v\sum_i \boldsymbol{u}_i(\boldsymbol{u}_igrad)\delta(\boldsymbol{r}-\boldsymbol{r}_i)\ . \tag{73}$$

The inertial term is

$$\begin{aligned}(\boldsymbol{u}grad)\boldsymbol{u}&=v^2\textstyle\sum_{ij}\boldsymbol{u}_j\delta(\boldsymbol{r}-\boldsymbol{r}_i)(\boldsymbol{u}_igrad)\delta(\boldsymbol{r}-\boldsymbol{r}_j)=\\&=v\textstyle\sum_i \boldsymbol{u}_i(\boldsymbol{u}_igrad)\delta(\boldsymbol{r}-\boldsymbol{r}_i)\ .\end{aligned} \tag{74}$$

On comparing equations (73) and (74), we can see that the inertial (Euler's) term disappears from equation. This is expected, since the δ-function equals the variable $\boldsymbol{r}$ to the function $\boldsymbol{r}_i(t)$, which amounts to Lagrange's approach. The term on the left in equation (72) becomes

$$\rho\tfrac{\partial \boldsymbol{u}}{\partial t}+\rho(\boldsymbol{u}grad)\boldsymbol{u}=M\textstyle\sum_i \dot{\boldsymbol{u}}_i\delta(\boldsymbol{r}-\boldsymbol{r}_i)\ \ , \tag{75}$$

and equation (72) reads now

$$\begin{aligned}M\textstyle\sum_i \dot{\boldsymbol{u}}_i\delta(\boldsymbol{r}-\boldsymbol{r}_i)&=-v\textstyle\sum_i p_igrad\delta(\boldsymbol{r}-\boldsymbol{r}_i)+\\&+\eta v\textstyle\sum_i \boldsymbol{u}_i\Delta\delta(\boldsymbol{r}-\boldsymbol{r}_i)\ \ ,\end{aligned} \tag{76}$$

where p_i is the pressure at the position $\boldsymbol{r}_i$. The pressure term in equation (76) is a force per unit volume acting upon the vortex placed at $\boldsymbol{r}_i$; it may arise from the pressure exerted by the background fluid particles. This term may be written as

$$\sum_i \boldsymbol{f}_i\delta(\boldsymbol{r}-\boldsymbol{r}_i)\ \ , \tag{77}$$

where $\boldsymbol{f}_i$ is the force acting at $\boldsymbol{r}_i$. The factor $\Delta\delta(\boldsymbol{r}-\boldsymbol{r}_i)$ is of the order $-\frac{1}{a^2}\delta(\boldsymbol{r}-\boldsymbol{r}_i)$, where a is of the order of the dimension of the vortex ($v=a^3$); consequently, we may replace the viscosity terms in equation (76) by

$$-\frac{\eta v}{a^2}\sum_i \boldsymbol{u}_i\delta(\boldsymbol{r}-\boldsymbol{r}_i)\ . \tag{78}$$

A similar contribution brings the ζ-term. The equation of motion (76) describes a set of independent particles with mass M, subjected to an external force $\boldsymbol{f}_i$ and a friction force; the equation of motion of each such particle can be written as

$$M\dot{\boldsymbol{u}}_i=\boldsymbol{f}_i-\eta a\boldsymbol{u}_i\ \ , \tag{79}$$

which is Newton's law of motion. The damping coefficient caused by the friction force is very small, such that we can consider the ensemble of singularities as a (quasi) ideal classical gas of independent, identical, pointlike particles. Therefore, a (singular) fully developed turbulence may be viewed as the (quasi) thermodynamic equilibrium of such a gas (or a solution of singularities in the background fluid). We can define a temperature of turbulence, which is approximately the mean kinetic energy of the translational motion of a singularity. Also, we can estimate a chemical potential by evaluating $\overline{v^2}/2$. We note that the density ρ_v and the dimension a of the singularities remain undetermined; these parameters can be estimated from experiment.

Also, it is worth noting that we may consider the equations of the fluid mechanics for this new gas of singularities, viewed as a continuous medium, at a higher scale.

8 Vorticial liquids

8.1 Vortex

For an incompressible fluid with an isentropic flow we consider a velocity field given by

$$\boldsymbol{v} = \boldsymbol{\omega} \times gradf(r) \ , \tag{80}$$

where $\boldsymbol{\omega}$ is a vector which may depend only on the time and the function $f(r)$ is smooth everywhere, except, possibly, at the origin $\boldsymbol{r} \neq 0$, and vanishing rapidly at infinity. The velocity $\boldsymbol{v}$ rotates about $\boldsymbol{\omega}$; such a velocity field defines a vortex.

We can ckeck that $div\boldsymbol{v} = 0$ and

$$\boldsymbol{v} = -curl\left[\boldsymbol{\omega} f(r)\right] \ , \tag{81}$$

such that we can define a vector potential $\boldsymbol{A} = -\boldsymbol{\omega} f(r)$ ($\boldsymbol{v} = curl\boldsymbol{A}$). We note that $div\boldsymbol{A} \neq 0$ and the vorticity differs from $\Delta\boldsymbol{A}$, in general,

$$curl\boldsymbol{v} = \Delta\left[\boldsymbol{\omega} f(r)\right] - grad\, div\left[\boldsymbol{\omega} f(r)\right] \neq -\Delta\boldsymbol{A} \ . \tag{82}$$

If the velocity given by equation (81) satisfies the second equation (40), we should have $\boldsymbol{\omega} \sim e^{-\nu\lambda t}$ and $\Delta f + \lambda f = 0$, *i.e.* $f \sim e^{\pm i\sqrt{\lambda} r}/r$, where λ is, in general, complex. In two dimensions f is a Bessel function. The Navier-Stokes equation gives the pressure.

The most common example of a velocity field given by equation (80) is the filamentary vortex (the "cyclon"), with $\boldsymbol{\omega} = const$, ($\lambda = 0$), $f(r) = -\ln r$ and the two-dimensional position vector $\boldsymbol{r}$ perpendicular to $\boldsymbol{\omega}$. In this case $\boldsymbol{v} = \boldsymbol{r} \times \boldsymbol{\omega}/r^2$, $\boldsymbol{A} = \boldsymbol{\omega} \ln r$ and $curl\boldsymbol{v} = -2\pi\boldsymbol{\omega}\delta(\boldsymbol{r})$. The velocity is singular at $\boldsymbol{r} = 0$. For the filamentary vortex the Navier-Stokes equation can be satisfied for $p = -\rho\omega^2/2r^2$, *i.e.* for an external potential $\varphi = p/\rho = -\omega^2/2r^2$. This can be provided by the gravitational field for a fluid with a free surface.

For convenience, we give the expression of the various terms in the Navier-Stokes equation for the vortex given by equation (80),

$$\begin{gathered}
\frac{\partial \boldsymbol{v}}{\partial t} = (\dot{\boldsymbol{\omega}} \times \boldsymbol{r}) \frac{f'}{r} \ , \\
(\boldsymbol{v} grad)\boldsymbol{v} = \left[\boldsymbol{\omega}(\boldsymbol{\omega}\boldsymbol{r}) - \omega^2 \boldsymbol{r}\right] \frac{f'^2}{r^2} \ , \\
\Delta \boldsymbol{v} = (\boldsymbol{\omega} \times \boldsymbol{r}) \frac{(f'' + 2f'/r)'}{r} \ ,
\end{gathered} \tag{83}$$

where the primes denote the derivatives of the function f; the second equation (83) is derived by using the identity

$$(\boldsymbol{v} grad)\boldsymbol{v} = -\boldsymbol{v} \times curl\boldsymbol{v} + grad(v^2/2) \ . \tag{84}$$

For a filamentary vortex the above expressions become

$$\begin{gathered}
(\boldsymbol{v} grad)\boldsymbol{v} = -\omega^2 \boldsymbol{r} \frac{f'^2}{r^2} \ , \\
\Delta \boldsymbol{v} = (\boldsymbol{\omega} \times \boldsymbol{r}) \frac{(f'' + f'/r)'}{r} \ .
\end{gathered} \tag{85}$$

In general, the vortex given by equation (80) does not satisfy the Navier-Stokes equation; it develops internal (Euler's) forces which cannot be compensated; the vortex is unstable. Such vortices are examples of singular velocities.

8.2 Vorticial liquid

A set of vectors $\boldsymbol{\omega}_i$ placed at $\boldsymbol{r}_i$ form a vorticial liquid. The velocity field is

$$\boldsymbol{v} = \sum_i \boldsymbol{\omega}_i \times gradf_i(R_i) \ , \tag{86}$$

where $\boldsymbol{R}_i = \boldsymbol{r} - \boldsymbol{r}_i$. This velocity field is a superpositon of independent vortices. Each i-th vortex develops an internal force, while the inertial term generates an interaction which brings an additional force; this additional force depends on all the other j-th vortices, $j \neq i$. Therefore, the Navier-Stokes equation is not satisfied by the velocity field given by equation (86), in the sense that there is no physical external pressure to compensate the Euler force.

We assume randomly fluctuating vectors $\boldsymbol{\omega}_i$, as a distinctive feature of a fully developed vorticial turbulence. It is likely that the fluid develops fluctuations as a reaction to its uncompensated internal forces. Specifically, we assume $\overline{\boldsymbol{\omega}}_i = 0$ and $\overline{\omega_i^\alpha \omega_j^\beta} = \frac{1}{3}\overline{\omega_i^2}\delta_{ij}\delta^{\alpha\beta}$, where α, $\beta = 1, 2, 3$ are the cartesian labels of the components of the vectors $\boldsymbol{\omega}_i$, and the overbar indicates the average over time. Then, the average velocity is zero ($\overline{\boldsymbol{v}} = 0$) and we are left with the inertial term

$$\overline{(\boldsymbol{v}grad)\boldsymbol{v}} = \overline{v_j \partial_j v_i} \ . \tag{87}$$

The calculation of this term is straightforward; we get

$$\begin{aligned} \overline{(\boldsymbol{v}grad)\boldsymbol{v}} = \tfrac{1}{3}\textstyle\sum_i \omega_i^2 [\tfrac{1}{2} grad\,(gradf_i(R_i))^2 - \\ -gradf_i(R_i) \cdot \Delta f_i(R_i)] \ . \end{aligned} \tag{88}$$

The averaged Euler forces are

$$\begin{aligned} -\overline{\boldsymbol{v} \times curl\boldsymbol{v}} = -\tfrac{1}{3}\textstyle\sum_i \omega_i^2 [\tfrac{1}{2} grad\,(gradf_i(R_i))^2 + \\ +gradf_i(R_i) \cdot \Delta f_i(R_i)] \ , \\ \overline{grad(v^2/2)} = \tfrac{1}{3}\textstyle\sum_i \omega_i^2 grad\,(gradf_i(R_i))^2 \ . \end{aligned} \tag{89}$$

By using the spherical symmetry of the function $f_i(R_i)$, these expressions can be cast in the form

$$\begin{aligned} \overline{(\boldsymbol{v}grad)\boldsymbol{v}} = -\tfrac{2}{3}\textstyle\sum_i \omega_i^2 f_i'^2 \frac{\boldsymbol{R}_i}{R_i^2} \ , \\ -\overline{\boldsymbol{v} \times curl\boldsymbol{v}} = -\tfrac{2}{3}\textstyle\sum_i \omega_i^2 f_i' \left(f_i'' + \frac{1}{R_i} f_i'\right) \frac{\boldsymbol{R}_i}{R_i} \ , \\ \overline{grad(v^2/2)} = \tfrac{2}{3}\textstyle\sum_i \omega_i^2 f_i' f_i'' \frac{\boldsymbol{R}_i}{R_i} \ . \end{aligned} \tag{90}$$

We can see that even on average the Navier-Stokes equation is not satisfied, in the sense discussed above. Even on average the vorticial liquid develops internal forces which are not equilibrated; it is unstable. We shall give specific examples of such an instability below.

Now, let us assume that the vorticial liquid is sufficiently dense, *i.e.* if we can define a density ρ_v of points $\boldsymbol{r}_i$; further, we assume that the liquid is homogeneous and isotropic, *i.e.* this density is constant and the ω_i and the functions f_i can be replaced in the above equations by uniform functions $\omega_i = \omega$ and $f_i = f$. We assume that this is another distinctive feature of a fully developed turbulence. Then, we may transform the summation over i in equations (90) into an integral, like in equation (65). By choosing the origin at $\boldsymbol{r} = \boldsymbol{r}_i$ for a fixed $\boldsymbol{r}_i$, and using the notation $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_i$, we get

$$\overline{(\boldsymbol{v}grad)\boldsymbol{v}} = -\frac{2}{3}\rho_v\omega^2 \int_0^\infty dR \cdot Rf'^2(R) \int do(\boldsymbol{R}/R) \; ; \tag{91}$$

the result of integration in this equation is zero, due to the integration over the solid angle o, providing the radial integration is finite. In this case the Navier-Stokes equation is satisfied trivially.

Let us assume that the integral over R is singular at $R = 0$, as another distinctive feature of a fully developed turbulence (the function f is assumed to decrease sufficiently rapid at infinity to have a finite integral in this limit). The integration outside a small region around the i-th point is zero, while the integration over such a small region is indefinite. This singularity implies

$$f(R) \sim (a/R)^n \; , \; n > 0 \tag{92}$$

for $R \ll a$, where a is a small characteristic distance (compare with equation (66)). Therefore, we are left with a discrete set of points $\boldsymbol{r}_i$, where the function $f(R_i)$ and the velocity ($v \sim 1/R_i^{n+1}$) are singular. We have now a small region of dimension a, around each point $\boldsymbol{r}_i$, which includes a mass of fluid, say, M, where the inertial term given by equation (91) is not defined. Since the positions $\boldsymbol{r}_i$ may change in time, we are left with a classical gas of particle-like vortices (or a solution of vortices in the background fluid), as discussed above. Of course, we may have also a mixture of vorticial gases, each characterized by a dimension a and a mass M.

The equation of energy conservation (equation (67))

$$\frac{\partial}{\partial t}\left(\frac{1}{2}v^2\right) + \boldsymbol{v} \cdot (\boldsymbol{v}grad)\boldsymbol{v} = \nu\boldsymbol{v}\Delta\boldsymbol{v} \; , \tag{93}$$

averaged over the fluctuating vectors $\boldsymbol{\omega}_i$, is reduced to the viscosity term

$$\nu\overline{\boldsymbol{v}\Delta\boldsymbol{v}} = \frac{2\nu}{3}\sum_i \omega_i^2 f_i' \left(f_i'' + 2f_i'/R_i\right)' \; . \tag{94}$$

This term should be computed outside the regions with dimension a, where the motion is defined. The non-vanishing value of this term indicates an energy loss, which should be compensated from the outside. The vorticial gas is in quasi equilibrium.

The above considerations are valid for spherical-symmetric functions $f_i(R_i)$; if the vortices have a lower (internal) symmetry, the unit vector $\boldsymbol{R}/R$ in equation (91) is replaced by functions which do not have a spherical symmetry, and the inertial term is not vanishing, in general. The vortices are unstable, and, likely, they could tend to acquire a spherical symmetry, which ensures a (quasi)-equilibrium.

8.3 Filamentary liquid

Let us consider a set of rectilinear, parallel filaments, directed along the z-axis, placed at positions $\boldsymbol{r}_i$ in the (x, y)-plane, with vorticities $\boldsymbol{\omega}_i$. This is a two-dimensional vorticial liquid of "cyclons". The velocity field is given by

$$\boldsymbol{v} = -\sum_i \boldsymbol{\omega}_i \times grad \ln R_i = \sum_i \frac{\boldsymbol{R}_i \times \boldsymbol{\omega}_i}{R_i^2} \; , \tag{95}$$

where $\boldsymbol{R}_i = \boldsymbol{r} - \boldsymbol{r}_i$.

Equation (95) shows that the velocity is derived from a vector potential $\boldsymbol{A}$, through $\boldsymbol{v} = curl\boldsymbol{A}$. By taking the *curl* in this equation, we get

$$\Delta \boldsymbol{A} = -2\boldsymbol{\omega} \tag{96}$$

(providing $div\boldsymbol{A} = 0$ and $div\boldsymbol{\omega} = 0$), where we introduce the notation $curl\boldsymbol{v} = 2\boldsymbol{\omega}$; $\boldsymbol{\omega}$ is called vorticity. The vorticity distribution corresponding to equation (95)

$$\boldsymbol{\omega}(\boldsymbol{r}) = \frac{1}{2}curl\boldsymbol{v} = -\pi \sum_i \boldsymbol{\omega}_i \delta(\boldsymbol{R}_i) \ , \tag{97}$$

gives the vector potential

$$\boldsymbol{A} = \sum_i \boldsymbol{\omega}_i \ln R_i \ . \tag{98}$$

According to equation (84), the inertial term has the components

$$\boldsymbol{f} = -\boldsymbol{v} \times curl\boldsymbol{v} = -2\pi \sum_{i \neq j} \omega_i \omega_j grad_i \ln R_{ij} \cdot \delta(\boldsymbol{R}_i) \ , \tag{99}$$

where $\boldsymbol{R}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$, and $grade$, where

$$e = \frac{1}{2}v^2 = \sum_{i \neq j} \omega_i \omega_j \frac{\boldsymbol{R}_i \boldsymbol{R}_j}{2R_i^2 R_j^2} \tag{100}$$

is a density of kinetic energy (per unit mass). Apart from the force $\boldsymbol{f}$, which acts at the positions of the vortices, there exist internal forces given by $grade$, which make the liquid unstable. The motion and the statistics of parallel, rectilinear filaments have been extensively investigated,[24]-[31] the instability being associated with a negative temperature in an attempt of a statistical theory.[25, 27]

The total force given by equation (99)

$$\boldsymbol{F} = \int d\boldsymbol{r}\boldsymbol{f} = -2\pi \sum_{i \neq j} \omega_i \omega_j grad_i \ln R_{ij} \tag{101}$$

is zero. We can see that a force

$$\boldsymbol{f}_{ij} = -\boldsymbol{f}_{ji} = -2\pi \omega_i \omega_j grad_i \ln R_{ij} \tag{102}$$

acts between any pair (ij) of vortices. This force derives from a potential

$$U_{ij} = 2\pi \omega_i \omega_j \ln R_{ij} \ , \tag{103}$$

such that

$$\boldsymbol{F} = -\sum_{i \neq j} grad_i U_{ij} \tag{104}$$

The density of kinetic energy e can be written as

$$e = \frac{1}{2}v^2 = \frac{1}{2}\boldsymbol{v} curl\boldsymbol{A} = -\frac{1}{2}div(\boldsymbol{v} \times \boldsymbol{A}) + \boldsymbol{A}\boldsymbol{\omega} \ . \tag{105}$$

The first term in equation (105) is singular at $\boldsymbol{r} = \boldsymbol{r}_i$; we integrate this term over the whole space, transform it into surface integrals, both at infinity and over small circles around each filament, and neglect their contributions. The result of such integrations is a self-energy (or a self-force), which may be left aside. We call this procedure a "renormalization".[32] By doing so, we are left with a total kinetic energy

$$E = \int d\boldsymbol{r}e = \int d\boldsymbol{r}\boldsymbol{A}\boldsymbol{\omega} = -\pi \sum_{i\neq j} \omega_i\omega_j \ln R_{ij} = \\ = -\frac{1}{2}\sum_{i\neq j} U_{ij} = -U \ \ , \tag{106}$$

where U is the total potential energy. We can see that the total energy is conserved, *i.e.* $E+U = const$. Also, the total force $\boldsymbol{F} = 0$, such that the total momentum is conserved. The total angular momentum is $-2\int d\boldsymbol{r}\boldsymbol{A}$; it is proportional to $\sum_i \boldsymbol{\omega}_i$. The total torque

$$2\pi \sum_{i\neq j} \omega_i\omega_j \frac{\boldsymbol{r}_i \times \boldsymbol{r}_j}{R_{ij}^2} \tag{107}$$

is zero. By this "renormalization" procedure the points $\boldsymbol{r_i}$ are completely decoupled from the fluid, and they may have their own motion.

The average over fluctuating vorticities can be computed straightforwardly, by using $\overline{\boldsymbol{\omega}_i^2} = \frac{1}{2}\omega_i^2$; it is given by

$$\overline{(\boldsymbol{v}grad)\boldsymbol{v}} = -\sum_i \omega_i^2 \frac{\boldsymbol{R}_i}{2R_i^4} \ \ . \tag{108}$$

We can see that for a sufficiently dense, homogeneous and isotropic liquid we get a gas of (singular) vortices, as discussed above.

8.4 Coulombian liquid

For $f_i(R_i) = -1/R_i$ in equation (86) we get a coulombian vorticial liquid with the velocity field

$$\boldsymbol{v} = -\sum_i \boldsymbol{\omega}_i \times grad(1/R_i) = \sum_i \frac{\boldsymbol{\omega}_i \times \boldsymbol{R}_i}{R_i^3} \tag{109}$$

and the vector potential

$$\boldsymbol{A} = \sum_i \frac{\boldsymbol{\omega}_i}{R_i} \ \ . \tag{110}$$

The equation $\Delta\boldsymbol{A} = -2\boldsymbol{\omega}$ is satisfied for

$$\boldsymbol{\omega}(\boldsymbol{r}) = 2\pi \sum_i \boldsymbol{\omega}_i \delta(\boldsymbol{R}_i) \ \ , \tag{111}$$

but this vorticity differs from

$$\frac{1}{2}curl\boldsymbol{v} = 2\pi \sum_i \boldsymbol{\omega}_i\delta(\boldsymbol{R}_i) - \frac{1}{2}grad\, div \sum_i (\boldsymbol{\omega}_i/R_i) \ \ . \tag{112}$$

By applying the "renormalization" procedure we get the force

$$\boldsymbol{F} = \int d\boldsymbol{r}\boldsymbol{f} = -\int d\boldsymbol{r}\boldsymbol{v} \times curl\boldsymbol{v} = 4\pi \sum_{i\neq j} grad_i \frac{\boldsymbol{\omega}_i\boldsymbol{\omega}_j}{R_{ij}} \tag{113}$$

and the energy

$$E = \int d\boldsymbol{r}e = \frac{1}{2}\int d\boldsymbol{r}v^2 = 2\pi \sum_{i\neq j} \frac{\boldsymbol{\omega}_i\boldsymbol{\omega}_j}{R_{ij}} \ \ , \tag{114}$$

where, in both cases $\boldsymbol{v} = curl\boldsymbol{A}$ is used. The potential from equation (103) is now $U_{ij} = -4\pi\frac{\boldsymbol{\omega}_i\boldsymbol{\omega}_j}{R_{ij}}$. The total energy is conserved, the total force is zero, the total angular momentum is proportional to $\sum_i \boldsymbol{\omega}_i$ and the total torque (zero) is

$$4\pi\sum_{i\neq j}\boldsymbol{\omega}_i\boldsymbol{\omega}_j\frac{\boldsymbol{r}_i\times\boldsymbol{r}_j}{R_{ij}^3} \ . \tag{115}$$

The average of the inertial term over fluctuating vortices is obtained from equation (90)

$$\overline{(\boldsymbol{v}grad)\boldsymbol{v}} = -\frac{2}{3}\sum_i\omega_i^2\frac{\boldsymbol{R}_i}{R_i^6} \ , \tag{116}$$

such that we may get a gas of singular vortices.

8.5 Dipolar liquid

A dipolar liquid is defined by the vorticity

$$\boldsymbol{\omega} = -2\pi\sum_i\boldsymbol{m}_i\times grad\delta(\boldsymbol{r}-\boldsymbol{r}_i) \ , \tag{117}$$

where the vectors $\boldsymbol{m}_i$ may depend on the time, at most. We get the vector potential

$$\boldsymbol{A} = \sum_i\frac{\boldsymbol{m}_i\times\boldsymbol{R}_i}{R_i^3} = -\sum_i\boldsymbol{m}_i\times grad(1/R_i) \tag{118}$$

and the velocity field

$$\begin{aligned}\boldsymbol{v}(\boldsymbol{r}) &= \textstyle\sum_i\left[-\boldsymbol{m}_i/R_i^3+3\boldsymbol{R}_i(\boldsymbol{m}_i\boldsymbol{R}_i)/R_i^5\right] = \\ &= \textstyle\sum_i grad\left[\boldsymbol{m}_i grad(1/R_i)\right] \ .\end{aligned} \tag{119}$$

We recognize in these equations magnetic (dipole) moments $\boldsymbol{m}_i$, a dipolar vector potential $\boldsymbol{A}$ and a magnetic field $\boldsymbol{v}$.

The inertial term has the components

$$\begin{aligned}\boldsymbol{f} &= -\boldsymbol{v}\times curl\boldsymbol{v} = 2\boldsymbol{\omega}\times\boldsymbol{v} = \\ &= 4\pi\textstyle\sum_{i\neq j} grad\left[\boldsymbol{m}_j grad(1/R_j)\right]\times\left[\boldsymbol{m}_i\times grad\delta(\boldsymbol{R}_i)\right]\end{aligned} \tag{120}$$

and $grade$, where

$$\begin{aligned}e = \tfrac{1}{2}v^2 = \tfrac{1}{2}\textstyle\sum_{i\neq j}\Big[&\frac{\boldsymbol{m}_i\boldsymbol{m}_j}{R_i^3R_j^3}-\frac{3(\boldsymbol{m}_i\boldsymbol{R}_j)(\boldsymbol{m}_j\boldsymbol{R}_j)}{R_i^3R_j^5}- \\ &-\frac{3(\boldsymbol{m}_j\boldsymbol{R}_i)(\boldsymbol{m}_i\boldsymbol{R}_i)}{R_i^5R_j^3}+ \\ &+\frac{9(\boldsymbol{R}_i\boldsymbol{R}_j)(\boldsymbol{m}_i\boldsymbol{R}_i)(\boldsymbol{m}_j\boldsymbol{R}_j)}{R_i^5R_j^5}\Big] \ .\end{aligned} \tag{121}$$

We can see that the dipolar liquid is unstable.

The total force and the total kinetic energy are

$$\begin{aligned}\boldsymbol{F} &= \textstyle\int d\boldsymbol{r}\boldsymbol{f} = 2\int d\boldsymbol{r}\boldsymbol{\omega}\times\boldsymbol{v} = -\sum_{i\neq j} grad_i U_{ij} \ , \\ E &= \textstyle\int d\boldsymbol{r}e = \int d\boldsymbol{r}\boldsymbol{\omega}\boldsymbol{A} = -\frac{1}{2}\sum_{i\neq j}U_{ij} \ ,\end{aligned} \tag{122}$$

where

$$U_{ij} = -4\pi \boldsymbol{m}_i grad_i \left[\boldsymbol{m}_j grad_i (1/R_{ij}) \right] \ . \tag{123}$$

By this "renormalization" procedure, the liquid is reduced to a set of interacting particle-like vortices. We can see that the energy is conserved, the total force is zero, the total angular momentum and the total torque are zero.

The time average over vorticities in equation (121) leads to

$$\overline{e} = \frac{1}{2} \sum_i \left[\frac{m_i^2}{R_i^6} + \frac{3\overline{(\boldsymbol{m}_i \boldsymbol{R}_i)^2}}{R_i^8} \right] = \sum_i \frac{m_i^2}{R_i^6} \ , \tag{124}$$

which gives a force

$$grad\overline{e} = -6 \sum_i \frac{m_i^2 \boldsymbol{R}_i}{R_i^8} \ . \tag{125}$$

For a dense, homogeneous and isotropic liquid the force given by equation (125) is zero (for $\boldsymbol{R}_i \neq 0$); also, the force $\boldsymbol{f}$ is zero for $\boldsymbol{R}_i \neq 0$, and the Navier-Stokes equation is satisfied trivially (on average). We note that $\Delta \boldsymbol{v}$ is zero for $\boldsymbol{R}_i \neq 0$ (equation (119)), such that the viscosity contribution is zero. We are left with a set of positions $\boldsymbol{r}_i$, each surrounded by a small region, where the motion is not defined. According to the above discussion, such a structure may be viewed as a (quasi) ideal classical gas of vortices (or a solution of vortices in the background fluid).

9 Concluding remarks

In fairly general conditions we have given in this paper an explicit (smooth) solution for the potential flow. We have shown that, rigorously speaking, the equations of the fluid mechanics have not rotational solutions. However, usually we may neglect the variations of the density and the temperature, such that, in these conditions, the Navier-Stokes equation may exhibit (approximate) vorticial solutions, governed by the viscosity. We give arguments that the vortices are unstable. On the other hand, for large variations of the velocity over small distances, the fluid velocity exhibits turbulent, highly fluctuating instabilities, controlled by viscosity. Such a fully developed turbulence occurs as a consequence of the insufficiency of the viscosity to dissipate heat. We represent the fully developed turbulence as a superposition of highly fluctuating velocities, associated to a discrete distrbution of turbulence centres, and are interested in the temporal average of this velocity field. A regular mean flow may be added. It is shown that the Navier-Stokes equation is not satisfied on average. However, for a homogeneous and isotropic distribution of (non-singular) turbulence centres (as another distinctive feature of a fully developed turbulence), the temporal average of both the fluctuating velocity and the inertial term is zero, such that the Navier-Stokes equation is satisfied trivially. If the velocity is singular at the turbulence centres we are left with a quasi ideal classical gas of singularities (or a solution of singularities in the background fluid), in thermal quasi equilibrium, as an example of emergent dynamics. The Navier-Stokes equation for this fluid of singularities is reduced to Newton's law of motion (with a small friction). At a higher scale, equations of fluid mechanics can be considerd for this gas, as a continuous medium. We have illustrated all the above considerations with three examples of (singular) vorticial liquids.

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