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Transit and Delay Times in Waves Propagation

M. Apostol

Department of Theoretical Physics, Institute of Atomic Physics,

Magurele-Bucharest MG-6, POBox MG-35, Romania

email: apoma@theory.nipne.ro

Dedicated to our friend dr. N. Marinescu, in reverence for his belief in theoretical physics.

Let

$$\int d\mathbf{k} \cdot e^{i(\mathbf{k}\mathbf{r}-\omega t)} \quad (1)$$

be a wave packet of width $|\Delta\mathbf{r}|$, centered on some wavevector \mathbf{k} , over a $|\Delta\mathbf{k}| \sim 1/|\Delta\mathbf{r}|$ range; one may view it as a rather narrow packet, with a small $|\Delta\mathbf{r}|$ and a large $|\Delta\mathbf{k}|$; the amplitudes of the waves are not relevant for what follows; it is localized on $\mathbf{r} = 0$, as a result of the wave interference; if the packet is centered on $\mathbf{k}_0 = 0$, it may look approximately as a δ -function.

All this is valid at $t = 0$, or for a constant frequency ω . If the frequency depends on the wavevectors, *i.e.* for a function $\omega = \omega(\mathbf{k})$, then the packet decays, in general, because the interference affects the origin too, and the waves propagate with distinct velocities; we say that the waves are then dispersed. There is one exception, for $\omega = \mathbf{c}\mathbf{k}$, where the packet moves as a whole with velocity \mathbf{c} . We say that all the waves propagate in-phase then along \mathbf{c} , with the phase velocity $c = \omega/k$, and the "lateral" waves $\mathbf{k} \perp \mathbf{c}$ do not contribute.

It follows that the main contribution to (1) comes from the linear part of the phase $\Phi = \mathbf{k}\mathbf{r} - \omega t$, *i.e.* from

$$\partial\Phi/\partial\mathbf{k} = \mathbf{r} - (\partial\omega/\partial\mathbf{k})t = 0 ; \quad (2)$$

this is the principle of the stationary phase, and the basic equation of wave packets propagation. It tells that every \mathbf{k} -wave in the packet propagates with a velocity

$$\mathbf{v}_g = \partial\omega/\partial\mathbf{k} \quad (3)$$

called group velocity, along the gradient of ω , *i.e.* perpendicular to the curves $\omega(\mathbf{k}) = \text{const}$. The natural trajectories of the wave propagation in a packet are given by $\text{grad}\omega = \text{const}$, and along these trajectories the \mathbf{k} moves freely; indeed, from $v_g = r/t$ derived from (2) one gets $\partial k/\partial t + v_g \partial k/\partial r = 0$ along the trajectory. In addition, if a gaussian superposition is allowed in (1) for \mathbf{k} -waves close to that \mathbf{k} given by (2) one obtains, by power expansion,

$$\int dk \cdot e^{-\lambda k^2 + ik(x - v_{gx}t)} \quad (4)$$

for the wave packet along x -direction, say, which yields $\exp[-(x - v_{gx}t)^2/4\lambda]$, *i.e.* a peak localized for a while over $\sim \lambda$, moving with v_{gx} ; hence, the term group velocity for \mathbf{v}_g ; all these motions are

free, since the wave equations are linear, but higher-order terms in the expansion of ω destroy the packet in time; they are diffusive contributions.

The actual trajectories may differ from the natural trajectories, so that (2) must be read on these actual trajectories, and v_g is the component of the group velocity along such actual trajectories. The group velocities are not only different for different waves, giving waves dispersion, but they also differ in general from the phase velocity $v_\Phi = \omega/k$ for the same wave, where k is the component of \mathbf{k} along the trajectory; the latter point leads to a delay, or advance, in propagating waves.

Indeed, let s be the length along the trajectory; then, the wave transits this length in a transit time

$$\tau_t = s/v_g \quad , \quad (5)$$

or, in general,

$$\tau_t = \int ds/v_g \quad . \quad (6)$$

At the same time, along the same trajectory, the wave phase does itself propagate in a time

$$\tau_\Phi = \int ds/v_\Phi = \int ds/(\omega/k) \quad , \quad (7)$$

the difference

$$\tau_d = \tau_t - \tau_\Phi = \int ds/v_g - \int ds/(\omega/k) \quad (8)$$

being the delay time; it may take positive or negative values. One can see easily that $\partial\Phi/\partial\mathbf{k} = (\partial\Phi/\partial\omega)\mathbf{v}_g$, so that $\partial\Phi/\partial\mathbf{k}$ in (2) vanishes together with $\partial\Phi/\partial\omega$, since $\partial\omega/\partial\mathbf{k} = \partial\omega/\partial k$ along the trajectory. If we write up $\Phi = \mathbf{k}\mathbf{r} - \omega\tau_\Phi$, *i.e.* $\Phi = 0$, then $\partial\Phi/\partial\omega = \int ds/v_g - \tau_\Phi = \tau_d$; therefore the delay time is also given by

$$\tau_d = \partial\Phi/\partial\omega \text{ for } \Phi = 0 \quad , \quad (9)$$

which is analogous to the transit time being given by $\partial\Phi/\partial\omega = 0$. Therefore, the transit time is given by the extrema of the phase along the trajectory, while the delay time is given by the slope of the vanishing phase. It is worth noting that $\omega(\mathbf{k})$ along the trajectory is inversable, *i.e.* there exists $k(\omega)$.

Let $\omega = \sqrt{\omega_c^2 + c^2k^2}$ be the usual frequency in a waveguide, for instance; the group velocity is $v_g = c^2k/\omega$, while the phase velocity is $v_\Phi = \omega/k$; the transit time is $\tau_t = s\omega/c^2k = s\omega/c\sqrt{\omega^2 - \omega_c^2}$, and the phase time is $\tau_\Phi = sk/\omega = s\sqrt{\omega^2 - \omega_c^2}/c\omega$; the delay time is $\tau_d = s\omega_c^2/c\omega\sqrt{\omega^2 - \omega_c^2}$.

In general, the frequency increases with increasing the wavevector, so that the high-frequency waves in a packet are propagated faster in comparison with the long-wavelengths waves. One may take for a "tunneling" time the longest time of transit, which corresponds to the slowest wave; usually, it is the lower bound of the wave packet, *i.e.*

$$\tau_{tunn} = \int ds/v_g \text{ for } \mathbf{k} = \mathbf{k}_0 - |\Delta\mathbf{k}|/2 \quad . \quad (10)$$

However, the so-called problem of tunneling is usually put in other terms, requiring the time needed for a particle, or a wave, to pass through a potential barrier. There is a close analogy between quantal tunneling of a particle through a potential barrier and the waves propagating in the evanescent region of a waveguide. Indeed, in the latter case the waveguide may have a narrow region which pushes the frequency below the cutoff; a similar push down is done by changing the dielectric constant, and the wave enters thereby an evanescent region; in general, for a still propagating wave, the wavevectors change in such situations, as the frequency rests constant; the

passing through an evanescent region of a wave is similar with a potential barrier for a quantal particle.[1] In any case, however, the "tunneling time problem" is not a problem, because the continuity of the waves ensures both their non-locality as well as their sudden tunneling through any potential barrier, *i.e.* the waves pass through a potential barrier in no time. Indeed, it is easy to see a train of waves coming, say, from the left and reaching the barrier wall at some moment of time; then, as the wave increases abruptly at the wall, there will appear a similar increase inside the barrier, and at the other barrier wall, so that both the reflected and the transmitted waves appear simultaneously with the incoming wave reaching the barrier, and all this due to the continuity of the waves. An explicit investigation of this "phenomenon" has been suggested by Condon,[2] and done by MacColl[3] with gaussian wave packets; gaussian wave packets have much been used in the early days of quantal mechanics, in order to visualise the distinction between the quantal motion and the classical motion.[4] In spite of all this, many authors[5] still talk nowadays about the "delay", "transit", "dwelling", "flight", "tunneling" or all kind of whatever times in passing particles or waves through a potential barrier; some of them associate the phase shifts at the barrier walls with such times, which, of course, contradicts (9), to say the least.

References

- [1] See the work of N. Marinescu on this problem, following a Feynman suggestion.
- [2] E. U. Condon, *Revs. Mod. Phys.* **3** 43 (1931).
- [3] L. A. MacColl, *Phys. Rev.* **40** 621 (1932).
- [4] See, for instance, the references in Ref.1.
- [5] See, for instance, Th. Martin and R. Landauer, *Phys. Rev.* **A45** 2611 (1992).