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Scattering of longitudinal waves (sound) by defects in fluids. Rough surface

Research Article

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Abstract: The classical theory of scattering of longitudinal waves (sound) by small inhomogeneities (scatterers) in an ideal fluid is generalized to a distribution of scatterers and such as to include the effect of the inhomogeneities on the elastic properties of the fluid. The results are obtained by a new method of solving the wave equation with spatial restrictions (caused by the presence of the scatterers), which can also be applied to other types of inhomogeneities (like surface roughness, for instance). A coherent forward scattering is identified for a uniform distribution of scatterers (practically equivalent with a mean-field approach), which is due to the fact that our treatment does not include multiple scattering. The reflected wave is obtained for a half-space (semi-infinite fluid) of uniformly distributed scatterers, as well as the field diffracted by a perfect lattice of scatterers. The same method is applied to a (inhomogeneous) rough surface of a semi-infinite ideal fluid. A perturbation-theoretical scheme is devised, with the roughness function as a perturbation parameter, for computing the waves scattered by the surface roughness. The waves scattered by the rough surface are both waves localized (and propagating only) on the surface (two-dimensional waves) and waves reflected back in the fluid. They exhibit directional effects, slowness, attenuation or resonance phenomena, depending on the spatial characteristics of the roughness function. The reflection coefficients and the energy carried on by these waves are calculated both for fixed and free surfaces. In some cases, the surface roughness may generate waves confined to the surface (damped, rough-surface waves).

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1. Introduction

The scattering of longitudinal waves (sound) by small inhomogeneities (scatterers) in an ideal fluid is a well-known subject (see, for instance, Ref. [1]). We derive here these classical results by a new method, which allows a generalization. The generalization consists in including the effect the inhomogeneities may have upon the elastic properties of the fluid localized on them (parameter η in this paper) and to get the scattered field arising from any distribution of scattering centers. The method can be applied also to other types of scatterers (like, for instance, a surface roughness). There is a great deal of interest today in scattering of sound, especially in random media (by using Foldy's theory and its recent developments) [2–4], and,

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in general, in complex media, where serious mathematical difficulties are encountered [5]. Though it does not include the multiple scattering, the model put forward here leads to definite results, such as the field reflected by a half-space of uniformly distributed scatterers, or the field diffracted by a perfect lattice of scatterers.

The effect of a rough, solid surface on the fluid dynamics, in particular the waves (sound) scattered by the surface roughness, enjoy also a great deal of interest [6–25] .The interaction between a solid wall and the fluid flow, as well as the action of a solid interface on the fluid dynamics have been emphasized recently [26, 27]. A rough surface shares, to some extent, the properties of a porous medium [28]. The surface roughness was modelled as an inhomogeneous fluid layer on a rigid plate and the scattering of acoustic waves was considered within a radiative regime by means of coupled integral equations [29]. A great deal of insight into the scattering mechanism by rough surfaces has been achieved [30] by means of Biot's theory and its recent developments [31–33]. The general characteristics of the waves scattered by a rough surface are directional effects, slowness and attenuation, as well as possible resonances for surface gratings (corrugations). The main difficulty in getting more definite results in this problem resides in modelling conveniently the inhomogeneities and the surface roughness, such as to arrive at mathematically operational approaches [34, 35] .

The method devised for the scattering of sound by small inhomogeneiteis is applied here to the scattering of sound by an inhomogeneous rough surface of a semi-infinite (half-space) fluid. The rough surface is modelled as a surface whose elastic properties differ from the ones of the semi-infinite (half-space) fluid bulk, in contrast with a homogeneous rough surface which has the same elastic properties as the bulk. In general, a surface, especially a rough one, acts like a source for scattered waves. We devise here a theoretical-perturbation scheme for treating the wave equation for longitudinal (sound) waves proapagating in a semi-infinite solid with a rough surface. The perturbation parameter is the roughness function, *i.e.* the deviation of the surface from a plane. It is shown that the scattered waves appear in the first-order approximation for a fixed surface, while for a free surface they appear only in the second-order approximation. Two kinds of scattered waves are identified: waves localized (and propagating only) on the surface (two-dimensional waves) and waves reflected back in the fluid. In some cases, the latter waves may get confined to the surface (damped, rough-surface waves). For a homogeneous roughness only the waves localized on the surface survive. The reflection coefficients (and the energy carried on by these waves) are calculated and various characteristics like slowness, attenuation or possible resonance phenomena are discussed.

2. Background

We consider a homogeneous, isotropic, ideal fluid of infinite extension. A small displacement field $\mathbf{u}(\mathbf{r}, t)$, where $\mathbf{r}$ denotes the position and t denotes the time, gives rise to a density imbalance $\delta n = -n div\mathbf{u}$ in the fluid density n, a local change of volume $\delta V = V div\mathbf{u}$ and a local change of pressure δp, depending on the equation of state of the fluid; for an adiabatic change, $\delta p = (\partial p/\partial n)_S \delta n = -n(\partial p/\partial n)_S div\mathbf{u}$, where S denotes the entropy. As it is well known [1], such a fluid supports longitudinal waves (sound), described by the equation of motion

$$\frac{1}{c^2}\ddot{\mathbf{u}} - grad \cdot div\mathbf{u} = 0 \ , \tag{1}$$

where c is the sound velocity. Indeed, by taking the div in equation (1), we get the wave equation for free waves propagating with velocity c. The displacement field is subjected to the condition $curl\mathbf{u} = 0$. Therefore, it is convenient to introduce the potential function $\Phi = div\mathbf{u}$ (proportional to the pressure) and write equation (1) as

$$\frac{1}{c^2}\ddot{\Phi} - \Delta\Phi = 0 \ . \tag{2}$$

The sound propagation in fluids is also described by means of another potential function Ψ, defined by $\delta p = -\rho\partial\Psi/\partial t$ and $\mathbf{v} = \dot{\mathbf{u}} = grad\Psi$, where ρ is the (mass) density and $\mathbf{v}$ is the fluid velocity [1]. Then, Euler's equation $\rho\partial\mathbf{v}/\partial t + grad\delta p = 0$ (for small velocities $\mathbf{v}$) is satisfied identically, and the continuity equation $\partial\delta\rho/\partial t + \rho div\mathbf{v} = 0$ becomes the wave equation $\partial^2\Psi/\partial t^2 - c^2\Delta\Psi = 0$, through $\delta p = (\partial p/\partial\rho)_S\delta\rho$, with the sound velocity given by $c^2 = (\partial p/\partial\rho)_S$. The connection between the two potential function Ψ and Φ is given by

$$\begin{aligned}\delta p = & -\rho\partial\Psi/\partial t = (\partial p/\partial\rho)_S\delta\rho = -\rho(\partial p/\partial\rho)_S div\mathbf{u} \\ = & -\rho c^2\Phi \ ,\end{aligned} \tag{3}$$

or

$$\frac{\partial\Psi}{\partial t} = c^2\Phi \ ; \tag{4}$$

for a monochromatic wave $\Psi = (ic^2/\omega)\Phi$. According to equation (1), the energy density (per unit mass) carried on by the longitudinal waves in a fluid is given by

$$e = \frac{1}{2}\dot{\mathbf{u}}^2 + \frac{1}{2}c^2\Phi^2 = \frac{1}{2}\frac{c^4}{\omega^2}(grad\Phi)^2 + \frac{1}{2}c^2\Phi^2 \ , \tag{5}$$

where equation (4) is used for a monochromatic wave. For a plane wave, equation (5) gives $e = c^2\Phi^2$.

3. Small inhomogenities

We assume a small inhomogeneity (foreign body, impurity) in an ideal fluid, placed at a fixed position $\mathbf{r}_i$, of a mean radius h_i (a scatterer). For h_i much smaller than the relevant wavelengths of the disturbances propagating in the fluid we can write the potential function Φ as

$$\Phi(\mathbf{r},\ t) = \varphi(\mathbf{r},\ t)\theta(|\mathbf{r}-\mathbf{r}_i| - h_i) \simeq \\ \simeq \varphi(\mathbf{r},\ t)\theta(|\mathbf{r}-\mathbf{r}_i|) - h_i\varphi(\mathbf{r},\ t)\delta(|\mathbf{r}-\mathbf{r}_i|)\ , \tag{6}$$

where $\theta(x) = 1$ for $x > 0$, $\theta(x) = 0$ for $x < 0$ is the step function and δ is the Dirac function, or

$$\Phi = \varphi + \delta\Phi\ ,\ \ \delta\Phi = -h_i\varphi(\mathbf{r}_i,\ t)\delta(|\mathbf{r}-\mathbf{r}_i|)\ . \tag{7}$$

The potential Φ satisfies the free wave equation (with specific boundary conditions at the surface of the inhomogeneity). According to our decomposition given by equation (6) we can see that φ satisfies the free wave equation in the whole space, while $\delta\Phi$ generates a source-term (a force), localized on the inhomogenity, which may give scattered waves. We introduce the potential Φ_1 for describing these scattered waves. It should obey the wave equation

$$\frac{1}{c^2}\ddot{\Phi}_1 - \Delta\Phi_1 = f\ , \tag{8}$$

where the force f is given by

$$f = \frac{1}{c^2}\delta\ddot{\Phi} - \Delta\delta\Phi\ . \tag{9}$$

Equation (8) is merely a re-writing of the wave equation for $\delta\Phi$. The force f is the difference between the inertial force $\delta\ddot{\Phi}/c^2$ and the elastic force $\Delta\delta\Phi$; it represents the distinct way the inhomogeneity responds to (follows) the wave motion in comparison with the fluid bulk. For waves localized on the inhomogeneity, equation (8) has the solution $\Phi_1 = \delta\Phi$. Another solutions are given by the waves scattered in the fluid by the inhomogeneity, *i.e.* waves generated in equation (8) by the source term f (a particular solution of equation (8)). We generalize this model of inhomogeneity by introducing a different "sound" velocity $\overline{c}$ in equation (9). The force is then written as

$$f = \frac{1}{\overline{c}^2}\delta\ddot{\Phi} - \Delta\delta\Phi\ . \tag{10}$$

Such a generalization amounts to assuming that the elastic properties of the fluid localized on the inhomogeneity are different than the elastic properties of the fluid bulk. For instance, the spatial variations of the scatterer shape may affect the elastic properties of the fluid in its neighbourhood. It is convenient to introduce the parameter $\eta = 1 - c^2/\overline{c}^2$ for describing such an "inhomogeneous" scatterer. A homogeneous scatterer (*i.e.*, the absence of the scatterer) would correspond to $\eta = 0$. A perfectly rigid scatterer would have $\overline{c} \to \infty$ and $\eta \to 1$.

Obviously, according to equations (6) and (7), the scheme of calculation put forwad here is a perturbation-theoretical scheme, with the mean radius h_i as the perturbation parameter. In view of the small magnitude of the mean radius h_i, we limit ourselves here to the first order of the perturbation theory.

We consider an incident plane wave $\varphi = \varphi_0 e^{-i\omega t + i\mathbf{k}\mathbf{r}}$, where $\omega = ck$. Then, the source-term becomes

$$\delta\Phi = -h_i\varphi_0\delta(|\mathbf{r}-\mathbf{r}_i|)e^{-i\omega t + i\mathbf{k}\mathbf{r}_i} \tag{11}$$

and the force given by equation (10) reads

$$f = h_i\varphi_0\left[\frac{\omega^2}{\overline{c}^2}\delta(|\mathbf{r}-\mathbf{r}_i|) - \Delta\delta(|\mathbf{r}-\mathbf{r}_i|)\right]e^{-i\omega t + i\mathbf{k}\mathbf{r}_i}\ , \tag{12}$$

or

$$f = -\eta\frac{h_i\varphi_0\omega^2}{c^2}\delta(|\mathbf{r}-\mathbf{r}_i|)e^{-i\omega t + i\mathbf{k}\mathbf{r}_i} + \\ + h_i\varphi_0\left[\frac{\omega^2}{c^2}\delta(|\mathbf{r}-\mathbf{r}_i|) - \Delta\delta(|\mathbf{r}-\mathbf{r}_i|)\right]e^{-i\omega t + i\mathbf{k}\mathbf{r}_i}\ . \tag{13}$$

As it is well known, the solution of equation (8) is given by

$$\Phi_1(\mathbf{r},t) = \frac{1}{4\pi}\int d\mathbf{r}'\frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}f(\mathbf{r}',t)\ , \tag{14}$$

with f given by equation (13). The second term in the *rhs* of equation (13) can be integrated by parts in equation (14), and we get the laplacian applied to the Green function (spherical wave) of the Helmholtz equation. This way, we get the localized waves

$$\Phi_{1l} = -h_i\varphi_0\int d\mathbf{r}'\delta(\mathbf{r}-\mathbf{r}')\delta(|\mathbf{r}'-\mathbf{r}_i|) = -h_i\varphi_0\delta(|\mathbf{r}-\mathbf{r}_i|)\ , \tag{15}$$

which are precisely the localized waves $\Phi_{1l} = \delta\Phi$ given by equation (11), as expected (we leave aside the exponential factor $e^{-i\omega t + i\mathbf{k}\mathbf{r}_i}$). The first term in the *rhs* of equation (13) gives the scattered waves

$$\Phi_{1s} = -\eta\frac{h_i\varphi_0\omega^2}{4\pi c^2}e^{-i\omega t + i\mathbf{k}\mathbf{r}_i}\int d\mathbf{r}'\frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\delta(|\mathbf{r}'-\mathbf{r}_i|)\ . \tag{16}$$

We assume that the δ-function in equation (16) extends over the small distance h_i, *i.e.* $\delta(|\mathbf{r}-\mathbf{r}_i|) \simeq 1/h_i$ for $|\mathbf{r}-\mathbf{r}_i| < h_i$. Then, the integral in equation (16) is evaluated easily. We get

$$\Phi_{1s} \simeq -\eta \frac{v_i \varphi_0 \omega^2}{4\pi c^2} e^{-i\omega t + i\mathbf{k}\mathbf{r}_i} \frac{e^{ik|\mathbf{r}-\mathbf{r}_i|}}{|\mathbf{r}-\mathbf{r}_i|} \quad , \tag{17}$$

where v_i is the (mean) volume of the scatterer.

We can see that the validity of the perturbation-theoretical scheme requires $h_i \ll \lambda$, where $\lambda = c/\omega$ is the wavelength of the incident wave. According to equation (17), the scatterer generates spherical waves, the dfferential cross-section being given by

$$d\sigma = \eta^2 \frac{v_i^2 \omega^4}{(4\pi)^2 c^4} d\Omega \quad , \tag{18}$$

where Ω denotes the solid angle. As it is well known, it is proportional to the square volume of the scatterer and the fourth power of the frequency. The energy flux (per unit mass) $d\Gamma = cer^2 d\Omega$ (with the origin of the reference frame at $\mathbf{r}_i$) is given by

$$d\Gamma = \eta^2 \frac{v_i^2 \varphi_0^2 \omega^4}{(4\pi)^2 c} d\Omega \quad , \tag{19}$$

which is to be compared with the energy flux $c^3 \varphi_0^2$ per unit cross-sectional area in the incident wave (cross-section). The scattered field given by equation (17) does not exhibit directional effects, because the fluid velocity $\mathbf{v} = (ic^2/\omega) grad \delta\Phi$ in the source-term given by equation (11) is isotropic. We can say that the scattered field given by equation (17) arises from a "monopole" scatterer.

There is another solution of the free waves equation in the presence of a small inhomogenity placed at $\mathbf{r}_i$: it is given by $\Phi' \sim \delta(\mathbf{r}-\mathbf{r}_i)$. Indeed, it satisfies trivially the free waves equation for any $\mathbf{r} \neq \mathbf{r}_i$. This potential function should carry in front of the δ-function a factor proportional to the volume v_i. Since $2\pi h_i^2 \delta(\mathbf{r}) = \delta(r)$, it is easy to see that this factor is $3v_i/2$. Under the action of an incident wave Φ_0 this solution changes by an amount which can be derived from $\partial\delta\Phi' \partial t = \mathbf{v} grad \Phi' = (3/2) v_i \mathbf{v} grad \delta(\mathbf{r}-\mathbf{r}_i)$, where $\mathbf{v} = grad \Psi_0$ is the velocity of the fluid particles. For a monochromatic wave, making use of $\Psi_0 = (ic^2/\omega)\Phi_0$, we get $\delta\Phi' = (3/2) v_i (c^2/\omega^2) grad \Phi_0 grad \delta(\mathbf{r}-\mathbf{r}_i)$, or, for a plane wave,

$$\begin{aligned} \delta\Phi' &= \tfrac{3 i v_i c^2 \varphi_0}{2\omega^2} e^{-i\omega t + i\mathbf{k}\mathbf{r}_i} \mathbf{k}\, grad \delta(\mathbf{r}-\mathbf{r}_i) = \\ &= \tfrac{3 i v_i \varphi_0}{2k^2} e^{-i\omega t + i\mathbf{k}\mathbf{r}_i} \mathbf{k}\, grad \delta(\mathbf{r}-\mathbf{r}_i) \end{aligned} \tag{20}$$

(the multiplication should be done for complex conjugate quantities). This change in the potential function gives rise to a force, similar with the force given above by equation (12). Introduced in equation (14), it generates a localized wave equal to $\delta\Phi'$, as expected, and a scattered wave given by

$$\Phi'_{1s} = -i\eta \frac{3 v_i \varphi_0 \omega^2}{8\pi c^2} e^{-i\omega t + i\mathbf{k}\mathbf{r}_i} \frac{\mathbf{k}}{k^2} \int d\mathbf{r}' \frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} grad \delta(|\mathbf{r}'-\mathbf{r}_i|), \tag{21}$$

or

$$\Phi'_{1s} \simeq \eta \frac{3 v_i \varphi_0 \omega}{8\pi c} e^{-i\omega t + i\mathbf{k}\mathbf{r}_i} \frac{\mathbf{k}(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|} \frac{e^{ik|\mathbf{r}-\mathbf{r}_i|}}{|\mathbf{r}-\mathbf{r}_i|} \tag{22}$$

(leading approximation). We can see that these scattered waves exhibit directional effects, as arising from a "dipole" scatterer.

The total scattered field is obtained by adding equations (17) and (22). We get

$$\Phi_s = -\eta \frac{v_i \varphi_0 k}{4\pi} \left(k - \frac{3}{2} \mathbf{k}\mathbf{n}_i \right) e^{-i\omega t + i\mathbf{k}\mathbf{r}_i} \frac{e^{ik|\mathbf{r}-\mathbf{r}_i|}}{|\mathbf{r}-\mathbf{r}_i|} \quad , \tag{23}$$

where $\mathbf{n}_i = (\mathbf{r}-\mathbf{r}_i)/|\mathbf{r}-\mathbf{r}_i|$ is the unit vector from the scatterer to the observation point. The cross section is given by

$$d\sigma = \eta^2 \frac{v_i^2 \omega^4}{(4\pi)^2 c^4} \left(1 - \frac{3}{2} \cos\theta_i \right)^2 d\Omega \quad , \tag{24}$$

where θ_i is the angle between the direction of propagation of the incident wave and the direction of observation from the scatterer. For a perfectly rigid scatterer we may take $\overline{c} \to \infty$ and $\eta \to 1$. It is worth noting that there is a scattering angle given by $\cos\theta = 2/3$ where the scattered field is vanishing. This is a well-known, classical result for the scattering of sound.[1]

4. Distributions of scatterers

For a distribution of inhomogeneities equation (23) gives the scattered field

$$\Phi_s = -\eta \frac{\varphi_0 k}{4\pi} e^{-i\omega t} \sum_i v_i \left(k - \frac{3}{2} \mathbf{k}\mathbf{n}_i \right) e^{i\mathbf{k}\mathbf{r}_i} \frac{e^{ik|\mathbf{r}-\mathbf{r}_i|}}{|\mathbf{r}-\mathbf{r}_i|} \quad . \tag{25}$$

Let us assume a uniform distribution of identical scatterers ($v_i = v$), with a density σ, and take the origin as the observation point ($\mathbf{r} = 0$). The summation in equation (25) becomes an integral,

$$\Phi_s = -\eta\sigma v \frac{\varphi_0 k^2}{4\pi} e^{-i\omega t} \int d\mathbf{r} \left(1 + \frac{3}{2}\cos\theta\right) e^{ikr\cos\theta} \frac{e^{ikr}}{r} \ , \tag{26}$$

where θ is the angle between the propagation vector $\mathbf{k}$ and the position $\mathbf{r}_i$ of the inhomogeneity. It is easy to see that the integral in equation (26) can be put in the form

$$\int d\mathbf{r} \left(1 + \frac{3}{2}\cos\theta\right) e^{ikr\cos\theta} \frac{e^{ikr}}{r} = -\frac{\pi}{k^2} \int_{-1}^{1} du \frac{2 + 3u}{(1+u)^2} \ , \tag{27}$$

where $u = \cos\theta$. We can see that this integral has a singularity for $\theta = \pi$, arising from the backward scatterers (forward scattering). Indeed, we can see easily that for $\theta_i = \pi$ in equation (25) (a line of scatterers), we get a logarithmic singularity. This is an example of coherent forward scattering, corresponding to a vanishing phase $\mathbf{k}\mathbf{r}_i + kr_i = 0$ in equation (25), an expected result for a uniform distribution of scatterers without multiple scattering, which is equivalent with a mean-field approach for a uniform medium. It is to be compared with the scattering by one scatterer (placed at $\mathbf{r}_i = 0$), where the maximum of the scattered field lies in the backward direction. This singularity arises from the fact that our approach does not include multiple scattering (for instance, forward and backward scattering).

Equation (25) gives reflected waves. Indeed, let us assume that we have a uniform distribution of identical scatterers in a half-space defined by $z > d$. The scattered field given by equation (25) can be written as

$$\Phi_s = -\eta v \frac{\varphi_0 k^2}{4\pi} e^{-i\omega t + i\mathbf{k}\mathbf{r}} I \ , \tag{28}$$

where

$$I = -\frac{1}{2} \sum_i \frac{1}{r_i} e^{ikr_i(1+\cos\theta_i)} - \frac{3i}{2} \frac{\partial}{\partial k} \sum_i \frac{1}{r_i^2} e^{ikr_i(1+\cos\theta_i)} \ . \tag{29}$$

The summation in equation (29) is peformed oevr the half-space. It is convenient to introduce $\mathbf{k} = (\mathbf{k}_\perp, \kappa)$, where $\mathbf{k}_\perp$ is the wavevector parallel to the surface of the half-space and κ is the component of the wavevector perpendicular to this surface. It is also convenient to use cylindrical coordinates $\mathbf{r}_i = (\mathbf{r}_{i\perp}, z_i)$. The calculations are straightforward; they imply the known integral[36]

$$\int_{|z|} dr J_0(k_\perp \sqrt{r^2 - z^2}) e^{ikr} = \frac{i}{\kappa} e^{i\kappa|z|} \ . \tag{30}$$

We get the leading contribution to the scattered field

$$\Phi_s \simeq \eta\sigma v \frac{\varphi_0 k^2}{4\kappa^2} e^{-i\omega t + i\mathbf{k}_\perp \mathbf{r}_\perp} e^{-i\kappa z} \ , \tag{31}$$

which is the reflected field. The reflection coefficient (the ratio of the scattered amplitude to the amplitude of the incident wave) is $R = \eta\sigma v k^2/4\kappa^2$.

It is worth discussing a laticial distribution of identical scatterers. The force which generates the scattered field in this case contains a factor which has the latice periodicity. For instance, this force in equation (13) can be written as

$$-\eta \frac{h\varphi_0\omega^2}{c^2} e^{i\mathbf{k}\mathbf{r}} \sum_i \delta(|\mathbf{r} - \mathbf{r}_i|) e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}_i)} \tag{32}$$

(where the factor $e^{-i\omega t}$ is left aside). We can see that the summation over i is a periodic function of $\mathbf{r}$ with the lattice periodicity. Therefore, it can be expanded in a Fourier series involving only the reciprocal vectors $\mathbf{g}$ of the lattice. The scattered field given by equation (8) can also be expanded in a Fourier series of wavevectors $\mathbf{k}+\mathbf{g}$. We get the final result for the scattered field at large distances

$$\Phi_s = \eta\sigma v\varphi_0 \frac{e^{ikr}}{8\pi r} \sum_{\mathbf{g}} (k^2 + 3\mathbf{k}\mathbf{g}) \int d\mathbf{r}' e^{i(\mathbf{k}-\mathbf{k}'+\mathbf{g})\mathbf{r}'} \ , \tag{33}$$

where $\mathbf{k}' = k\mathbf{r}/r$ is the wavevector of the scattered wave and integration is performed over the scatterers sample. We can see that the wave exhibits diffraction spots, provided the well-known Laue-Bragg diffraction condition $\mathbf{k} - \mathbf{k}' + \mathbf{g} = 0$ ($g^2 + 2\mathbf{k}\mathbf{g} = 0$) is satisfied. The cross-section for a diffraction spot is given by $d\sigma = (\pi/8)(\eta\sigma v)^2 V (k^2 + 3\mathbf{k}\mathbf{g})^2 d\Omega$, where V is the volume of the sample (compare with equation (24)). It can also be written as $d\sigma = (\pi/8)(\eta\sigma v k^2)^2 V (2 - 3\cos\theta)^2 d\Omega$, where θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$.

5. Fluid with a rough surface

We consider a semi-infinite (half-space) homogeneous, isotropic, ideal fluid, extending boundlessly along the $\mathbf{r} = (x, y)$ directions and limited along the z-direction by a surface $z = h(\mathbf{r})$, where $h(\mathbf{r}) > 0$ is a function to be further specified (roughness function). The fluid occupies the region $z < h(\mathbf{r})$. As before, we introduce a small displacement field $\mathbf{u}(\mathbf{r}, z, t)$ (where t denotes the time) which gives rise to a density imbalance $\delta n = -n div\mathbf{u}$ in the fluid density n, a local change of volume $\delta V = V div\mathbf{u}$ and a local change of pressure δp, as described in Section 2. The displacement field $\mathbf{u}$ satisfies equation (1) and the potential field $\Phi = div\mathbf{u}$ satisfies equation (2).

For a semi-infinite fluid with a surface described by equation $z = h(\mathbf{r})$ and extending in the region $z < h(\mathbf{r})$, the potential Φ can be written as

$$\Phi = \varphi(\mathbf{r}, z, t)\theta[h(\mathbf{r}) - z] \ , \tag{34}$$

where $\theta(z)$ is the step function. We assume that the magnitude of the roughness function $h(\mathbf{r})$ is small in comparison with the relevant wavelengths of the elastic disturbances propagating in the fluid, so that we may write

$$\Phi \simeq \Phi_0 + \delta\Phi_0 \ , \tag{35}$$

where

$$\Phi_0 = \varphi\theta(-z) \ , \ \delta\Phi_0 = h(\mathbf{r})\varphi\delta(z) + \frac{1}{2}h^2(\mathbf{r})\varphi\delta^{'}(z) + \dots \ , \tag{36}$$

where $\delta(z)$ is the Dirac function (and the prime means differentiation with respect to the variable z). The specific conditions of validity for this approximation will be discussed on the final results. We assume that the potential φ satisfies the wave equation

$$\frac{1}{c^2}\ddot{\varphi} - \Delta\varphi = 0 \tag{37}$$

with specific boundary conditions at $z = 0$. This equation describes the incident and (specularly) reflected waves propagating in a fluid with a plane surface $z = 0$. It is easy to see that for a fixed surface $\partial\varphi/\partial z|_{z=0} = 0$, so that we have the plane waves

$$\varphi = 2\varphi_0 \cos\kappa_0 z \cdot e^{-i\omega t + i\mathbf{k}_0\mathbf{r}} \ , \tag{38}$$

where ω is the frequency, $\mathbf{k}_0$ is the in-plane wavevector and $\kappa_0 = \sqrt{\omega^2/c^2 - k_0^2}$. In this case we can limit ourselves to the first order in h in the second equation (36), and get

$$\delta\Phi_0 = 2h(\mathbf{r})\varphi_0\delta(z)e^{-i\omega t + i\mathbf{k}_0\mathbf{r}} \ . \tag{39}$$

For a free surface $\varphi|_{z=0} = 0$, so we have

$$\varphi = 2i\varphi_0 \sin\kappa_0 z \cdot e^{-i\omega t + i\mathbf{k}_0\mathbf{r}} \ ; \tag{40}$$

in this case, the first-order contribution to the second equation (36) is vanishing and we get

$$\delta\Phi_0 = -ih^2(\mathbf{r})\kappa_0\varphi_0\delta(z)e^{-i\omega t + i\mathbf{k}_0\mathbf{r}} \ . \tag{41}$$

We can see that $\delta\Phi_0$ acts as a source-term (a force) localized on the surface, which can generate scattered waves. We denote the potential function associated with these waves byΦ_1; it satisfies the wave equation

$$\frac{1}{c^2}\ddot{\Phi}_1 - \Delta\Phi_1 = f \ , \tag{42}$$

where the force f is given by

$$f = \frac{1}{c^2}\delta\ddot{\Phi}_0 - \Delta\delta\Phi_0 \ . \tag{43}$$

Equation (42) is merely a re-writing of the wave equation for $\delta\Phi_0$. The force f is the difference between the inertial force $\delta\ddot{\Phi}_0/c^2$ and the elastic force $\Delta\delta\Phi_0$; it represents the distinct way the surface follows the wave motion in comparison with the bulk. For localized waves equation (42) has the solution $\Phi_1 = \delta\Phi_0$. Another solutions are given by the waves scattered back in the fluid by the surface roughness, *i.e.* waves generated in equation (42) by the source term f (a particular solution of equation (42)). We generalize this model of surface roughness by introducing a different "sound" velocity $\overline{c}$ in equation (43). The force is then written as

$$f = \frac{1}{\overline{c}^2}\delta\ddot{\Phi}_0 - \Delta\delta\Phi_0 \ . \tag{44}$$

Such a generalization amounts to assuming that the elastic properties of the fluid localized on the rough surface are different than the elastic properties of the fluid bulk, *i.e.* the surface roughness is inhomogeneous in comparison with the bulk. This may correspond either to a surface whose physical properties have been changed, or to a fluid homogeneous everywhere, including its rough surface. Indeed, in the latter case, it is precisely the spatial variations of the rough surface which affect its elastic properties, viewed as a homogeneous medium, and render it, in fact, a rough surface which is inhomogeneous with respect to the bulk. It is convenient to introduce the parameter $\eta = 1 - c^2/\overline{c}^2$ for describing the inhomogeneous roughness. A homogeneous roughness corresponds to $\eta = 0$.

Obviously, according to equations (35) and (36), the scheme of calculation put forwad here is a perturbation-theoretical scheme, with the roughness function $h(\mathbf{r})$ as the perturbation parameter. We limit ourselves here to the first relevant orders of the perturbation theory. We can see that for a fixed surface the first-order approximation is sufficient for getting scattered waves, while for a free surface we have to go to the second-order approximation. This implies already a double scattering by the surface roughness. Higher-orders of the perturbation theory will give multiple scattering.

6. Waves scattered by the rough surface

We use the potential $\delta\Phi_0$ given by equations (39) and (41) to compute the force given by equation (44). The calculations are easily performed for one Fourier component $h(\mathbf{q})e^{i\mathbf{qr}}$of the roughness function $h(\mathbf{r})$, corresponding to the wavevector $\mathbf{q}$ (for simplicity we drop the argument $\mathbf{q}$ in $h(\mathbf{q})$). For a fixed surface, making use of equation (39), we get

$$f = -2h\varphi_0\left[\overline{\kappa}^2\delta(z) + \delta''(z)\right]e^{-i\omega t + i\mathbf{kr}} \ , \qquad (45)$$

where $\mathbf{k} = \mathbf{k}_0 + \mathbf{q}$ and $\overline{\kappa} = \sqrt{\omega^2/\overline{c}^2 - k^2}$. The solution of equation (42) is of the form $\Phi_1 = \varphi_1(z)\theta(-z)e^{-i\omega t + i\mathbf{kr}}$, so that equation (42) becomes

$$\frac{\partial^2\varphi_1}{\partial z^2} + \kappa^2\varphi_1 = \left.\frac{\partial\varphi_1}{\partial z}\right|_{z=0}\delta(z) + \left.\varphi_1\right|_{z=0}\delta'(z) + 2h\varphi_0\left[\overline{\kappa}^2\delta(z) + \delta''(z)\right] \ , \qquad (46)$$

where $\kappa = \sqrt{\omega^2/c^2 - k^2}$. We note that $k = \sqrt{k_0^2 + 2\mathbf{k}_0\mathbf{q} + q^2}$ and $\kappa = \sqrt{\kappa_0^2 - 2\mathbf{k}_0\mathbf{q} - q^2}$. The combination of the wavevectors $\mathbf{k}_0$ and $\mathbf{q}$ in $\mathbf{k} = \mathbf{k}_0 + \mathbf{q}$ is the source of directional effects, included both in k and κ. As it is well-known, the Green function of equation (46) (one-dimensional Helmholtz equation) is

$$G(z - z') = \frac{1}{2i\kappa}e^{i\kappa|z - z'|} \ , \qquad (47)$$

so that the solution of equation (46) is given by

$$\varphi_1(z) = \int dz' S(z')G(z - z') \ , \qquad (48)$$

where the source S denotes the *rhs* of equation (46). The calculations are straightforward. We get a localized solution $\varphi_{1l} = 2h\varphi_0\delta(z)$, which corresponds to $\delta\Phi_0$ given by equation (39), as expected, and a wave reflected back in the fluid, given by

$$\varphi_{1r} = -\frac{ih\varphi_0}{2\kappa}(\overline{\kappa}^2 - \kappa^2)e^{-i\kappa z} \ , \qquad (49)$$

or

$$\Phi_{1r} = i\eta\frac{h\varphi_0\omega^2}{2c^2\kappa}e^{-i\omega t + i\mathbf{kr} - i\kappa z} \qquad (50)$$

(for $z < 0$).

Likewise, for a free surface, making use of equation (41), we get a localized wave

$$\Phi_{1l} = -ih_2\varphi_0\kappa_0\delta(z)e^{-i\omega t + i\mathbf{kr}} \ , \qquad (51)$$

which coincides with $\delta\Phi_0$ given by equation (41), and a reflected wave

$$\Phi_{1r} = \eta\frac{h_2\varphi_0\kappa_0\omega^2}{4c^2\kappa}e^{-i\omega t + i\mathbf{kr} - i\kappa z} \ , \qquad (52)$$

where

$$h_2 = \int d\mathbf{r}h^2(\mathbf{r})e^{-i\mathbf{qr}} \qquad (53)$$

(which depends on $\mathbf{q}$) is the Fourier transform of the roughness function squared (the integration is performed in equation (53) over the unit area).

From the results derived above we can say, qualitativey, that the perturbation-theoretical scheme of calculation is valid for the magnitude of the roughness function much smaller than the relevant wavelengths. For instance, from equation (50) we have $h\omega^2/c^2\kappa \ll 1$, or $h \ll \lambda\cos\theta_r$, where λ is the wavelength of the scattered wave and θ_r is its reflection angle. From equation (52), we can see that the waves scattered by a free surface is a second-order effect, implying multiple (double) scattering, within this approximation, as expected. For the scattered waves localized on the surface, we may represent the δ-function as extending over a distance of the order of $h_m = \max h(\mathbf{r})$, and the perturbation-theory criterion is satisfied for $\overline{h}(\mathbf{r}) \ll h_m$, where $\overline{h}(\mathbf{r})$ is the average of the roughness function (the roughness function should have a few "spikes" only). For a constant roughness function ($\mathbf{q} = 0$), the criterion of series expansion is not satisfied for localized waves, while the scattered waves are reflected back along the original Φ_0-waves; this particular case should be included in the original formulation of the problem for the Φ_0-waves.

It is also worth noting that we have waves Φ_{1r} scattered back in the fluid only for an inhomogeneous roughness ($\eta \neq 0$); for a homogeneous roughness we have only the waves Φ_{1l} localized on the surface.

7. Discussion

The localized waves Φ_{1l} have the general form of the incoming wave $e^{-i\omega t + i\mathbf{k}_0\mathbf{r}}$
modulated by the roughness function $h(\mathbf{r})$ (for a fixed surface) or $h^2(\mathbf{r})$ (for a free surface). If $\mathbf{q}$ is a characteristic wavevector of these roughness functions and $\mathbf{k} = \mathbf{k}_0 + \mathbf{q}$, the velocity of the localized waves is given by

$c_s = \omega/k = ck_0/k\sin\theta$, where θ is the incidence angle of the incoming wave. The directional effects are clearly seen from the presence of $k == \sqrt{k_0^2 + 2\mathbf{k}_0\mathbf{q} + q^2}$ in the denominator of this relation. It is worth noting that for $\mathbf{q} = \pm\mathbf{k}_0$, *i.e.* for roughness functions ($h(\mathbf{r})$ or $h^2(\mathbf{r})$) modulated with the same wavelength as the original Φ_0-wave, there appear scattered waves with half the wavelength of the original Φ_0-waves (wavevector $2\mathbf{k}_0$) and, in addition, the whole surface suffers a vibration (independent of the coordinate $\mathbf{r}$), corresponding to $\mathbf{k} = 0$, a characteristic resonance phenomenon. The waves corresponding to the wavevector $2\mathbf{k}_0$ have a velocity $\omega/2k_0$, which is twice as small as the original velocity on the surface. This is indicative of the slowness phenomenon, associated with rough surfaces.

The $\mathbf{q} = \pm\mathbf{k}_0$ resonance phenomenon is exhibited also by the waves scattered back in the fluid. Another resonance phenomenon may appear for $\pm 2\mathbf{k}_0\mathbf{q} + q^2 = 0$, which is the well-known Laue-Bragg condition for the X-rays diffraction in crystalline bodies (or surface gratings) [14, 15, 37]. In this case $k = k_0$, $\kappa = \kappa_0$, and for $\mathbf{k}_0$ and $\mathbf{q}$ antiparallel the scattered waves propagate in opposite direction with respect to the original incident Φ_0-waves.

A worth noting case corresponds to $q \gg k_0$, when the wavevector κ may become purely imaginary ($\kappa \simeq -q$) and the scattered waves are confined to the surface. According to equations (19) and (21), the reflected waves are now damped ($\sim e^{qz}$) and their amplitudes are proportional to the roughness functions $h(\mathbf{r})$ or $h^2(\mathbf{r})$. These surface waves are generated by the rough surface; they may be called rough-surface waves.

As it is well known, the energy of the incident wave is transferred to the reflected waves. In the present case, it is transferred both to the specularly reflected waves as well as to the scattered waves, including the waves localized on the surface and the waves scattered back in the fluid. Within our approximation, in the limit $h \to 0$, equation (5) gives the main contribution

$$e_l \simeq \frac{2c^4}{\omega^2} h^2 \varphi_0^2 \delta'^2(z) \tag{54}$$

for waves localized on a fixed surface and

$$e_l \simeq \frac{c^4}{2\omega^2} h_2^2 \varphi_0^2 \kappa_0^2 \delta'^2(z) \tag{55}$$

for waves localized on a free surface. We can see that the localized waves can store an appreciable energy, especially for a fixed surface, arising from the component of the fluid velocity perpendicular to the surface. Indeed, taking approximately $\delta'^2(z) \simeq 1/h_m^4$ (and $\delta(z) \simeq 1/h_m$), we get the ratio of the energy density stored on a fixed surface (equation (54)) to the energy density of the incident wave of the order of $\simeq h^2\lambda^2/h_m^4$, which may achieve large values even for $h/h_m \ll 1$, for wavelengths λ much longer than the extension h_m of the surface roughness. This result reflects the large kinetic energy of the fluid particles acting upon a fixed surface.

Using equations (50) and (52), we can calculate the reflection coefficients of the scattered waves (the ratio of their amplitude to the amplitude φ_0 of the incident wave): $R = i\eta h\omega^2/2c^2\kappa$ for a fixed surface and $R = \eta h_2\kappa_0\omega^2/4c^2\kappa$ for a free surface. It is worth noting the directionality effects exhibited by these reflection coefficients, through κ appearing in the denominator. The energy density carried on by the scattered waves is the square of these reflection coefficients. We can see that the total amount of energy carried on diffusively by the waves scattered by the surface roughness implies sums of the form $\sum_{\mathbf{q}} |h(\mathbf{q})|^2/\kappa^2(\mathbf{q})$, or $\sum_{\mathbf{q}} |h_2(\mathbf{q})|^2/\kappa^2(\mathbf{q})$, where $h(\mathbf{q})$ and $h_2(\mathbf{q})$ are the Fourier transform of the roughness function $h(\mathbf{r})$ and, respecticvely, $h^2(\mathbf{r})$ and $\kappa(\mathbf{q}) = \sqrt{\kappa_0^2 - 2\mathbf{k}_0\mathbf{q} - q^2}$. In order to maximize this energy, it is necessary, apart from particular cases of gratings (one, or a few wavevectors $\mathbf{q}$), to include as many Fourier components as possible, *i.e.* the surface should be as "rough" as possible in order to have a good attenuation, a reasonably expected result.

Finally, it is worth noting that, in a formally rigourous treatment, we should "renormalize" the amplitudes of the reflected original Φ_0-waves such as to include (accomodate) the scattered waves in the boundary conditions, which is a well-known procedure specific to theoretical-perturbation calculations.

8. Conclusion

Finally, we may say that a new model of small inhomogeneities (scatterers) in an ideal fluid has been introduced here, which allows for including the effect the inhomogeneities may have on the elastic properties of the fluid (parameter η). The classical results for one scatterer have been re-derived by a new method of solving the wave equation and the wave reflected by a half-space of uniformly distributed scatterers, as well as the wave diffracted by a perfect lattice of scatterers have been derived. The model can be extended to other types of inhomogeneities, like, for instance, a rough surface, and may also be useful in the complex problem of multiple scattering.

Further, we have introduced a model of inhomogeneous surface roughness for a semi-infinite (half-space) homogeneous, isotropic, ideal fluid and solved the wave equation for the waves scattered by this surface roughness in the leading orders of approximation with respect to the roughness magnitude. For a fixed surface, the scattered waves appear in the first-order aproximation, while for a free surface they appear in the second-order approximation. The scattered waves are of two kinds: waves localized (and propagating only) on the surface (two-dimensional waves) and scattered waves reflected back in the fluid by the surface roughness. In some cases, the latter waves may become confined to the surface (rough-surface waves). The reflected waves are absent for a homogeneous roughness, where there exist only localized waves.

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