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Time-Magnitude Correlations and Time Variation of the Gutenberg–Richter Parameter in Foreshock Sequences

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Abstract—The time dependence of the parameter of the Gutenberg–Richter (GR) magnitude distribution is identified for foreshock sequences of earthquakes, correlated with the main shock, by using the geometric-growth model of earthquake focus, the magnitude distribution of correlated earthquakes and the time-magnitude correlations, derived recently. It is shown that this parameter decreases in time in the foreshock sequence, from the background values down to the main shock. If correlations are present, this time dependence and the time-magnitude correlations provide a tool of monitoring the foreshock seismic activity. We analyze the relevance of such a procedure for the occurrence moment and the magnitude of a main shock. The limitations of such an analysis are discussed.

Keywords: Gutenberg–Richter parameters, foreshock–aftershock sequences, correlated earthquakes, main shock, occurrence time.

1. Introduction

Recently, Gulia and Wiemer (2019) suggested that the difference between the parameters (β) of the Gutenberg–Richter (GR) magnitude distribution of the aftershocks and the foreshocks can be used to estimate the occurrence of main shocks. Accompanying earthquake sequences have been analyzed by these authors for the Amatrice-Norcia earthquakes (24 August 2016, magnitude 6.2; 30 October 2016, magnitude 6.6) and the Kumamoto earthquakes (15 April 2016, magnitude 6.5 and 7.3). They found that the foreshock parameter β is lower than the background value (e.g., by 10%), while the aftershock parameter is higher than the background value (e.g., by 20%). A similar decrease in the parameter β has been reported for the foreshocks of the L'Aquila earthquake (6 April 2009, magnitude 6.3) by Gulia et al. (2016) and the Colfiorito, Umbria-Marche, earthquake (26 September 1997, magnitude 6) by De Santis et al. (2011). The analysis method employed by Gulia and Wiemer (2019) was recently questioned (Dascher-Cousineau et al., 2020, 2021; see also Gulia & Wiemer, 2021). We discuss in this paper the variability of the GR parameter in sequences of foreshocks and aftershocks, which may have relevance for estimating the occurrence time and the magnitude of the main shocks. The method is based on the correlations which may exist between foreshocks and the main shock.

The standard GR magnitude distribution is $P(M) = \beta e^{-\beta M}$ (moment magnitude M), where the parameter β varies in the range 1.15-−3.45 (in decimal basis 0.5-−1.5); the mean value $\beta = 2.3$ (in decimal basis $\beta = 1$) is usualy accepted as a reference value (Stein & Wysession, 2003; Udias, 1999; Lay & Wallace, 1995; Frohlich & Davis, 1993). It has been shown (Apostol, 2006) that $\beta = br$, where $b = 3.45$ (in decimal basis 3/2) and r is a parameter characterizing the earthquake focus (see "Appendix"). The parameter r is a statistical parameter, related to the geometry of the focal region (e.g., the effective number of dimensions of the focus); it reflects mainly the structural condition of the focal region. We expect the parameter r to vary between $r = 1/3$ and $r = 1$, with a mean value $r = 2/3$ ($\beta = 2.3$). The standard cumulative (excedence) GR distribution (earthquakes with magnitude greater than M) is $P_{ex}(M) = e^{-\beta M}$; it is used in its logarithmic form $\ln N(M) = \ln N(0) - \beta M$, where $N(M)$ is the

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number of earthquakes with magnitude greater than M.

According to these standard formulae, an increase in β indicates the occurrence of more small-magnitude earthquakes, which may appear in the aftershock region, while a decrease in β indicates, comparatively, more greater-magnitude earthquakes. A decrease in β in the foreshock region has been reported in many instances (see, e.g., Gulia et al. (2016) and References therein), as well as an increase in the aftershock region (Gulia et al., 2018). In principle, a statistical description of the accompanying seismic activity implies a symmetric distribution in the foreshock–aftershock regions. However, in the proximity of a main shock, especially after a main shock, the condition of the seismic region may change appreciably, such that it is not likely that the foreshocks and the aftershocks are members of the same statistical ensemble.

2. Correlations

Earthquakes which occur closely in time and space, like the earthquake sequence accompanying a main shock, may be correlated with the main shock (see "Appendix"). The magnitude distribution of the correlated earthquakes differs from the standard Gutenberg–Richter distribution discussed above (Apostol, 2021). Judged by their time-dependence shape, the first part of the foreshock distribution indicated by Gulia and Wiemer (2019) may exhibit correlations, but correlations cannot be definitely assessed in the aftershocks distribution; a change in the seismicity conditions may be present for accompanying events. We discuss below a possible relevance of a correlated foreshock sequence for the occurrence of a main shock.

The earthquake correlations, as identified in Apostol (2021), are time correlations (or purely dynamical correlations), purely statistical and time-magnitude correlations. The time correlations (called also "causal" correlations) imply a sharing of the accumulation time. They lead to modified statistical distributions, as the modified GR distribution given below. They may appear by a static or a dyamical stress, a change in the seismicity conditions of the focal region, a triggering mechanism, etc. We note that these correlations imply the same statistical ensemble. Mathematical conditions (constraints) imposed on the statistical variables give rise to purely statistical correlations. Time-magnitude correlations, which are discussed in this paper, are dynamical correlations arising from the non-linearity of the law of energy accumulation in the focus. This law allows an energy sharing between two (or more) earthquakes, which makes an earthquake to depend on the other, i.e. it generates a correlation between these earthquakes (see "Appendix").

The correlation-modified magnitude distribution (modified GR distribution, Apostol, 2021) is

$$P^c(M) = \beta e^{-\beta M} \frac{2}{(1+e^{-\beta M})^2}; \tag{1}$$

without other specifications, this distribution includes the so-called purely dynamical correlations, which affect mainly the small-magnitude earthquakes. We expect such correlations to be present mainly in foreshock sequences. From Eq. (1) we get the correlation-modified cumulative distribution

$$P^c_{ex}(M) = e^{-\beta M} \frac{2}{1+e^{-\beta M}}. \tag{2}$$

The logarithmic form of this distribution

$$\ln N^c(M) = \ln N(0) + \ln 2 - \ln\left(1+e^{\beta M}\right) \tag{3}$$

should be compared to the standard logarithmic form

$$\ln N(M) = \ln N(0) - \beta M. \tag{4}$$

We can see that the modified GR distributions (Eqs. 1 and 2) differ from the standard GR distributions, as shown in Figs. 1 and 2. It seems that such a qualitative difference has been found for southern California earthquakes recorded between 1945–1985 and 1986–1992 (Jones, 1994). The difference arises mainly in the small-magnitude region $M \lesssim 1$, where the distributions are flattened. For instance, in this region the parameter β of the cumulative distribution tends to $\beta/2$, according to Eqs. (1) and (2) (see "Appendix"). This deviation, known as the roll-off effect (Bhattacharya et al., 2009; Pelletier, 2000), is assigned usually to an insufficient determination of the small-magnitude data. We can see that it may also be due to correlations, at least partially. For large

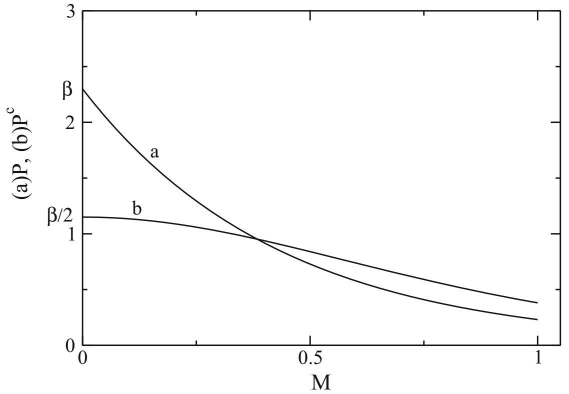

Fig. 1
The standard GR distribution $P = \beta e^{-\beta M}$ (curve a) compared to the correlation-modified GR distribution P^c, Eq. (1) (curve b)

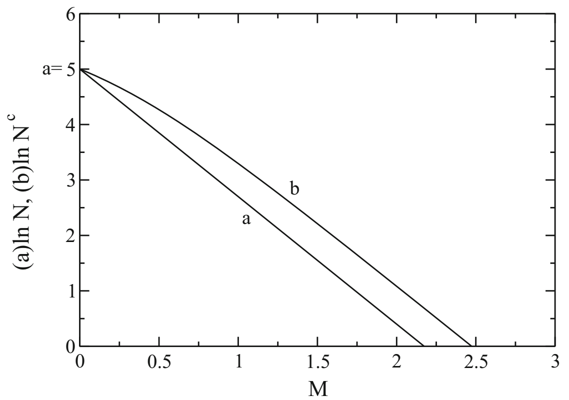

Fig. 2
The standard cumulative GR distribution $\ln N = \ln N(0) - \beta M$ (curve a) compared to the correlation-modified cumulative GR distribution $\ln N^c$, Eq. (3) (curve b) for $\beta = 2.3$ and an arbitrary value $\ln N(0) = 5$

magnitudes the logarithmic cumulative distribution is shifted upwards by $\ln 2$ (Eq. 3), while its slope is very close to the slope of the standard cumulative GR distribution (β).

The correlation-modified cumulative distribution given by Eq. (2) can be used to identify a correlated sequence of foreshocks. We consider a seismic region with a background of earthquakes (regular earthquakes) extended over a long period of time T, interrupted from time to time by (rare) big seismic events. We may assume that some of these large earthquakes are main shocks in accompanying sequences of foreshocks (and aftershocks), correlated with the main shock. For moderate and large magnitudes we may fit the seismic activity by the standard cumulative GR distribution given by Eq. $\ln N(M) = \ln N(0) - \beta M$. Usually, such fits are done by using a small-magnitude cutoff, such that the slope of the distribution (β) is not affected by correlated small-magnitude earthquakes [difficulties in determining the β-parameter by finite sets of data and the related completeness magnitude are discussed recently by Marzocchi et al. (2020) and Lombardi (2021)]. A proper fitting of the full (modified) GR distribution given by Eq. (3) leads to very small differences in the parameter β. It is convenient to introduce the parameter $t_0 = T/N(0)$; its inverse is a seismicity rate. Due to the small-magnitude cutoff, this seismicity-rate parameter becomes a fitting parameter (Apostol, 2021). The standard GR cumulative distribution reads

$$\ln[N(M)/T] = -\ln t_0 - \beta M. \tag{5}$$

By fitting this law to the empirical data we get the parameters β (and r) and t_0. For instance, such a fit, done for a set of 3640 earthquakes with magnitude $M \geq 3$ which occurred in Vrancea during 1981–2018, leads to $-\ln t_0 = 11.32$ (t_0 measured in years) and $\beta = 2.26$ ($r = 0.65$), with an estimated 15% error. We note that the value $\beta = 2.26$ is close to the reference value given above (2.3). (The data for Vrancea have been taken from the Romanian Earthquake Catalog (2023), http://www.infp.ro/data/romplus.txt. A completeness magnitude $M = 2.2$ to $M = 2.8$ is usually accepted (Enescu et al., 2008 and References therein); a more conservative figure would be $M = 3$. The magnitude average error is $\Delta M = 0.1$). A similar fit, with slightly modified parameters, is valid for 8455 Vrancea earthquakes with magnitude $M \geq 2$ (period 1980–2019). This way, we get the parameters of the background seismic activity for Vrancea (β, r, t_0).

3. Time-Magnitude Formula

Let us assume now that we are in the proximity of a main shock with magnitude M_0, at time τ until its occurence, and we monitor the sequence of correlated foreshocks. It was shown (Apostol, 2021) that the magnitudes of the (correlated) foreshocks M ($<M_0$) are related to the time τ by

$$M = \frac{1}{b}\ln(\tau/\tau_0), \tag{6}$$

where

$$\tau_0 = rt_0 e^{-b(1-r)M_0} \tag{7}$$

is a cutoff time, which depends on the magnitude of the main shock, the seismicity-rate parameter t_0 and the parameter $r = \beta/b$ (see "Appendix"). The parameters t_0 and r are provided by the analysis of the background seismic activity. The small threshold time τ_0 corresponds to a very short quiescence time (Ogata & Tsuruoka, 2016) before the occurrence of the main shock. In addition, the time τ should be cut off by an upper threshold, corresponding to the magnitude of the main shock ($\tau < \tau_0 e^{bM_0}$). We limit ourselves to small and moderate magnitudes M in the accompanying seismic activity, such that the magnitude of the main shock may be viewed as being sufficiently large (in this respect, the so-called purely statistical correlations discussed by Apostol (2021), are not included). Equation (6) is derived by analyzing the time-magnitude correlations predicted by the geometric-growth model of earthquake focus (Apostol, 2006, see "Appendix"). According to this model the accumulation time of an earthquake with energy E is $t = t_0(E/E_0)^r = t_0 e^{\beta M}$, where E_0 is a cutoff energy. By means of this model, Bath's law is derived and the occurrence time of the Bath partner is calculated, as well as the cumulative magnitude distribution of the accompanying seismic activity.

The distribution given by Eq. (2) indicates a change in the parameter β of the standard GR distribution. We denote by B the modified parameter β; it is given by

$$e^{-\beta M}\frac{2}{1+e^{-\beta M}} = e^{-BM}, \tag{8}$$

where B is a function of M ($B(M)$). It is convenient to introduce the ratio $R = B/b$ (similar to $r = \beta/b$ given above), such that Eq. (8) becomes

$$R = \frac{1}{\ln\theta}\ln\left[\frac{1}{2}(1+\theta^r)\right], \tag{9}$$

where $\theta = \tau/\tau_0$ from Eq. (6). The parameter R varies from $R = r$ for large values of the variable θ to $R = r/2$ for $\theta \to 1$ ($\tau \to \tau_0$). The function $R(\theta)$ is plotted in Fig. 3 vs $\log\theta$ for $r = 2/3$. The decrease of the function $R(\theta)$ for $\theta \longrightarrow 1$ indicates correlations.

According to Eq. (8), the modified GR parameter B is given approximately by

$$B(M) \simeq \beta - \frac{\ln 2}{M}, \tag{10}$$

or

$$R(\tau) \simeq r - \frac{\ln 2}{\ln(\tau/\tau_0)} \tag{11}$$

for a reasonable range of foreshock magnitudes $M > 1$. Equations (9)–(11) show the decrease of the GR parameter in a foreshock sequence. For instance, a 10% decrease is achieved for $M = 3$, or $\tau/\tau_0 \simeq 3.6\times 10^4$ ($\beta = 2.3$, $r = 2/3$). It is worth noting that smaller magnitudes occur in the sequence of correlated foreshocks for shorter times, measured from the occurrence of the main shock (the nearer main shock, the smaller correlated foreshocks).

On the other hand, the time-magnitude correlations expressed by Eq. (6) lead to $\tau = \tau_0 e^{bM}$ for the accumulation time elapsed from the main shock to an aftershock. This relation shows a change in the seismicity conditions, where t_0 is replaced by τ_0 and β is replaced by b in the regular accumulation time $t = t_0 e^{\beta M}$. The magnitude distribution $(t_0/t^2)dt = \beta e^{-\beta M} dM$, which follows from this

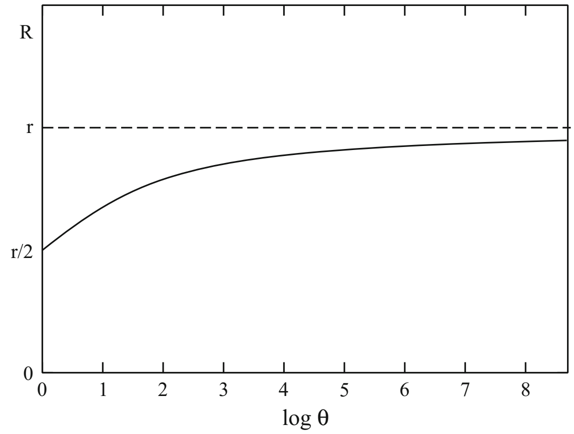

Fig. 3
Function $R(\theta)$ vs $\log\theta$ for $r = 2/3$ (Eq. 9; Apostol & Cune, 2023)

accumulation time (Apostol, 2006, see "Appendix"), is changed in this case to $be^{-bM}dM$, which indicates an increase in the GR parameter ($b = 3.45$) with respect to its background value β. Such a deviation holds up to a cutoff magnitude M_c where the two distributions become equal, such that we may estimate an average increase in the parameter β as $(b-\beta)/2\beta = 25\%$ for $\beta = 2.3$. The cutoff magnitude is given by $be^{-bM_c} = \beta e^{-\beta M_c}$, whence $M_c = 0.36$ for $r = 2/3$, $b = 3.45$ ($\beta = 2.3$). Both these estimated deviations of the GR parameter for foreshocks and aftershocks are in quantitative agreement with data reported by Gulia et al. (2016, 2018) and Gulia and Wiemer (2019).

The logarithmic law expressed by Eq. (6) for the time-magnitude correlated foreshocks provides a means of estimating the occurrence time of the main shock. Indeed, if we update the slope B of the cumulative distribution $\ln[N(M)/N(0)] = -BM$ at various successive times t (Eq. 8), and if this B fits Eq. (10), then we may say that we are in the presence of a correlated sequence of foreshocks which may announce a main shock at the moment $t_{ms} = t + \tau$. (In particular, the probability of occurrence of a main shock with magnitude M_0 increases in this case by a factor $\frac{\overline{B}}{\beta} e^{(\beta-\overline{B})M_0}$, where $\overline{B}$ is the average value of the parameter B). For practical use it is more convenient to use directly Eq. (6), which leads to the time dependence

$$M(t) = \frac{1}{b} \ln \frac{t_{ms} - t}{\tau_0} \tag{12}$$

of the foreshock magnitudes, for $(1-r)t_{ms} < t < t_{ms} - \tau_0$ $(0 < M < M_0)$. This formula provides an estimate of the occurrence moment of the main shock t_{ms} from the correlated-foreshock magnitudes $M(t)$ and the background seismicity parameter τ_0; the occurrence time is given by

$$t_{ms} = t + \tau_0 e^{bM(t)}. \tag{13}$$

It is worth noting that the time t_{ms} depends on the magnitude of the main shock, as expected (M_0, which enters τ_0, Eq. (7)). For instance, a magnitude M indicates a time $\tau = \tau_0 e^{bM}$ up to the main shock (Eq. 6). Let us assume that we are interested in a main shock with magnitude $M_0 = 7$; then by using $t_0 = e^{-11.32}$ (years, for Vrancea) and $r = 2/3$ given above, we get $\tau_0 = \frac{2}{3} 10^{-8.42}$ (years); a foreshock with magnitude $M = 5$ would indicate that we are at $\tau = \frac{2}{3} 10^{-8.42} 10^{7.5} = 0.079$ years, i.e. $\simeq 29$ days, from that main shock. The time t_{ms} of the occurrence of the main shock is obtained from Eq. (12) as a fitting parameter of the correlated-foreshock magnitudes $M(t)$. In practice, it is also convenient to view τ_0 as a fitting parameter. Since, for moderate magnitudes, the variation of the parameter R is small (Eqs. 10 and 11), we may use the background value for r in the expression of τ_0 (e.g., $r = 2/3$), which leads to an estimate of the expected main-shock magnitude M_0 from the fitting parameter τ_0. However, a reliable estimation of the time t_{ms} provided by Eq. (13) requires a very high slope of the decreasing magnitudes $M(t)$ in the neighbourhood of t_{ms}, which can only be attained by a special data set, including, ideally, many small-magnitude foreshocks whose magnitudes fall rapidly to zero.

4. *Discussion and Concluding Remarks*

The procedure described above has been applied to foreshock sequences of a few Vrancea earthquakes, l'Aquila, Yangbi (Yunnan) and Izmit earthquakes. Details of the analysis are given in Apostol and Cune (2023). For the Vrancea earthquake of 30 August 1986, magnitude $M = 7.1$, the sequence of foreshocks from 16 to 24 August (seven earthquakes) indicated the occurrence of a main shock with magnitude 4.4 on 24 August. For the Vrancea earthquake of 30 November 2021, with magnitude $M = 3.8$, the sequence of foreshocks from 24 to 29 November 2021 (seven earthquakes) indicated a main shock with magitude 4.5 on 1 December. Two magnitude-descending foreshock sequences were identified for the l'Aquila earthquake (6 April 2009, local magnitude 5.9), one on 2 April (seven earthquakes), another on 6 April (five earthquakes). The analysis showed the occurence of main shocks on 3 April and on 6 April, very close to the occurrence time of the l'Aquila earthquake. Similar results were obtained for the other two earthquakes in the above list. On the other hand, there was two big Vrancea earthquakes on 30–31 May 1990 ($M = 6.9$--6.4) where magnitude-descending foreshock sequences were absent. An interesting case is the Norcia earthquake of 30 October 2016, with magnitude $M = 6.6$, which was preceded by a large number of rapidly succeeding, magnitude-decreasing and very short foreshock sequences, a few hours before its occurrence (data from Gulia and Wiemer (2019), SourceDataFig. 1). All these foreshock sequences indicate the proximity of a main shock. Particularly interesting are the sequences $M = 2.4$ to $M = 1$ from 6/4/7.76 to 6/12/43.08 (hours/min/sec) and $M = 2$ to $M = 1.5$ from 6/31/19.89 to 6/38/23.49 (the main shock occurred at 6/40/17.36). Unfortunately, such sequences are too poor to be useful for quantitative results.

The application of Eqs. (12) and (13) to fitting the correlated foreshocks involves certain particularities. First, we should note that not all the precursory seismic events are foreshocks correlated with the main shock. Second, small clusters of precursory events may exist, which may include second-order (an even higher-order) correlated earthquakes, i.e. events which accompany precursory events, according to the epidemic-type model [see, for instance, Ogata (1988, 1998), as well as Helmstetter and Sornette (2003) and Saichev and Sornette (2005)]. These secondary events have little relevance upon a forthcoming main shock, such that they may be left aside. We limit ourselves to the highest foreshocks occurring in short periods of time (though an average magnitude for each small cluster may also be used). Third, the relevant part of the logarithmic curve given by Eq. (12) (or the exponential in Eq. (13)) is its abrupt decrease in the immediate proximity of t_{ms} (of the order of days for Vrancea), such that the most relevant foreshock sequence is the one which occurs in the immediate proximity of the main shock. This circumstance is related to the very small values of the parameter t_0 and the large magnitude M_0, of interest for the main shock (small values of the parameter τ_0). In this regard, a reliable estimation of the parameters t_{ms} and τ_0 would be conditioned by a rich seismic activity in the immediate vicinity of the occurrence moment of a main shock (a very short-time prediction, e.g., of the order of days). This is an ideal situation, which is not achieved in practice, since the number of small-magnitude foreshocks is small in the immediate vicinity of the main shock, precisely due to decrease of the parameter β (B, R). Therefore, such fits are necessarily of poor quality.

According to theory, the "philosophy" of our approach is as follows. We monitor daily (sometimes even hourly!) the seismic activity in a given seismic region. We need this continuous surveyance because the theory implies a short-term prediction. This short-time character implies necessarily rather short sequences of earthquakes. Consequently, our procedure is not a standard statistical procedure, which would require a large set of data. Any time we see a sequence of earthquakes descending in magnitude, we fit it with our equations. If that sequence happens to be a sequence of correlated foreshocks, the fit would be satisfactory, and the fitting parameters give the occurrence time of a main shock (if the fit is poor, very likely the sequence is not correlated). More, if we know the parameters of the background seismicity of that region, we are able to predict even the magnitude of the main shock. It may happen that the prediction fails. This may appear, very likely, by a change in the structural seismic conditions. It is reasonable to assume that after a sequence of foreshocks the structural conditions may change. Then, we have a false positive. The seismic activity is resumed, and we continue our analysis. It may happen that no sequence descending in magnitude appears before a main shock (as for Vrancea earthquakes of 30–31 May 1990). Then we have a false negative. As we can see the method may fail, but also it may succeed, as shown in some cases like those discussed above.

In conclusion, the GR distributions modified by correlations in the foreshock region and the time dependence of the foreshock magnitudes (Apostol, 2021) can be used, in principle, to estimate the moment of occurrence of the main shock and its magnitude, although with limitations. The main source of errors arises from the quality of the fit $B(t)$

vs $M(t)$ (Eq. 10), or, equivalently, the fit of the function $R(\theta)$ given by Eq. (9), or the fit given by Eqs. (12) and (13). These fits are necessarily of a poor quality, due to the abrupt decrease of the function $M(t)$ near the occurrence time t_{ms} of the main shock (Eq. 12), or, equivalently, the abrupt decrease of the parameters $B(M)$ and $R(\tau)$ for small values of the variables M and τ. Another source of errors arises from the background parameters t_0 and r (β), which may affect considerably the exponentials in the formula of the time cutoff τ_0 (Eq. 7). The procedure described above is based on the assumption that the foreshock magnitudes are ordered in time according to the law given by Eq. (6). However, according to the epidemic-type model, the time-ordered magnitudes may be accompanied by smaller-magnitudes earthquakes, such that the law given by Eq. (6) may exhibit lower-side oscillations, and the slope given by Eq. (11) may exhibit upper-side oscillations. Several subsets of correlated foreshocks may be identified (in accordance with the epidemic-type model), as well as the absence of correlations. In spite of all these limitations, a continuous monitoring of the foreshock seismic activity by means of the procedure described in this paper may give interesting information about a possible main shock. Also, the decrease of the GR parameter in the correlated foreshock sequences and its increase in aftershock sequences, as identified in the previous works (e.g., Gulia and Wiemer (2019)), as well as in the present one, is a valuable piece of information.

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Declarations

Conflict of interest The author declares that no conflict of interest.

Appendix

Geometric-Growth Model

A typical earthquake is characterized by a small focal region (Apostol, 2006). Since the relevant distance scale is much larger than the focal dimension, we may view the focus as a point in an elastic body. The seismic energy is accumulated in the focal by the movement of the tectonic plates. This energy accumulation is described by the continuity equation

$$\frac{\partial E}{\partial t} = -\boldsymbol{v} gradE, \tag{14}$$

where E is the energy, t denotes the time and $\boldsymbol{v}$ is an accumulation velocity. Since the focal region is localized, the derivatives in this equation may be replaced by ratios of small, finite differences. As an example, we write $\Delta E/\Delta x$ instead of $\partial E/\partial x$ for the coordinate x. At the borders of the focus the energy tends to zero, such that $\Delta E = -E$. We may assume that the coordinates of the borders move uniformly, according to $\Delta x = u_x t$, etc, where we denote by $\boldsymbol{u}$ a small velocity of the medium. By using these assumptions, we get from Eq. (14)

$$\frac{\partial E}{\partial t} = \left(\frac{v_x}{u_x} + \frac{v_y}{u_y} + \frac{v_z}{u_z}\right)\frac{E}{t}. \tag{15}$$

For a uniform motion the two velocities are equal ($\boldsymbol{v} = \boldsymbol{u}$), and we get a coefficient 3 in Eq. (15). If the motion is one dimensional, the coefficient is 1. Therefore, the above equation can be written as

$$\frac{\partial E}{\partial t} = \frac{1}{r}\frac{E}{t}, \tag{16}$$

where the parameter r varies in the range 1/3–1. For a shearing fault we have $u_x = v_x$ and $u_y = 2v_y$, $v_z = 0$, because, apart form the x-direction, the energy is accumulated also along two opposite perpendicular directions (y-directions), in order to conserve the mass. We get $r = 2/3$, which corresponds to the mean value $\beta = br = 2.3$ ($b = 3.45$), accepted as the reference value (see the main text). Therefore, the parameter r is a statistical parameter, related, mainly, to the effective number of dimensions of the focus. The above model is called the geometric-growth model of seismic energy accumulation.

In order to integrate Eq. (16) we need a cutoff energy and a cutoff time. Therefore, a small amount of energy E_0 is accumulated in a short time t_0. This energy may be lost in the next time t_0, or the accumulation process may continue. These processes are called fundamental processes, or E_0-seismic events. From equtaion (16) we get the law of energy accumulation

$$t/t_0 = (E/E_0)^r. \tag{17}$$

The energy E may be released in an earthquake, which occurs after time t.

Gutenberg–Richter Law

It is well known that seismic moment $\overline{M}$ is related to the magnitude M through the Hanks-Kanamori empirical law

$$\ln \overline{M} = const + bM, \tag{18}$$

where $b = 3.45$ ($\frac{3}{2}$ for base 10). The seismic moment can be defined as $\overline{M} = \left(\sum_{ij} M_{ij}^2\right)^{1/2}$, where M_{ij} is the tensor of the seismic moment; it is related to the energy through $\overline{M} = 2\sqrt{2}E$ (Apostol, 2019), such that Eq. (18) can be written as

$$\ln E = const + bM \tag{19}$$

(by using another *const*). This relation can be cast in the form

$$E/E_0 = e^{bM}, \tag{20}$$

where E_0 is a cutoff energy. Now, we can make use of Eq. (17), and get

$$t = t_0 e^{brM} = t_0 e^{\beta M}, \tag{21}$$

where $\beta = br$. From this equation we derive the useful relations $dt = \beta t_0 e^{\beta M} dM$, or $dt = \beta t dM$. In the well-known Gutenberg–Richter distribution we have

$$dP = \beta e^{-\beta M} dM. \tag{22}$$

On the other hand, from Eq. (21) we get $dt = \beta t_0 e^{\beta M} dM$, or $dt = \beta t dM$. By using this result, Eq. (22) becomes

$$dP = \beta \frac{t_0}{t} \frac{1}{\beta t} dt = \frac{t_0}{t^2} dt. \tag{23}$$

This is the time distribution of independent earthquakes; it gives the probability $dP = t_0 dt/t^2$ for an earthquake to occur between t and $t + dt$. Since t is the accumulation time, this earthquake has energy E and magnitude M, related by the above formulae. An equivalent derivation of the time probability can be obtained from the definition of the probability of the fundamental E_0-seismic events ($dP = -\frac{\partial}{\partial t}\frac{t_0}{t} dt$; Apostol (2021)).

Correlations: Time-Magnitude Correlations

Two (or more) earthquakes may depend on one another, by various mutual influences. We say that those earthquakes are correlated. We limit ourselves to two-earthquake (pair) correlations, which bring the main contribution. Two earthquakes may share their energy; then we have time-magnitude correlations, as shown below. Also, two earthquakes may share their accumulation time. Then, we have time correlations (also called purely dynamical correlations), as shown in the next Appendix (Apostol, 2021). The statistical distributions are affected by both these correlations. Other types of (statistical) correlations may appear, due, for instance, to additional constraints imposed upon the statistical variables (for instance, the magnitudes of the accompanying events be smaller than the magnitude of the main shock).

An energy E, accumulated in time t, may be released in two successive earthquakes, with energies $E_{1,2}$. The two earthquakes share the seismic energy. We may write $E = E_1 + E_2$ and

$$\begin{aligned} t/t_0 = (E/E_0)^r = (E_1/E_0 + E_2/E_0)^r = \\ = (E_1/E_0)^r (1 + E_2/E_1)^r \end{aligned} \tag{24}$$

(Eq. 17). This equation may bewritten as

$$t = t_1 \left[1 + e^{b(M_2 - M_1)}\right]^r, \tag{25}$$

where $t_1 = t_0 (E_1/E_0)^r$ is the accumulation time of the earthquake with energy E_1 and magnitude M_1, and

M_2 is the magnitude of the earthquake with energy E_2. Equation (25) leads to

$$b(M_2 - M_1) = \ln\left[(1 + \tau/t_1)^{1/r} - 1\right], \qquad (26)$$

where $t = t_1 + \tau$, τ being the time elapsed from the occurrence of the earthquake 1 until the occurrence of the earthquake 2. In foreshock–main shock–aftershock sequence $\tau/t_1 \ll 1$, such that we get from the above equation

$$M_2 = \frac{1}{b}\ln\frac{\tau}{\tau_0},\ \tau_0 == rt_0 e^{-b(1-r)M_1}. \qquad (27)$$

We can see that τ given by this equation differs from the accumulation time of the M_2-earthquake (compare with Eq. 21). The difference arises from parameters which depend on the M_1-earthquake, as expected for correlated earthquakes. We may view the M_1-earthquake as a main shock and the M_2-earthquake as a foreshock or an aftershock. These accompanying earthquakes are correlated to the main shock. These are the time-magnitude correlations.

Time Correlations

Two earthquakes may share their accumulation time $t = t_1 + t_2$, such that an earthquake appears in time t_1, followed by another which appears in time t_2. From Eq. (23) the probability density of such an event is given by

$$-\frac{\partial}{\partial t_2}\frac{t_0}{(t_1 + t_2)^2} = \frac{2t_0}{(t_1 + t_2)^3} \qquad (28)$$

(where $t_0 < t_1 < +\infty$, $0 < t_2 < +\infty$). We may pass in this formula to magnitude distributions ($t_{1,2} = t_0 e^{\beta M_{1,2}}$), and get the probability

$$d^2P = 4\beta^2 \frac{e^{\beta(M_1+M_2)}}{(e^{\beta M_1} + e^{\beta M_2})^3} dM_1 dM_2 \qquad (29)$$

(where $0 < M_{1,2} < +\infty$, corresponding to $t_0 < t_{1,2} < +\infty$, which introduces a factor 2 in Eq. (28)). This is a pair, bivariate statistical distribution (Apostol, 2021). By integrating with respect to M_2, we get the so-called marginal distribution, i.e. the distribution of a correlated earthquake,

$$dP = \beta e^{-\beta M_1} \frac{2}{(1 + e^{-\beta M_1})^2} dM_1; \qquad (30)$$

By integrating further this distribution from $M_1 = M$ to $+\infty$, we get the correlated cumulative distribution

$$P(M) = \int_M^\infty dP = e^{-\beta M} \frac{2}{1 + e^{-\beta M}}. \qquad (31)$$

For $M \gg 1$ the correlated cumulative distribution becomes $P(M) \simeq 2e^{-\beta M}$ and $\ln P(M) \simeq \ln 2 - \beta M$. Therefore, the slope β of the logarithm of the independent cumulative distribution (Gutenberg–Richter, standard distribution $e^{-\beta M}$) is not changed (for moderate and large magnitudes); the distribution is only shifted upwards by $\ln 2$. For small magnitudes ($M \ll 1$) the slope of the correlated cumulative distribution becomes $\beta/2$ (by the series expansion $P(M) \simeq 1 - \frac{1}{2}\beta M + \cdots$ of Eq. 31); this result differs from the slope β of the standard Gutenberg–Richter distribution ($e^{-\beta M} \simeq 1 - \beta M + \cdots$). The correlations modify the slope of the Gutenberg–Richter standard distribution for small magnitudes. This is the roll-off effect referred to in the main text.

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