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Vibrations of an Elastic Half-Space

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Abstract: We report on the resolution of the vibration problem for a homogeneous and isotropic elastic half-space (the Lamb problem), with application to the seismic tensorial force. We assume a homogeneous and isotropic half-space with a localized force which produces vibrations. The solution is achieved by introducing vector plane-wave functions. Explicit results are given for an isotropic tensorial force and a half-space with free surface. The contribution of the Rayleigh surface waves to vibrations is analyzed in the special case of a temporal-impulse force, where the solution exhibits unphysical features, as expected: it extends over the entire free surface and time domain, with a (scissor-like) double-wall propagating both in the future and the past.

Keywords: elastic half-space; vibrations; vector plane-wave functions; Rayleigh surface waves

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1. Introduction

The vibration problem of a homogeneous and isotropic elastic half-space (the Lamb problem) is a long-standing problem [1–5]. Traditionally, it is related to the seismic waves and the seismic main shock propagating during earthquakes on Earth's surface. Fourier and Laplace transform techniques have been employed to solve this problem [6–9]. Recent results are reported in Refs. [10–15] and the references therein. By exploiting the properties of the membrane equation, surface waves of arbitrary profile have been studied [16] (see also Ref. [17]). The reciprocity theorem [18] and hyperbolic-elliptic models [19,20] have been applied to surface waves. Though its main feature—the Rayleigh surface waves—has been long known [21], the progress towards an explicit full solution requires, on one side, a physically sound force source and, on the other, a consistent differentiation between vibrations and propagating waves. Recently, a general, formal scheme of solution has been analyzed and the difference between waves and vibrations has been emphasized [22]. We report here on the resolution of this problem.

The seismic tensorial point force governed by the seismic moment tensor was introduced in Refs. [23,24]. The static deformations produced by this force in a homogeneous and isotropic elastic half-space have been calculated and the waves generated in a homogeneous and isotropic body have been derived for a temporal-impulse seismic moment. The P and S seismic waves have been obtained and the main shock generated by these waves on the free surface of the half-space has been calculated. The seismic waves are spherical shells (proportional to the derivative of the δ-function) and the main shock has the shape of an abrupt wall with a long tail, propagating on the surface and vanishing beyond the position of the wall. The vibration problem is different from the wave-like problem. We present in this paper the full solution to the vibration problem, with explicit results for an (isotropic) tensorial point force in a homogeneous and isotropic half-space with a (free) plane surface. The solution is obtained by introducing vector plane-wave functions.

The seismic focus is localized over distances much smaller than our scale distance; therefore, a point focal force is justified. The seismic activity in an earthquake focus lasts a short time compared to our scale time; therefore, a temporal-impulse of the seismic focal force is also justified. Moreover, the depth of the earthquake foci is very small in comparison with the Earth's radius; therefore, the approximation of a half-space with a plane surface for the seismic effects is justified, at least for not very long propagation times (transient regime). Under these conditions, the seismic waves are localized (singular) and obey the causality principle, i.e., they have the shape of propagating spherical shells which appear only after the seismic activity started in the focus; they come to any location and pass over rather quickly. Once arrived at the surface of the half-space, these waves generate secondary waves, according to Huygens principle, which, on the surface, propagate as an abrupt wall-like main shock, behind the seismic waves (actually two main shocks, for different components of the displacement) [24]. A vibration problem, which requires necessarily boundary conditions, is obviously a different problem, since the waves are not present permanently on the entire surface, and, for the main shock, the surface is the location of the wave sources. The wave propagation in these conditions is a transient phenomenon, in contrast to vibrations, which are stationary (standing) waves.

For the vibration problem, we consider a wave source which lasts an indefinite time. Such a source may generate delocalized waves, which propagate both in the future and in the past. These waves should obey boundary conditions. This is a valid vibration problem (of forced vibrations), which we solve in this paper for a half-space. The solution includes the contribution of the Rayleigh surface waves. In the special limit of a temporal-impulse source, the solution extends over the entire free-surface, exhibiting a (scissor-like) double-wall, which propagates in both time directions. Apart from being unphysical, as a result of an improper treatment, this solution is different from the abrupt wall-like seismic main shock. After the temporal-impulse seismic force ceases its action, we may have free oscillations (vibrations), which, for the half-space, are governed by the Rayleigh surface wave frequency.

The current approach to Lamb's problem [2–9], in a simplified description, envisages a spherical wave propagating from the focus (incident wave) and scattered (reflected) by the plane surface of the half-space; the condition of scattering is usually a free surface. The fulfilment of the boundary conditions is questionable, since the spherical wave is a singular wave, proportional to $\delta'(R-ct)$ (or, sometimes, $\delta(R-ct)$), where R is the distance from the focus, c is the velocity of the wave, t denotes the time and δ is the Dirac function. We can see that the wave is zero everywhere, except at $R=ct$, where it is infinite, such that the wave and its derivatives, implied by the boundary conditions, are not usual functions (they are distributions). An image wave is sometimes used, arising from a fictitious focus, which is the focus reflected with respect to the surface; the image wave compensates the incident wave (as for a rigid surface). In an attempt to overcome such difficulties, the incident wave is expanded in plane, or conical waves, and boundary conditions are imposed upon these elementary waves; this amounts to introducing reflected waves with reflection coefficients. These waves are produced by points on the surface where the incident wave is zero, and their superposition does not correspond to localized sources, such that the difficulty persists. The proper mechanism of producing the reflected waves by a singular wave incident on the surface is the Huygens principle. The circle of intersection of the wavefront of the incident wave with the plane surface expands with a higher velocity than the elastic wave velocity. Consequently, the incident wave leaves behind sources of secondary waves, which generates the seismic main shock [24]. In any case, the generation of the secondary waves by the incident spherical waves is a different problem than the vibration problem.

A typical seismogram exhibits a faible tremor, consisting of the P and S primary waves, followed by a main shock, which looks like a wall, or two walls for the two primary waves [25–27]. The primary waves are propagating, localized, scissor-like waves, while the main shock is a propagating, abrupt wall-like disturbance with a long tail. Following Rayleigh [21], the main shock is attributed to vibrations, in the form of the well-known surface waves. However, for vibrations, we need to have waves which are permanently present on the surface, and extending over all the surface, obeying boundary conditions. This is not the case for the seismic motion, as shown in seismograms. The Lamb problem was (and is) motivated by the seismic motion, which is a wave-like, propagating motion, treated as a vibration problem, which is a stationary motion. Nevertheless, in this context, the vibration problem for the elastic half-space is an interesting and relevant problem in itself. This is the primary motivation for undertaking the investigation reported in this paper.

As described above, the seismic main shock is known as the Lamb problem. The seismic main shock is not a vibration problem—it is a wave-like problem for the primary waves, and a wave scattering problem for the main shock, as shown in our previous works [24]. As discussed above, the primary waves arriving at the Earth's surface generate wave sources which produce secondary waves, in the form of the main shock.

The current strategy of approaching the Lamb problem is to expand a localized primary wave in plane waves, viewed as an incident wave, and to look for reflected (and refracted) plane waves which satisfy the boundary conditions, as for surface waves. Conical waves in the Weyl–Sommerfeld expansion may be used, and Laplace transforms, as in the Cagniard–de Hoop method. The solution is provided by the re-summation of the resulting plane waves. Unfortunately, the properties of the plane waves are not shared by their summation. For example, a plane wave extends over the whole space, while the summation of plane waves with equal weights is a localized delta-function. We give in this paper an example of such an unphysical situation. The proper way of dealing with vibrations is to use expansion in eigenfunctions, a method well known, for example, for vector spherical, or cylindrical harmonics. For a half-space, such eigenfunctions are not known yet. We provide them in the present paper. The vibration problem has nothing to do with the seismic effects, but it is of importance in itself. We hope that the investigation presented in this paper may throw a new light upon the current methodology in seismology and structural engineering.

2. Vibration Equation

The elastic vibrations of a homogeneous and isotropic body are described by the Navier–Cauchy equation

$$c_2^2 curl\, curl \mathbf{u} - c_1^2 grad\, div \mathbf{u} - \omega^2 \mathbf{u} = \boldsymbol{F}, \tag{1}$$

where $\boldsymbol{u}$ is the time Fourier transform of the local displacement, $c_{1,2}$ are the velocities of the elastic waves, ω is the frequency and $\boldsymbol{F}$ is the time Fourier transform of the force (per unit mass) [28]. We consider this equation in a half-space $z < 0$, with a free, or fixed, plane surface $z = 0$, for the seismic tensorial force with components

$$F_i = m_{ij} \partial_j \delta(\boldsymbol{R} - \boldsymbol{R}_0), \tag{2}$$

placed at $\boldsymbol{R}_0 = (0, 0, z_0)$, $z_0 \leq 0$, where m_{ij} are the Cartesian components of the seismic (symmetric) tensor ($i, j = x, y, z$; index-summation convention is adopted throughout this paper) [23,24]. The position vector is $\boldsymbol{R} = (\boldsymbol{r}, z)$, where $\boldsymbol{r} = (x, y)$ is the in-plane position vector (parallel to the surface $z = 0$), with horizontal coordinates x, y, while z is the perpendicular-to-surface coordinate (vertical coordinate, $z < 0$). In these equations, the unknown function $\boldsymbol{u}$ depends on ω and $\boldsymbol{R}$ ($\boldsymbol{u}=\boldsymbol{u}(\omega;\boldsymbol{R})$) and the seismic tensor depends on ω ($m_{ij}=m_{ij}(\omega)$). For simplicity, we omit occasionally the position-coordinates variables and the arguments of the Fourier transforms, which can be read easily from the context; also, we denote by the same symbol the Fourier transforms.

We introduce the orthogonal vector plane waves

$$\begin{gathered}\boldsymbol{Z}(\boldsymbol{k};\boldsymbol{r})=\boldsymbol{e}_z\frac{e^{i\boldsymbol{kr}}}{2\pi},\\ \boldsymbol{G}(\boldsymbol{k};\boldsymbol{r})=\frac{i}{k}(k_x\boldsymbol{e}_x+k_y\boldsymbol{e}_y)\frac{e^{i\boldsymbol{kr}}}{2\pi}\ ,\ \boldsymbol{C}(\boldsymbol{k};\boldsymbol{r})=\frac{i}{k}(k_y\boldsymbol{e}_x-k_x\boldsymbol{e}_y)\frac{e^{i\boldsymbol{kr}}}{2\pi},\end{gathered} \tag{3}$$

where $\boldsymbol{k}$ is the in-plane wavevector and $\boldsymbol{e}_i$, $i=x,y,z$, are the unit vectors along the x,y,z-directions, and use the decompositions

$$\begin{gathered}\boldsymbol{u}(\omega,\boldsymbol{R})=\int d\boldsymbol{k}[f(\omega,\boldsymbol{k};z)\boldsymbol{Z}(\boldsymbol{k},\boldsymbol{r})+\\ +g(\omega,\boldsymbol{k};z)\boldsymbol{G}(\boldsymbol{k},\boldsymbol{r})+h(\omega,\boldsymbol{k};z)\boldsymbol{C}(\boldsymbol{k},\boldsymbol{r})],\\ \boldsymbol{F}(\omega,\boldsymbol{R})=\int d\boldsymbol{k}[F_z(\omega,\boldsymbol{k};z)\boldsymbol{Z}(\boldsymbol{k},\boldsymbol{r})+\\ +F_g(\omega,\boldsymbol{k};z)\boldsymbol{G}(\boldsymbol{k},\boldsymbol{r})+F_c(\omega,\boldsymbol{k};z)\boldsymbol{C}(\boldsymbol{k},\boldsymbol{r})],\end{gathered} \tag{4}$$

where the coefficients f,g,h and $F_{z,g,c}$ are determined shortly. The function $\boldsymbol{G}$ is the gradient of a plane wave ($\boldsymbol{G}$ from "gradient") and the function $\boldsymbol{C}$ is the vertical component of the *curl* of a plane wave ($\boldsymbol{C}$ from "*curl*"). These functions are constructed following the example of the vector spherical and cylindrical harmonics [29–33]. Equation (1) becomes

$$\begin{gathered}c_1^2f''+(\omega^2-c_2^2k^2)f-(c_1^2-c_2^2)kg'=-F_z,\\ c_2^2g''+(\omega^2-c_1^2k^2)g+(c_1^2-c_2^2)kf'=-F_g,\\ c_2^2h''+(\omega^2-c_2^2k^2)h=-F_c,\end{gathered} \tag{5}$$

where the derivatives are taken with repect to the variable z and

$$\begin{gathered}F_z=\frac{i}{2\pi}m_{z\alpha}k_\alpha\delta(z-z_0)+\frac{1}{2\pi}m_{zz}\delta'(z-z_0),\\ F_g=\frac{1}{2\pi k}m_{\alpha\beta}k_\alpha k_\beta\delta(z-z_0)-\frac{i}{2\pi k}m_{z\alpha}k_\alpha\delta'(z-z_0),\\ F_c=\frac{1}{2\pi k}(m_{x\alpha}k_\alpha k_y-m_{y\alpha}k_\alpha k_x)\delta(z-z_0)-\\ -\frac{i}{2\pi k}(m_{xz}k_y-m_{yz}k_x)\delta'(z-z_0),\end{gathered} \tag{6}$$

where $\alpha,\beta=x,y$. An anisotropic seismic moment is specific to earthquakes, which produce seismic waves (by a shear faulting). Although the solution can be obtained for this general form (or for any other force with components $F_{z,g,c}$), for vibrations, we prefer an isotropic tensor $m_{ij}=-m\delta_{ij}$, which simplifies greatly the calculations. The salient features of the solution are not affected by this simplification. For an isotropic seismic moment, the force components become

$$\begin{gathered}F_z=-\frac{m}{2\pi}\delta'(z-z_0)\ ,\ F_g=-\frac{mk}{2\pi}\delta(z-z_0),\\ F_c=0.\end{gathered} \tag{7}$$

An average over the directions of the wavevector $\boldsymbol{k}$ in Equation (6) leads to the similar force components (with different coefficients).

The boundary conditions are $\sigma_{iz}\mid_{z=0}=P_i$, where $\sigma_{ij}=c_2^2u_{ij}+(c_1^2-2c_2^2)div\boldsymbol{u}\cdot\delta_{ij}$ is the stress tensor, $u_{ij}=\frac{1}{2}(\partial_i u_j+\partial_j u_i)$ is the strain tensor and P_i are the components of the force acting upon the surface (force per unit cross-section area divided by the density of the body) [28]. For a free surface ($P_i=0$), we obtain the boundary conditions

$$(g'+kf)\mid_{z=0}=0\ ,\ \left[c_1^2f'-(c_1^2-2c_2^2)kg\right]_{z=0}=0,$$
$$h'\mid_{z=0}=0; \tag{8}$$

for a fixed surface, the boundary conditions are $f,g,h\mid_{z=0}=0$ ($\boldsymbol{u}\mid_{z=0}=0$). For $F_c=0$ (free or fixed surface), the solution of the third Equation (5) is $h=0$. We are left with the equations

$$c_1^2f''+c_2^2\kappa_2^2f-(c_1^2-c_2^2)kg'=\frac{m}{2\pi}\delta'(z-z_0),$$
$$c_2^2g''+c_1^2\kappa_1^2g+(c_1^2-c_2^2)kf'=\frac{mk}{2\pi}\delta(z-z_0), \tag{9}$$

where $\kappa_{1,2}^2=\omega^2/c_{1,2}^2-k^2$. The system of homogeneous Equation (9) has the eigenvalues $\kappa_{1,2}^2$; therefore, the free solution is

$$f_0=A\kappa_1e^{-i\kappa_1z}+iBke^{-i\kappa_2z}+c.c,$$
$$g_0=iAke^{-i\kappa_1z}+B\kappa_2e^{-i\kappa_2z}+c.c, \tag{10}$$

where the (complex) coefficients A and B are determined by the boundary conditions ($\kappa_{1,2}\neq0$). We use the convention of replacing $\kappa_{1,2}=\sqrt{\omega^2/c_{1,2}^2-k^2}$, $\omega^2/c_{1,2}^2-k^2>0$, by $i\kappa_{1,2}$ for $\kappa_{1,2}^2=k^2-\omega^2/c_{1,2}^2>0$. A particular solution of Equation (9) is obtained by Fourier transforming these equations with respect to the variable z; it is given by

$$f_p=sgn(z-z_0)\frac{m}{4c_1^2}e^{i\kappa_1|z-z_0|}+c.c,$$
$$g_p=-i\frac{mk}{4\kappa_1c_1^2}e^{i\kappa_1|z-z_0|}+c.c. \tag{11}$$

Finally, by making use of the boundary conditions for the free surface (Equation (8)), we obtain the full solution ($f,g=f_0,g_0+f_p,g_p$)

$$f=\frac{(\kappa_2^2-k^2)^2-4\kappa_1\kappa_2k^2}{\Delta}\frac{m}{4c_1^2}e^{-i\kappa_1(z+z_0)}+\frac{(\kappa_2^2-k^2)k^2}{\Delta}\frac{m}{c_1^2}e^{-i\kappa_1z_0-i\kappa_2z}+$$
$$+sgn(z-z_0)\frac{m}{4c_1^2}e^{i\kappa_1|z-z_0|}+c.c,$$
$$g=\frac{i\left[(\kappa_2^2-k^2)^2-4\kappa_1\kappa_2k^2\right]k}{\kappa_1\Delta}\frac{m}{4c_1^2}e^{-i\kappa_1(z+z_0)}-i\frac{(\kappa_2^2-k^2)\kappa_2k}{\Delta}\frac{m}{c_1^2}e^{-i\kappa_1z_0-i\kappa_2z}-$$
$$-i\frac{mk}{4\kappa_1c_1^2}e^{i\kappa_1|z-z_0|}+c.c, \tag{12}$$

where

$$\Delta=(\kappa_2^2-k^2)^2+4\kappa_1\kappa_2k^2 \tag{13}$$

is the determinant of the system of equations arising from the boundary conditions. For a fixed surface, the solution is

$$\begin{aligned} f = {} & \tfrac{k^2-\kappa_1\kappa_2}{k^2+\kappa_1\kappa_2}\tfrac{m}{4c_1^2}e^{-i\kappa_1(z+z_0)} - \tfrac{k^2}{k^2+\kappa_1\kappa_2}\tfrac{m}{2c_1^2}e^{-i\kappa_1 z_0 - i\kappa_2 z} + \\ & + sgn(z-z_0)\tfrac{m}{4c_1^2}e^{i\kappa_1|z-z_0|} + c.c, \\ g = {} & \tfrac{i(k^2-\kappa_1\kappa_2)k}{\kappa_1(k^2+\kappa_1\kappa_2)}\tfrac{m}{4c_1^2}e^{-i\kappa_1(z+z_0)} + i\tfrac{\kappa_2 k}{k^2+\kappa_1\kappa_2}\tfrac{m}{2c_1^2}e^{-i\kappa_1 z_0 - i\kappa_2 z} - \\ & - i\tfrac{mk}{4\kappa_1 c_1^2}e^{i\kappa_1|z-z_0|} + c.c. \end{aligned} \tag{14}$$

The above formulae give the functions $f(\omega,k;z)$ and $g(\omega,k;z)$. By inserting these functions in the first Equation (4) and performing the $\boldsymbol{k}$-integrations, we obtain the solution $\boldsymbol{u}(\omega;\boldsymbol{R})$; it can be represented as

$$\begin{aligned} \boldsymbol{u}(\omega;\boldsymbol{R}) = {} & \boldsymbol{e}_z\int_0^\infty dkkf(\omega,k;z)J_0(kr) - \\ & -\boldsymbol{e}_r\int_0^\infty dkkg(\omega,k;z)J_1(kr) + c.c., \end{aligned} \tag{15}$$

where $J_{0,1}(kr)$ are Bessel functions and $\boldsymbol{e}_r$ is the radial unit vector. Finally, a frequency Fourier transform gives the desired solution $\boldsymbol{u}(t,\boldsymbol{R})$. This completes formally the solution of the forced vibration problem. We may perform first the ω-integration. If the force is a harmonic oscillation with frequency ω_0, i.e., if $m(t) = m\cos\omega_0 t$ ($m(\omega) = \pi m\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$), the solution looks like

$$\begin{aligned} \boldsymbol{u}(t,\boldsymbol{R}) = {} & [\boldsymbol{e}_z\int_0^\infty dkkf(k;z)J_0(kr) - \\ & -\boldsymbol{e}_r\int_0^\infty dkkg(k;z)J_1(kr)]\cos\omega_0 t + c.c., \end{aligned} \tag{16}$$

where $f(k;z) = f(\omega_0,k;z)$ and $g(k;z) = g(\omega_0,k;z)$ are analytic functions and $m(\omega_0)$ is replaced by m. If the functions $f(\omega,k;z)$ and $g(\omega,k;z)$ have poles at some frequency ω_s, then contributions of the form $\sim\delta(\omega_s\pm\omega_0)$ may appear, which indicate resonances (for $\omega_0 = \pm\omega_s$). The solution given by Equation (16) has the typical form of a stationary vibration (standing wave), with the time dependence separated from the position dependence.

The description given above can also be applied to a two-dimensional space (a half-plane), defined by the coordinates $\boldsymbol{r} = (x,z)$, $z<0$, with the source placed at $x=0$, $z=z_0\le 0$. The vector functions given by Equation (3) become $\boldsymbol{Z}(k;x) = \boldsymbol{e}_z\frac{e^{ikx}}{\sqrt{2\pi}}$ and $\boldsymbol{G}(k;x) = i\boldsymbol{e}_x\frac{e^{ikx}}{\sqrt{2\pi}}$ (the function $\boldsymbol{C}(k;x)$ is irrelevant). The displacement is given by

$$\begin{aligned} \boldsymbol{u}(\omega;\boldsymbol{r}) = {} & \boldsymbol{e}_z\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dkf(\omega,k;z)e^{ikx} + \\ & + \boldsymbol{e}_x\tfrac{i}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dkg(\omega,k;z)e^{ikx} + c.c. \end{aligned} \tag{17}$$

(a result which can be obtained straightforwardly from Equation (15) by integrating over the y-coordinate).

Also, the above approach can be used straightforwardly for the standard wave equation, leading to the well-known mirror-image terms (the so-called problems of radio in Ref. [34]).

3. Surface-Wave Contribution

Let us perform first the ω-integration in Equation (15), by assuming a smooth function $m(\omega)$. The integrals are determined by the zeros of the denominators in Equations (12) and (14), if they exist. The branch lines associated to $\kappa_{1,2} = \sqrt{\omega^2/c_{1,2}^2 - k^2}$ are from $-c_2k$ to c_2k and from c_1k to $+\infty$, $-\infty$ to $-c_1k$, such that they bring no special contribution ($c_2 < c_1$) [35–37]. For a fixed surface, the denominators in Equation (14) do not vanish ($\kappa_1 \neq 0$). The so-called "lateral waves" associated to $\kappa_{1,2} = 0$, i.e., waves which do not depend on the coordinate z, are not produced by a source, so the boundary conditions cannot be satisfied and the solution remains undetermined. As is well known, for a free surface the determinant Δ (Equation (13)) is vanishing for only two frequencies $\omega_s = \pm c_2\xi_0 k$, where ξ_0 varies between 0.87 and 0.95 (it depends on the ratio c_2/c_1, which varies between $1/\sqrt{2}$ and 0) [22,28]. We note that this solution corresponds to damped waves ($\kappa_{1,2} \to i\kappa_{1,2}$). These are the well-known Rayleigh surface waves [21,28]. We need to compute the residues of the ω-integration of the functions f and g given by Equation (12). To this end, we expand the determinant Δ in the neighbourhood of ω_s,

$$\Delta \simeq \frac{k^2}{\alpha c_2^2}(\omega^2 - \omega_s^2), \tag{18}$$

where $\alpha = \sqrt{1-\xi_0^2}$. Here and in all the subsequent computations, we use the (numerical) approximations $\xi_0 \simeq 1$ and $c_2^2/c_1^2 \ll 1$, wherever appropriate. In addition, we limit ourselves to surface vibrations ($z = 0$). We obtain for the residues of the functions f and g

$$\begin{aligned} f_s &\simeq g_s/2\alpha \simeq \\ &\simeq \tfrac{\alpha c_2 k}{2c_1^2} sgn(t) Im\left[m(ck)e^{ickt}\right] e^{-k|z_0|}, \end{aligned} \tag{19}$$

where $c = c_2\xi_0$ ($\omega_s = ck$). For a harmonic oscillation with frequency ω_0, we have $m(ck) \sim \delta(ck \pm \omega_0)$ and the vertical (z) and the radial (horizontal, r) surface displacements are of the form $u_{z,r} \sim \sin\omega_0 \mid t \mid e^{-\omega_0|z_0|/c} J_{0,1}(\omega_0 r/c)$, according to Equation (16). It is worth noting that Equation (19) includes waves propagating in both time directions, i.e., it has an acausal structure, specific to vibrations.

If the force is absent, the boundary conditions (Equation (8)) give a homogeneous system of equations for the coefficients A and B of the free solutions (Equation (10)), which leads to $A, B \sim \delta(\omega^2 - \omega_s^2)$ (arising from $\Delta(A,B) \sim (\omega^2 - \omega_s^2)(A,B) = 0$); the solutions are free oscillations (vibrations) proportional to $\cos\omega_s t$, $\sin\omega_s t$ whose spatial dependence remains undetermined (as expected). We note that these free oscillations are governed by the (normal mode) frequency ω_s.

Now, we examine the special case of a temporal-impulse source $m(t) = mT\delta(t)$, where T is the short duration of the impulse. This is an improper case, since a temporal-impulse source requires causal conditions, while the vibration solution obtained above is acausal. The result which is obtained below for this special case is unphysical. However, we include it here for the sake of completeness and for the interest it may arouse occasionally in regard to the seismic waves. The vertical displacement (f_s) becomes

$$u_z \simeq \frac{\alpha m c_2 T}{2c_1^2} \int_0^\infty dk k^2 \sin ck \mid t \mid e^{-k|z_0|} J_0(kr). \tag{20}$$

By analytic continuation of the Weyl–Sommerfeld integral [34], we obtain

$$\int_0^\infty dke^{-k(|z_0|-ic|t|)}J_0(kr) = \frac{1}{\sqrt{r^2+(|z_0|-ic|t|)^2}} = \\ = \left[(R_0^2-c^2t^2)^2+4c^2z_0^2t^2\right]^{-1/4}e^{i\chi/2}, \tag{21}$$

where $R_0^2 = r^2 + z_0^2$ and

$$\tan\chi = \frac{2c \mid z_0 t \mid}{R_0^2 - c^2t^2}. \tag{22}$$

The vertical displacement becomes

$$u_z \simeq -\frac{\alpha m T}{2c_2c_1^2}\frac{\partial^2}{\partial t^2}I(t,r), \tag{23}$$

where

$$I(t,r) = \left[(R_0^2-c^2t^2)^2+4c^2z_0^2t^2\right]^{-1/4}\sin\frac{\chi}{2}. \tag{24}$$

The function $I(t,r)$ is vanishing for $r\to\infty$ and any t, and for $\mid t \mid\to\infty$ and any r; it has a (scissor-like) double-wall shape in the vicinity of the circle $R_0 = c \mid t \mid$ (for $z_0 \neq 0$); this feature propagates on the surface with velocity c. The propagating double-wall comes from the surface waves and the temporal-impulse $m(t) = mT\delta(t)$. The infinite spatial extension of this solution and its dependence on $\mid t \mid$ are specific to vibrations, while its propagating character reflects waves. This dual, hybrid character, arising from treating a propagating-wave problem as a vibration problem, does not reflect a seismic main shock, which has a causal nature and exhibits an abrupt wall-like structure, vanishing for distances beyond the wall position.

As a consequence of the sharp jump of the angle χ from $\pi/2$ to $-\pi/2$ (Equation (22)) in the vicinity of $R_0 = c \mid t \mid$, the function $I(t,r)$ can be represented in this vicinity as $I = \frac{\sqrt{2}}{2}(2R_0 \mid z_0 \mid)^{-1/2}sgn(R_0 - c \mid t \mid)$, such that the vertical displacement is represented as

$$u_z \simeq -\frac{\alpha m c_2 T}{2c_1^2}\frac{1}{\sqrt{R_0 \mid z_0 \mid}}\delta'(R_0 - c \mid t \mid). \tag{25}$$

A similar representation is valid for the main contribution to the radial (horizontal) displacement

$$u_r \simeq -\frac{\alpha^2 m c_2 T}{c_1^2 r}\sqrt{\frac{\mid z_0 \mid}{R_0}}\delta'(R_0 - c \mid t \mid). \tag{26}$$

As expected, these displacements are very different from the propagating waves generated by a temporal-impulse, which go like $\sim \delta'(R_0 - c_{1,2}t)/R_0$ [24]. A similar calculation for a half-plane (according to Equation (17)) leads to local maxima for the displacement components, placed at $x = \pm\sqrt{z_0^2 + c^2t^2}$ and propagating with velocities $\pm c$. These solutions are very different from the propagating waves in two dimensions, generated by a temporal impulse, which are divergent (proportional to $\delta(r - c_{1,2}t)/\sqrt{c_{1,2}^2t^2 - r^2}$). Finally, we note that for $z_0 = 0$ (source placed on the surface), the surface displacement is zero.

For $z \neq 0$, the calculations are similar. If we denote the above displacements for $z = 0$ by $u(z_0)$, we obtain $u(z) = 2u(z_0) - u(z + z_0)$ for the displacements corresponding to $z \neq 0$.

4. Concluding Remarks

Vector plane-wave functions have been introduced herein for the vibration problem of a homogeneous and isotropic elastic half-space, by analogy with the vector spherical and cylindrical harmonics. By using these functions, the Navier–Cauchy equation is reduced to a system of ordinary differential equations with respect to the perpendicular-to-surface coordinate, which is readily solved. The solution is represented as temporal and in-plane Fourier transforms. The vibration source is the seismic tensorial point force, which we use here with an isotropic seismic moment (forced vibrations). For a free surface, the solution includes the contribution of the Rayleigh surface waves. We compute explicitly this contribution to the surface displacement in the special limit of a temporal-impulse source. It is shown that this surface displacement includes waves propagating both in the future and in the past, extending over the entire surface, with a propagating (scissor-like) double-wall structure, which is different from the causal, abrupt wall-like seismic surface main shock, vanishing beyond the wall-position. After the temporal-impulse force ceases its action, we may have free oscillations (vibrations), which, for the half-space, are governed by the Rayleigh surface wave frequency.

Finally, we note that the decomposition in vector plane waves given by Equation (4) can also be used for the propagating-wave problem. It can be checked easily that this method leads straightforwardly to the spherical-shell waves generated by the temporal-impulse tensorial point force given in Ref. [24]. Also, the vector plane waves can be used for other vibration problems with axial geometry, such as the vibrations of a plane-parallel slab [27] or the vibrations at the plane interface of two solids [38].

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