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ORIGINAL ARTICLE

An inverse problem in seismology: derivation of the seismic source parameters from P and S seismic waves

Bogdan Felix Apostol

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Abstract This paper presents the solution of an inverse problem in Seismology, which aims at deriving the seismic source parameters from P and S seismic waves. In particular, the paper gives the deduction of the seismic-moment tensor. The problem is tackled in this paper under three particular circumstances. First, we use the amplitude of the far-field (P and S) seismic waves as input data. We use the analytical expression of the seismic waves in a homogeneous isotropic body with a seismic-moment source of tensorial forces, the source being localized both in space and time. We assume that the position of the seismic source is known. The far-field waves provide three equations for the six unknown parameters of the general tensor of the seismic moment, such that the system of equations is under-determined. Second, the Kostrov vectorial (dyadic) representation of the seismic moment for a shear faulting is used. This representation relates the seismic moment to the focal displacement in the fault and the orientation of the fault (moment-displacement relation); it reduces the seismic moment to four unknown parameters. Third, the fourth missing equation is derived from the energy conservation and the covariance condition. The four equations derived here are solved and the seismic moment is determined, as well as other parameters of the seismic source, like focal volume, focal slip, fault orientation, and duration of the seismic activity in the source. It turns out that the seismic moment is traceless, its magnitude is of the order of the elastic energy stored in the focal region (as expected), and the solution is governed by the unit quadratic form associated with the seismic-moment tensor (related to the magnitude of the longitudinal displacement in the P wave). A useful picture of the seismic moment is the conic represented by the associated quadratic form, which is a hyperbola (seismic hyperbola). This hyperbola provides an image for the focal region: its asymptotes are oriented along the focal displacement and the normal to the fault. Also, the special case of an isotropic seismic moment is presented. Numerical examples are provided for this procedure, and the limitations are discussed.

Keywords Seismic source · Inverse problem · Seismic waves · Seismic moment · Elasticity · Seismic hyperbola

1 Introduction

The determination of the seismic source parameters aims at getting information about the nature and the structure of the forces acting in earthquake's focus from measurements of the seismic waves at distances far away from the earthquake focus (at Earth's

B. F. Apostol (✉)
Institute for Earth's Physics, PO Box MG-35, Magurele-Bucharest, Romania
e-mail: afelix@theory.nipne.ro

surface). Particularly interesting in this context is the determination of the seismic-moment tensor. This is currently called the inverse probem in Seismology. We present here the solution of the inverse problem in Seismology by means of the P and S seismic waves propagating in a homogeneous isotropic body with localized tensorial forces, the Kostrov vectorial representation of the seismic moment for a fault (moment-displacement relation) and the energy conservation together with the covariance condition. The energy conservation is derived by equating the energy carried by the far-field seismic waves and the mechanical work done by forces in the focal region.

The seismic moment and the seismic energy are basic concepts in the theory of earthquakes (Bath 1968; Ben-Menahem and Singh 1981; Udias 1999; Aki and Richards 2009). The seismic moment has emerged gradually in the first half of the twentieth century, the first estimation of a seismic moment being done by Aki in (1966). The relations between the seismic moment, seismic energy, the mean displacement in the focal region, the rate of the seismic slip, and the earthquake magnitude are recognized today as very convenient tools for characterizing the earthquakes (Brune 1968; Kostrov 1974; Kostrov and Das 1988).

Seismic-moment tensors are routinely determined from teleseismic data for stronger earthquakes (with magnitude higher than 5; Dziewonski and Anderson 1981, Sipkin 1982, Kawakatsu 1995). Regional data are needed for smaller earthquakes, especially long-period waveforms (Bernardi et al. 1995; Giardini 1992). Also, the simultaneous inversion of body and surface waves is used (Honda and Seno 1989), as well as intermediate-period surface waves (Arvidsson and Ekstrom 1998). Most conveniently, synthetic seismograms with fitting parameters (like, for instance, location coordinates) are compared with data recorded from several stations. The determination of the seismic-moment components M_{ij} $(i, j = 1, 2, 3)$ is performed by using information provided by far-field seismic waves at different locations and times (Gilbert 1973; Saikia and Herrmann 1985; Jost and Herrmann 1989; Shomali and Slunga 2000; Shomali 2001; Ekstrom et al. 2012; Vallee 2013), or free oscillations of the earth, long-period surface waves, supplemented with additional information (the so-called constraints, Ben-Menahem and Singh 1981 and references therein). Besides noise, the information used in these procedures may reflect particularities of the structure of the focal region and the focal mechanism which are not included in equations, like the structure factor of the focal region, both spatial and temporal, or deviations from homogeneity and isotropy. In addition, waves measured at different locations (or times) may lead to overdetermined systems of equations for the unknowns M_{ij}, and the solutions must be "compatibilized." A proper procedure of compatibilization may lead to redundant equations, if the covariance of the equations is not ensured. The covariance is understood in this paper as the invariance of the form of the equations to translations and rotations (independence of the reference frame). We may add that the normal modes of the pure free oscillations do not imply a source of waves, while surface waves, having sources on the surface, have an indirect connection to the body waves generated in the focal region. Surface displacement in the main shock of an earthquake is often used, which, also, has an indirect relevance for the earthquake source and mechanism.

We present here a way of determining (analytically) the seismic moment for a shear faulting (as well as for an isotropic source) by using the P and S far-field waves generated by a time-localized tensorial point source in a homogeneous isotropic body. Though the solution presented here has rather a theoretical character, it may serve as an instance of the difficulties presented by the inverse problem and may throw light upon the complex relationships which exist between earthquake characteristics and various parameters of the seismic source. The waves produced by extended sources imply additional information regarding the spatial and temporal structure factors; the inverse problem in this case is a more complex problem, which remains beyond the aim of the present paper.

The data considered in the present paper (input parameters) are the position of the focal source, the displacement vectors produced by the P and S seismic waves measured at Earth's surface and material constants of the body (density, wave velocities). The information provided by these data is the magnitude of the longitudinal (P-wave) displacement (one parameter) and the transverse-wave displacement vector (S-wave, two parameters; we assume that the direction of the earthquake focus is known). These data provide three independent parameters, related to the six components of the seismic moment by three equations. They may be viewed as a minimal set of independent data. On the other hand, according to Kostrov representation, the seismic moment is characterized by its magnitude and the fault orientation and the fault slip, which are two mutualy perpendicular unit vectors. This information includes four independent parameters. We can see, on the one hand, according to Kostrov representation, that only four out of six components of the seismic moment are independent and, on the other hand, we need a fourth equation in order to determine the four independent components of the seismic focus. We provide in this paper the fourth equation, which is the equation of energy conservation together with the covariance condition. The covariance condition reduces the four independent components of the seismic moment to three, which makes possible the determination of the seismic moment from the seismic-wave displacement. Also, we show that an image of the forces acting in the focal region and the geometry of the fault can be obtained by a so-called "seismic hyperbola." A similar procedure is presented for an isotropic seismic source.

2 Basic assumptions

It is widely assumed that typical tectonic earthquakes originate in a localized focal region (focus), with dimensions much shorter than the distance to the observation point (and the seismic wavelengths). The tensorial seismic force density

$$F_i = M_{ij}\partial_j\delta(\mathbf{R}-\mathbf{R}_0) \tag{2.1}$$

is used for the seismic focus (Ben-Menahem and Singh 1981; Aki and Richards 2009; Apostol 2017a) where M_{ij} is the tensor of the seismic moment, δ is the Dirac delta function and $\mathbf{R}_0$ is the position of the focus (hypocentre). We assume that the position $\mathbf{R}_0$ is a known parameter. The labels i, j denote the Cartesian axes and summation over repeating suffixes is assumed (throughout this paper). The seismic tensor M_{ij} is a symmetric tensor, which, in general, has six components. It may be decomposed into double-couple (shear faulting) and dipole components and an isotropic component; departure from double-couple components reflects a complex shear faulting, tensile faulting, volcanic morphology, etc. (Ben-Menahem and Singh 1981; Zahradnik et al. 2008a, b, Frohlich 1994; Julian et al. 1998; Ross et al. 2015). The force given by Eq. (2.1) is a generalization of the double-couple representation of the seismic force. Indeed, let us assume a force density $\mathbf{F}(\mathbf{R}) = \mathbf{f}g(\mathbf{R})$, where $\mathbf{f}$ is the force and $g(\mathbf{R})$ is a distribution function; a point couple associated with a force acting along the ith direction can be represented as

$$\begin{aligned} &f_i g(x_1+h_1, x_2+h_2, x_3+h_3) - f_i g(x_1, x_2, x_3) \\ &\simeq f_i h_j \partial_j g(x_1, x_2, x_3) \quad , \end{aligned} \tag{2.2}$$

where h_j, $j = 1, 2, 3$, are the components of an infinitesimal displacement $\mathbf{h}$; x_i, $i = 1, 2, 3$, are the coordinates of the position $\mathbf{R}$ and ∂_j denotes the derivative with respect to x_j. The force moment (torque) $t_{ij} = f_i h_j$ is generalized in Eq. (2.2) to a symmetric tensor M_{ij}, which is the seismic moment entering Eq. (2.1); in addition, the distribution $g(\mathbf{R})$ can be replaced by $\delta(\mathbf{R}-\mathbf{R}_0)$ for a spatially localized focal region. The δ-function used in Eq. (2.1) is an approximation for the shape of the focal region. In Eq. (2.1) the focus is viewed as being localized over a distance of order l (volume of order l^3), much shorter than the distance R to the observation point ($l \ll R$), according to the δ-function representation.

The seismic moment depends on the time t; we may write $M_{ij}(t) = M_{ij}h(t)$, where $h(t)$ is a positive function, localized at $t = 0$, which includes the time dependence of the seismic moment; we assume $max[h(t)] = h(0) = 1$ and denote by T the (short) duration of the seismic activity of the source; the time T is much shorter than any time of interest, such that we may view the function $h(t)$ as being represented by $T\delta(t)$. The particular case $h(t) = T\delta(t)$ is called an elementary earthquake (Apostol 2017a).

For a homogeneous isotropic body the seismic waves generated by the tensorial force given by Eq. (2.1) are governed by the equation of the elastic waves

$$\begin{aligned} &\ddot{u}_i - c_t^2\Delta u_i - (c_l^2 - c_t^2)\partial_i div\mathbf{u} \\ &= \frac{1}{\rho}M_{ij}(t)\partial_j\delta(\mathbf{R}) \quad , \end{aligned} \tag{2.3}$$

where u_i are the components of the displacement vector $\mathbf{u}$, $c_{l,t}$ are the velocities of the longitudinal and tranverse waves, ρ is the density and $\mathbf{R}$ is the position vector drawn from the focus (taken as the origin of the reference frame) to the observation point. The solution of this equation (Ben-Menahem and Singh 1981; Aki and Richards 2009; Apostol 2017a) can be written as $\mathbf{u} = \mathbf{u}^n + \mathbf{u}^f$, where

$$u_i^n = -\frac{1}{4\pi\rho c_t^2}\frac{M_{ij}x_j}{R^3}h(t-R/c_t) + \frac{1}{8\pi\rho R^3}\left(M_{jj}x_i + 4M_{ij}x_j - \frac{9M_{jk}x_ix_jx_k}{R^2}\right)\cdot \cdot\left[\frac{1}{c_l^2}h(t-R/c_l) - \frac{1}{c_t^2}h(t-R/c_t)\right] \tag{2.4}$$

is the near-field displacement (R comparable with l) and

$$u_i^f = -\frac{1}{4\pi\rho c_t^3}\frac{M_{ij}x_j}{R^2}h'(t-R/c_t) - \frac{1}{4\pi\rho}\frac{M_{jk}x_ix_jx_k}{R^4}\cdot \cdot\left[\frac{1}{c_l^3}h'(t-R/c_l) - \frac{1}{c_t^3}h'(t-R/c_t)\right] \tag{2.5}$$

is the far-field displacement ($R \gg l$). The near-field region is defined by distances R of the order l, while the far-field region is defined by distances R much larger than l. The short duration T of the seismic event (duration of activity of the focus) enters Eqs. (2.4) and (2.5) through $h(t)$ and the derivative $h'(t)$, which is of the order $1/T$. The displacement vectors given by Eqs. (2.4) and (2.5) include the longitudinal wave (denoted by suffix l), propagating with velocity c_l, and the transverse wave (suffix t), propagating with velocity c_t; in the far-field region the displacement vectors of the longitudinal wave (P wave) and the transverse wave (S wave) are mutually orthogonal (this is not so for the l, t-waves in the near-field region). As long as the function $h(t)$ may be viewed as a localized function, the magnitude of the displacement vectors varies as $1/R^2$ for the near-field wave and $1/R$ for the far-field waves. Their direction is determined by the tensor of the seismic moment M_{ij} (in particular the vector with components $M_{ij}x_j$). The far-field waves given in Eq. (2.5) are shell spherical waves with a thickness of the order $\Delta R \simeq c_{l,t}T$. A superposition of forces given by Eq. (2.1), localized at different positions $\mathbf{R}_0$ and different times, corresponds to a structured focus, and the elementary displacement given by Eqs. (2.4) and (2.5) gives access to the structure factor of the focal region (Apostol 2017a).

3 Far-field seismic waves

It is convenient to introduce the notations

$$M_i = M_{ij}n_j \ , \ \ M_0 = M_{ii} \ , \ \ M_4 = M_{ij}n_in_j \ , \tag{3.1}$$

where $\mathbf{n}$ is the unit vector along the radius drawn from the focus to the observation point (observation radius), $x_i = Rn_i$, and $h_{l,t} = h(t - R/c_{l,t})$; henceforth, we consider the unit vector $\mathbf{n}$ as a known vector. M_0 is the trace of the seismic-moment tensor and M_4 is the quadratic form associated to the seismic-moment tensor, constructed with the unit vector $\mathbf{n}$; we call it the unit quadratic form of the tensor. The vector $\mathbf{M}$ can be called the "projection" of the tensor along the focus-observation point direction (observation direction).

Making use of these notations, the seismic waves given by Eqs. (2.4) and (2.5) can be decomposed into l- and t-waves, written as $\mathbf{u}^n = \mathbf{u}_l^n + \mathbf{u}_t^n$,

$$\mathbf{u}_l^n = \frac{h_l}{8\pi\rho c_l^2 R^2}\left[(M_0 - 9M_4)\mathbf{n} + 4\mathbf{M}\right] \ , \qquad \mathbf{u}_t^n = -\frac{h_t}{8\pi\rho c_t^2 R^2}\left[(M_0 - 9M_4)\mathbf{n} + 6\mathbf{M}\right] \ , \tag{3.2}$$

and $\mathbf{u}^f = \mathbf{u}_l^f + \mathbf{u}_t^f$,

$$\mathbf{u}_l^f = -\frac{h_l'}{4\pi\rho c_l^3 R}M_4\mathbf{n} \ , \ \ \mathbf{u}_t^f = \frac{h_t'}{4\pi\rho c_t^3 R}\left(M_4\mathbf{n} - \mathbf{M}\right) \ . \tag{3.3}$$

For numerical purposes, we take the "maximum deviation" of the near-field displacement $\mathbf{u}_{l,t}^n$ (with its sign) for $t = R/c_{l,t}$, i.e., we take $h_{l,t}(0) = 1$. Equally well, we can take the average values of the vectors $\mathbf{u}_{l,t}^n$ over the support T of the functions $h_{l,t}$, or ΔR, which is of the order $c_{l,t}T$. Henceforth, $h_{l,t}$ in Eq. (3.2) are understood as $h_{l,t}(0) = 1$. The functions $h_{l,t}'$ are scissor-like functions ("double-shock" functions), with two sides with opposite signs (corresponding to $t > 0$ or $t < 0$), extending over T, or the distance ΔR; their "maximum deviations" are of the order $\pm 1/T$; for numerical estimations, it is convenient to introduce the notations $\mathbf{v}_{l,t} = \mathbf{u}_{l,t}^f / Th_{l,t}'$ and take the "maximum deviation" of these functions (with their sign), on any side of the functions $h_{l,t}'$, the same side for $\mathbf{v}_l$ and $\mathbf{v}_t$ ($\mathbf{v}_{l,t}$ may depend on the side of the functions $h_{l,t}'$, since the functions $h_{l,t}(t)$ are not necessarily symmetric with respect to $t = 0$). Similarly, we can take the average values of $\mathbf{v}_{l,t}$ over any side of the functions $h_{l,t}'$ (the same for $\mathbf{v_l}$ and $\mathbf{v}_t$). We consider that the displacement vectors $\mathbf{v}_{l,t}$ are accessible experimentally. We take them as input data for our problem. The amplitude vector of the P wave is represented by the vector $\mathbf{v}_l$, while the amplitude vector of the S wave is represented by the vector $\mathbf{v}_t$. Making use of these notations, Eq. (3.3) become

$$\mathbf{v}_l = -\frac{1}{4\pi\rho T c_l^3 R} M_4 \mathbf{n}, \mathbf{v}_t = \frac{1}{4\pi\rho T c_t^3 R}\left(M_4\mathbf{n} - \mathbf{M}\right). \tag{3.4}$$

We note that the vectors $R^2\mathbf{u}^n_{l,t}$ and $R\mathbf{v}_{l,t}$ depend on the density ρ, the duration T, the seismic moment and the elastic coefficients of the body (velocities of the elastic waves); if local deviations from this pattern are observed, the body is not locally homogeneous and isotropic (or the focus is not localized, or the seismic event is more complex).

The displacement in the far-field waves is determined by three independent parameters: the magnitude of the vectors $\mathbf{v}_{l,t}$ (two parameters) and the direction of the transverse vector $\mathbf{v}_t$ (one parameter). Consequently, we may view the equations

$$\mathbf{M} = -4\pi\rho T R\left(c_l^3\mathbf{v}_l + c_t^3\mathbf{v}_t\right) \quad , \tag{3.5}$$

derived from Eq. (3.3), as three independent equations for the six unknown components M_{ij} of the seismic moment; by multipling by n_i and summing over i, we get the first equation (3.3),

$$M_4 = M_{ij}n_in_j = -4\pi\rho T R c_l^3(\mathbf{v}_l\mathbf{n}) = -4\pi\rho T R c_l^3 v_l\,, \tag{3.6}$$

which is not independent of the three equations written above. We view $\mathbf{v}_{l,t}$ as (known) quantities measured experimentally, and ρ, R, $c_{l,t}$ as known parameters; duration T will be determined shortly. A simple observation would show that for given displacements $\mathbf{v}_{l,t}$ and given T we may solve equations (3.5) and get the three independent components of the seismic moment M_{ij}.. Unfortunately, leaving aside that the other three components are left as free parameters by such a procedure, the measurement of the duration T from $\Delta r/c_{l,t}$, where Δr is the projection of ΔR on Earth's surface, is dependent on the local frame, and, consequently, it does not provide a suitable input data for covariant equations.

We note in Eqs. (3.5) and (3.6) the consistency (compatibility) relation $M^2 > M_4^2$, derived from $v_t^2 > 0$ ($v_{l,t}$ denote the magnitudes of the vectors $\mathbf{v}_{l,t}$). The problem discussed in this paper consists in determining the tensor M_{ij} from the displacement $\mathbf{v}_{l,t}$ in the far-field (P and S) waves, making use of additional, model-related, information. The model we use is provided by the fault geometry of the focal zone. We can see that only three components of the seismic moment M_{ij} are independent. We determine the seismic-moment tensor by means of the vectors $\mathbf{M}$ and $\mathbf{n}$ (experimentally accessible). The special case of an isotropic moment is presented. We note that (3.4) are manifestly covariant. Also, we note that having known $\mathbf{M}$ and M_4 we can have access to the near-field diplacement given by Eq. (3.2), provided we know M_0.

4 Energy of earthquakes

If we multiply (2.3) by $\dot{u}_i$ and perform summation over the suffix i, we get the law of energy conservation

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\frac{1}{2}\rho\dot{u}_i^2 + \frac{1}{2}\rho c_t^2(\partial_j u_i)^2 + \frac{1}{2}\rho(c_l^2 - c_t^2)(\partial_i u_i)^2\right] \\ &-\rho c_t^2\partial_j(\dot{u}_i\partial_j u_i) - \rho(c_l^2 - c_t^2)\partial_j(\dot{u}_j\partial_i u_i) \\ &= \dot{u}_i M_{ij}(t)\partial_j\delta(\mathbf{R}) \ .\end{aligned} \tag{4.1}$$

According to this equation, the external force performs a mechanical work in the focus ($\dot{u}_i M_{ij}(t)\partial_j\delta(\mathbf{R})$ per unit volume and unit time). The corresponding energy is transferred to the waves (the term in the square brackets in Eq. (4.1)), which carry it through the space (the term including the div in Eq. (4.1)). It is worth noting that outside the focal region the force is vanishing. Also, the waves do not exist inside the focal region. Therefore, limiting ourselves to the displacement vector of the waves, we have not access to the mechanical work done by the external force in the focal region. This circumstance arises from the localized character of the focus.

In the far-field region, we can use the decomposition $\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$ into longitudinal and transverse waves, where $curl\mathbf{u}_l = 0$ and $div\mathbf{u}_t = 0$; this decomposition leads to

$$\frac{\partial e_{l,t}}{\partial t} + c_{l,t}div\mathbf{s}_{l,t} = 0 \quad , \tag{4.2}$$

where

$$e_{l,t} = \frac{1}{2}\rho\left(\dot{u}^f_{l,ti}\right)^2 + \frac{1}{2}\rho c_{l,t}^2\left(\partial_i u^f_{l,tj}\right)^2 \tag{4.3}$$

is the energy density and

$$s_{l,ti} = -\rho c_{l,t} \dot{u}^f_{l,tj} \partial_i u^f_{l,tj} \quad ; \tag{4.4}$$

$c_{l,t} s_{l,ti}$ are energy flux densities per unit time (energy flow). From Eq. (4.2) we can see that the energy is transported with velocities $c_{l,t}$ (as it is well known). The volume energy $E = \int d\mathbf{R}(e_l + e_t)$ is equal to the total energy flux

$$\begin{aligned} \Phi &= -\int dt d\mathbf{R} \left(c_l div \mathbf{s}_l + c_t div \mathbf{s}_t \right) \\ &= -\int dt \oint d\mathbf{S} \left(c_l \mathbf{s}_l + c_t \mathbf{s}_t \right) \quad . \end{aligned} \tag{4.5}$$

Making use of Eq. (3.3) and taking $h'' = -1/T^2$ as an order-of-magnitude estimate, we get

$$E = \Phi = \frac{4\pi\rho}{T} R^2 \left(c_l v_l^2 + c_t v_t^2 \right) \quad ; \tag{4.6}$$

this relation gives the energy released by the earthquake in terms of the displacement measured in the far-field region and the (short) duration of the earthquake. From Eq. (3.4) we get the relation

$$E = \frac{1}{4\pi\rho c_t^5 T^3} \left[M^2 - \left(1 - c_t^5 / c_l^5 \right) M_4^2 \right] \tag{4.7}$$

between energy and the seismic moment.

5 Geometry of the focal region

Let us consider a point torque $t_{ij} = f_i h_j$, where h_j are viewed as infinitesimal distances and f_i denote the components of a force $\mathbf{f}$; the force $\mathbf{f}$ originates in a volume force density $\partial_j \sigma_{ij}$, where σ_{ij} is the stress tensor; the latter can be expressed as $\sigma_{ij} = 2\mu u_{ij} + \lambda u_{kk} \delta_{ij}$, where μ and λ are the Lame coefficients ($c_l^2 = (2\mu + \lambda)/\rho$, $c_t^2 = \mu/\rho$), $u_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ is the strain tensor and $\mathbf{u}$, with components u_i, is the displacement vector (Landau and Lifshitz 1986). The components of the force can be written as

$$\begin{aligned} f_i &= \int d\mathbf{r} \partial_k \sigma_{ik} \\ &= \mu \int d\mathbf{r} \partial_k^2 u_i + (\mu + \lambda) \int d\mathbf{r} \partial_k \partial_i u_k \\ &= \mu \oint dS \cdot s_k \partial_k u_i + (\mu + \lambda) \oint dS \cdot s_k \partial_i u_k \ , \end{aligned} \tag{5.1}$$

where the $\mathbf{r}$-integration is performed over the focal volume surrounded by the surface S and $\mathbf{s}$ is the unit vector normal to this surface. In order to get the torque we multiply Eq. (5.1) by h_j and use $\frac{\Delta u_k}{\Delta x_i} h_j = \Delta u_k \delta_{ij} = u_k \delta_{ij}$, where u_k is the displacement on the surface. These equalities follow from the point-like nature of the torque. We note that $\mathbf{u}$ here is the focal displacement, which is distinct from the displacement in the waves. It follows

$$t_{ij} = \mu S \cdot \overline{s_j u_i} + (\mu + \lambda) S \cdot \overline{s_k u_k} \delta_{ij} \quad , \tag{5.2}$$

where the overbar denotes the average over the surface with area S. This relation acquires a useful form for a localized (plane) fault. We assume that the fault focal region includes two plane-parallel surfaces, each with (small) area S, separated by a (small) distance d, sliding against one another. The focal area is determined by two (small) lengths $l_{1,2}$, $S = l_1 l_2$. In general, the lengths l_1, l_2, d are distinct; in order to ensure the compatibility with the localization provided by the δ-function (used in deriving the waves), we assume $l_1 = l_2 = d = l$. For such a model of localized fault the product $\overline{s_j u_i}$ may be replaced by $2 s_j \overline{u}_i$, where the vector $\mathbf{s}$ is the unit vector normal to the fault (we note that the integration over the surfaces perpendicular to the fault is zero, due to the opposing (sliding) displacements). In view of the small extension of the focal region, we may drop the average bar over u_i. In addition, this model of fault-slip implies $s_k u_k = 0$, i.e., the normal to the fault $\mathbf{s}$ and the focal displacement (fault slip) $\mathbf{u}$ are mutually orthogonal vectors. In order to distinguish the focal displacement from the displacement in the seismic waves, we attach the superscript 0 to the focal displacement. The seismic moment is obtained by symmetrizing the expression given by Eq. (5.2); we get the seismic moment

$$M_{ij} = 2\mu S \left(s_i u_j^0 + s_j u_i^0 \right) = 2\mu S u^0 \left(s_i a_j + a_i s_j \right) \quad , \tag{5.3}$$

where we introduce the unit vector $\mathbf{a}$ along the direction of the focal displacement; we write $u_i = u^0 a_i$, where u^0 is the magnitude of the focal displacement and $a_i^2 = 1$. We can see that the seismic moment is represented in Eq. (5.3) by two orthogonal vectors ($\mathbf{as} = 0$): the unit vector $\mathbf{a}$ along the focal displacement $\mathbf{u}^0$ and the unit vector $\mathbf{s}$, which gives the orientation of the fault. This is the moment-displacement relation derived by Kostrov (1974) and Kostrov and Das (1988) for the slip along a (point-like) fault surface (see also Ben-Menahem and Singh 1981, Aki and Richards 2009); it can be called a vectorial, or dyadic, representation of the seismic moment. We note the invariant $M_0 = M_{ii} = 0$, which shows that the seismic moment in this representation is a traceless tensor. This particularity gives access to the near-field waves (Eq. (3.2)), which become

$$\mathbf{u}_l^n = \frac{h_l}{8\pi\rho c_l^2 R^2}\left(4\mathbf{M} - 9M_4\mathbf{n}\right) \ ,$$
$$\mathbf{u}_t^n = -\frac{3h_t}{8\pi\rho c_t^2 R^2}\left(2\mathbf{M} - 3M_4\mathbf{n}\right) \qquad (5.4)$$

($\mathbf{M}$ and M_4 are given by Eqs. (3.5) and (3.6)). In addition, we note the relations $M_4^0 = M_{ij}s_is_j = 0$ and $M_i^0 = M_{ij}s_j = 2\mu S u^0 a_i$; the former relation shows that the quadratic form associated to the seismic moment in the focal region is degenerate (it is represented by a conic), while the latter relation shows that the "force" in the focal region is directed along the focal displacement; both relations are expected from the Kostrov construction of the tensor of the fault seismic moment (Fig. 1).

The relations $M_0 = 0$ and $M_4^0 = 0$ reduce the number of independent parameters of the tensor M_{ij} from six to four, a circumstance which can be checked directly on Eq. (5.3).

It is worth noting an uncertainty (indeterminacy) of the dyadic construction of the seismic-moment tensor. We can see from Eq. (5.3) that the seismic moment is invariant under the inter-change $\mathbf{s} \longleftrightarrow \mathbf{a}$. This means that from the knowledge of the seismic moment M_{ij} we cannot distinguish between the two orthogonal vectors $\mathbf{s}$ and $\mathbf{a}$ (fault direction and fault slip). Another symmetry of the seismic moment given by Eq. (5.3) is $\mathbf{s} \longleftrightarrow -\mathbf{a}$ (and $\mathbf{s} \longleftrightarrow -\mathbf{s}$, $\mathbf{a} \longleftrightarrow -\mathbf{a}$), which means that we cannot distinguish between the signs of the vectors $\mathbf{s}$ and $\mathbf{a}$ (as expected from the construction of the seismic moment in Eq. (5.3)); this uncertainty is shown in Fig. 2.

In Eq. (5.3) the seismic moment is determined by four parameters: three components of the displacement vector $\mathbf{u}^0$ and one component of the (transverse) unit vector $\mathbf{s}$. By using this vectorial representation, the number of independent parameters of the seismic moment is reduced from six to four. We have, up to this moment, only the three equations (3.5) for these unknown parameters. The considerations made above for the vectorial representation of the seismic moment provide another equation, relating the mechanical work W done in the focal region to the magnitude of the focal diplacement.

Indeed, from Eq. (4.1) the mechanical work in the focal region is given by

$$W = \int dt \int d\mathbf{R}\dot{u}_i^0(t)M_{ij}(t)\partial_j\delta(\mathbf{R}) \ ; \qquad (5.5)$$

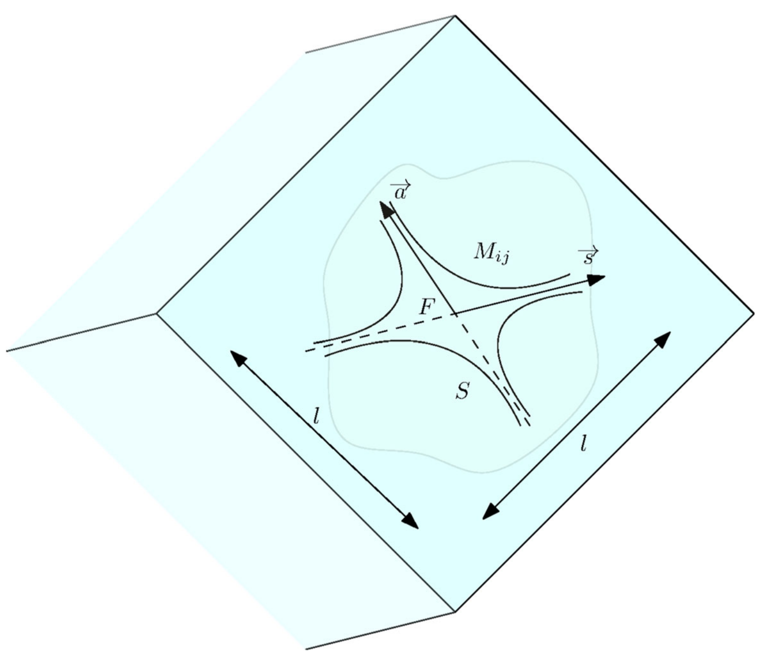

Fig. 1 A focal-fault cross-section with area S (dimension l, focus F); $\mathbf{s}$ is the unit vector normal to the fault and $\mathbf{a}$ is the unit vector of the focal displacement (in the plane of the fault); the seismic-moment tensor M_{ij} is represented by the rectangular hyperbola with the axes along the vectors $\mathbf{s}$ and $\mathbf{a}$

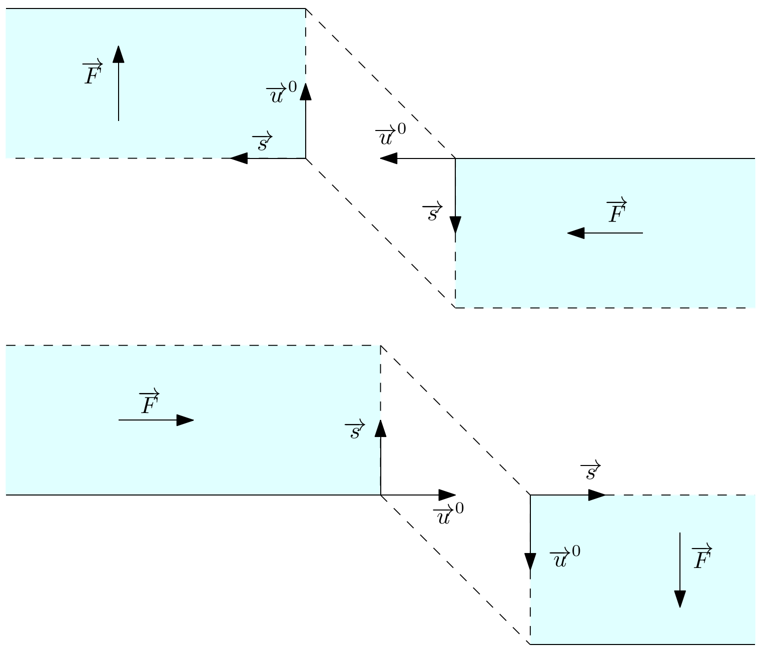

Fig. 2 Two couples of sliding displacements ($\mathbf{u}^0$) and two orthogonal orientations ($\mathbf{s}$) in a fault focal region, illustrating the indeterminacy in the Kostrov construction of the seismic moment; $\mathbf{F}$ denote the forces which give the torque

we may assume $\dot{u}_i^0(t) = \dot{h}(t)u_i^0$, and, since $M_{ij}(t) = M_{ij}h(t)$, we get

$$W = \frac{1}{2}\int d\mathbf{R}u_i^0 M_{ij}\partial_j\delta(\mathbf{R}) \ . \tag{5.6}$$

In this equation we may view the function $\delta(\mathbf{R})$ as corresponding to the shape of the focal surface, such that we may replace $\partial_j\delta(\mathbf{R})$ by s_j/l^4; using $V = l^3$ for the focal volume, we get $W \simeq \frac{1}{2l}u_i^0 M_{ij}s_j$. Here, we may take approximately u^0 for l, which leads to $W \simeq \frac{1}{2}a_i M_{ij}s_j$. Therefore, making use of Eq. (5.3), we get $W \simeq \mu S u^0 = \mu V$; we can see that the mechanical work done in the focal region is of the order of the elastic energy stored in the focal region, as expected. By equating W with energy E (and Φ) given by Eq. (4.6), we get

$$\mu V = \frac{4\pi\rho}{T}R^2\left(c_l v_l^2 + c_t v_t^2\right) \ , \tag{5.7}$$

which can also be written as

$$V = \frac{4\pi}{c_t^2 T}R^2\left(c_l v_l^2 + c_t v_t^2\right) \ . \tag{5.8}$$

Equation (5.8) gives the volume of the focal region in terms of the displacement in the far-field seismic waves (provided duration T is known); the seismic moment given by Eq. (5.3) can be written as

$$M_{ij} = 2\mu V\left(s_i a_j + a_i s_j\right) \ , \tag{5.9}$$

where μV can be inserted from Eq. (5.7). It remains to determine the vectors **a** and **s** by using Eq. (3.5) and the covariance condition, in order to solve completely the problem. We note that the elaborations done in Eq. (5.1) can be circumvented, in fact, since the torque can be immediately inferred from $t_{ij} = f_i h_j$ by $f_i \simeq 2\mu S u_i^0/l$ and $h_j \simeq l s_j$; we get $t_{ij} \simeq 2\mu V a_i s_j$.

We note here the representation

$$u_{ij}^0 = \frac{1}{2}\left(s_i a_j + a_i s_j\right) = \frac{1}{4\mu V}M_{ij} \tag{5.10}$$

for the focal strain, which follows immediately from the considerations made above on the geometry of the focal region. This equation relates the focal strain to the seismic moment; it may be used for assessing the accumulation rate of the seismic moment from measurements of the surface strain rate (Ward 1994; Savage and Simpson 1997).

It is worth noting that the estimations made above may be affected by errors in the numerical factors (of the order unity); such errors are related to the parameters T, l, the estimation of the derivatives $\partial_j\delta$, the assumption $l_1 = l_2 = d = l$, the volume $V = l^3$, etc. These errors affect the volume V in Eqs. (5.8) and (5.9). The errors in the seismic-moment parameters, especially those related to noise, have been analyzed recently (Mustac and Tkalcic 2016).

6 Determination of the seismic source parameters

Making use of the reduced moment $m_{ij} = M_{ij}/2\mu V$ and $m_i = M_i/2\mu V = M_{ij}n_j/2\mu V$, Eq. (5.9) leads to

$$s_i(\mathbf{na}) + a_i(\mathbf{ns}) = m_i \ ; \tag{6.1}$$

using Eqs. (3.5) and (5.7), the components m_i of the reduced moment are given by

$$m_i = -\frac{T^2}{2R}\cdot\frac{c_l^3 v_{li} + c_t^3 v_{ti}}{c_l v_l^2 + c_t v_t^2} \ . \tag{6.2}$$

We solve here the Eq. (6.1) for the unit vectors **a** and **s**, subject to the conditions

$$s_i^2 = a_i^2 = 1 \ , \ \ s_i a_i = 0 \ . \tag{6.3}$$

Since $M_0 = 0$ and $M^2 > M_4^2$, we have $m_0 = m_{ii} = 0$ and $m^2 > m_4^2$ (where $m_4 = m_{ij}n_i n_j$ and $m^2 = m_i^2$). From Eq. (6.2) we have $m_i < 0$. The compatibility condition $m^2 > m_4^2$ can be checked immediately from Eq. (6.2) (it arises from $v_t^2 > 0$). We write Eq. (6.1) as

$$\alpha\mathbf{s} + \beta\mathbf{a} = \mathbf{m} \ \ , \tag{6.4}$$

where we introduce two new notations $\alpha = (\mathbf{na})$ and $\beta = (\mathbf{ns})$. We assume that the vectors **s**, **a** and **n** lie in the same plane, i.e.,

$$\beta\mathbf{s} + \alpha\mathbf{a} = \mathbf{n} \ . \tag{6.5}$$

This condition determines the system of equations and ensures the covariance of the solution; Eq. (6.5) is the covariance condition. From Eqs. (6.4) and (6.5), we get

$$2\alpha\beta = m_4 \ , \ \ \alpha^2 + \beta^2 = m^2 = 1 \ . \tag{6.6}$$

The equality $m^2 = 1$ (covariance condition) has important consequences; it implies $M^2 = (2\mu V)^2$, such that we can write the seismic moment from Eq. (5.9) as

$$M_{ij} = M\left(s_i a_j + a_i s_j\right) \ ; \tag{6.7}$$

it follows the magnitude of the seismic moment $\left(M_{ij}{}^2\right)^{1/2} = \sqrt{2}M$ (Silver and Jordan 1982); M is the magnitude of the "projection" of the seismic-moment tensor along the observation radius. In addition, from $E = W = \mu V$ (5.6) we have $E = M/2 = \left(M_{ij}{}^2\right)^{1/2}/2\sqrt{2}$. The magnitude $\left(M_{ij}{}^2\right)^{1/2} = \sqrt{2}M = 2\sqrt{2}E$ may be used in the Hanks-Kanamori relation $\lg\left(M_{ij}{}^2\right)^{1/2} = 1.5M_w + 16.1$ (decimal logarithm), which defines the magnitude M_w of the earthquake (Hanks and Kanamori 1979; Gutenberg and Richter 1956); in terms of the earthquake energy this relation becomes $\lg E = 1.5(M_w - \lg 2) + 16.1$ (where $\lg 2 \simeq 0.3$). We note that an error of an order of magnitude in the seismic moment (M, E, $\left(M_{ij}{}^2\right)^{1/2}$) induces an error $\simeq 0.3$ in the magnitude M_w.

Further, from Eq. (6.2), the equality $m^2 = 1$ can be written as

$$\frac{T^4}{4R^2}\cdot\frac{c_l^6 v_l^2 + c_t^6 v_t^2}{\left(c_l v_l^2 + c_t v_t^2\right)^2} = 1 \ \ , \tag{6.8}$$

which gives the duration T in terms of the displacements $v_{l,t}$ measured at distance R. Inserting T in Eq. (5.8), we get

$$V^2 = \frac{8\pi^2 R^3}{c_t^4}\left(c_l v_l^2 + c_t v_t^2\right)\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/2} \tag{6.9}$$

and the magnitude of the seismic moment and the energy of the earthquake

$$\begin{aligned} M &= 2E = 2\mu V \\ &= 2\pi\rho(2R)^{3/2}\left(c_l v_l^2 + c_t v_t^2\right)^{1/2}\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/4} \end{aligned} \tag{6.10}$$

in terms of the displacements $v_{l,t}$ measured at distance R. In addition, eliminating R^2 between Eqs. (5.8) and (6.8) we can express the focal volume as

$$V = \frac{\pi T^3}{c_t^2}\cdot\frac{c_l^6 v_l^2 + c_t^6 v_t^2}{c_l v_l^2 + c_t v_t^2} \ . \tag{6.11}$$

The solutions of the system of Eq. (6.6) are given by

$$\alpha = \sqrt{\frac{1+\sqrt{1-m_4^2}}{2}} \ , \ \beta = sgn(m_4)\sqrt{\frac{1-\sqrt{1-m_4^2}}{2}} \tag{6.12}$$

and $\alpha \longleftrightarrow \pm\beta$, α, $\beta \longleftrightarrow -\alpha, -\beta$. Making use of Eqs. (6.2) and (6.8), the parameters m_i and m_4 are given by

$$m_i = -\frac{c_l^3 v_{li} + c_t^3 v_{ti}}{\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/2}} \ , \ m_4 = -\frac{c_l^3(\mathbf{v}_l\mathbf{n})}{\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/2}} . \tag{6.13}$$

Finally, we get the vectors

$$\begin{aligned} \mathbf{s} &= \frac{\alpha}{\alpha^2-\beta^2}\mathbf{m} - \frac{\beta}{\alpha^2-\beta^2}\mathbf{n} \ , \\ \mathbf{a} &= -\frac{\beta}{\alpha^2-\beta^2}\mathbf{m} + \frac{\alpha}{\alpha^2-\beta^2}\mathbf{n} \ ; \end{aligned} \tag{6.14}$$

from Eqs. (6.4) and (6.5); these solutions are symmetric under the operations $\mathbf{s} \longleftrightarrow \mathbf{a}$ ($\alpha \longleftrightarrow -\beta$) and $\mathbf{s} \longleftrightarrow -\mathbf{a}$ ($\alpha \longleftrightarrow \beta$, or α, $\beta \longleftrightarrow -\alpha$, $-\beta$). The seismic moment given by Eq. (6.7) is determined up to these symmetry operations.

We can see that the seismic-moment tensor given by Eq. (6.7) is determined by M (Eq. (6.10)) and the vectors $\mathbf{s}$ and $\mathbf{a}$ given by Eq. (6.14), with the coefficients α, β given by Eq. (6.12); the vector $\mathbf{n}$ is known and the vector $\mathbf{m}$ and the scalar m_4 are given by the experimental data (Eq. (6.13)). Equations (6.14) are manifestly covariant.

The eigenvalues of the seismic moment given by Eq. (6.7) are $\pm M$ (we leave aside the eigenvalue zero); the corresponding eigenvectors $\mathbf{w}$ are given by $\mathbf{aw} = \pm\mathbf{sw}$, which imply $\mathbf{mw} = \pm\mathbf{nw}$; the vectors $\mathbf{w}$ are directed along the bisectrices of the angles made by $\mathbf{s}$ and $\mathbf{a}$, or $\mathbf{m}$ and $\mathbf{n}$ ($\mathbf{w} \sim \mathbf{s} \pm \mathbf{a}$). The associated quadratic form $M_{ij}x_ix_j = const$ is a rectangular hyperbola in the reference frame defined by the vectors $\mathbf{s}$ and $\mathbf{a}$; by using the coordinates $u = \mathbf{sx}$ and $v = \mathbf{ax}$ in Eq. (6.7), the equation of this hyperbola is $uv = const/2M$. Actually, in the local frame (coordinates x_i), the quadratic form $M_{ij}x_ix_j = const$ is a degenerate hyperboloid, consisting of a family of parallel hyperbolas displaced along the third axis (perpendicular to the u- and v-axes). Making use of Eqs. (6.7) and (6.14), this quadratic form can also be written as

$$2\xi\eta - m_4\left(\xi^2 + \eta^2\right) = const \quad , \tag{6.15}$$

where the coordinates $\xi = m_ix_i$ and $\eta = n_ix_i$ are directed along the vectors $\mathbf{m}$ and $\mathbf{n}$, respectively. The asymptotes of this hyperbola are $\xi = m_4\eta/\left(1+\sqrt{1-m_4^2}\right)$ and $\eta = m_4\xi/\left(1+\sqrt{1-m_4^2}\right)$ (corresponding to the asymptotes $u = (\alpha\xi - \beta\eta)/(\alpha^2 - \beta^2) = 0$ and $v = (-\beta\xi + \alpha\eta)/(\alpha^2 - \beta^2) = 0$) (Fig. 3).

Finally, by making use of Eq. (6.14) in Eq. (6.7), we get the solution for the seismic moment

$$M_{ij} = \frac{M}{1-m_4^2}\left[m_in_j + m_jn_i - m_4\left(m_im_j + n_in_j\right)\right] , \tag{6.16}$$

where M is given by Eq. (6.10) and m_i, m_4 are given by Eq. (6.13); the focal strain is $u_{ij}^0 = M_{ij}/2M$ (Eq. (5.10)). In Eq. (6.16) there are only three independent components of the seismic tensor, according to the equations $m_{ij}n_j = m_i$ ($m_{ij} = M_{ij}/M$): it is assumed that the vectors $\mathbf{n}$ and $\mathbf{m}$ are known (6.13) from experimental data, such that these equations can

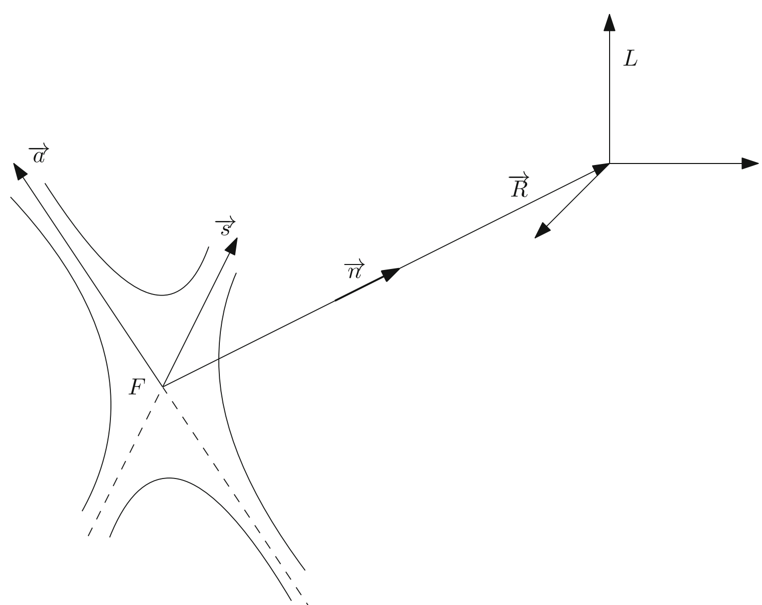

Fig. 3 The hyperbola of the displacement (**a**) in the fault plane (fault direction **s**) at the focus (F), seen from the local frame L

be viewed as three conditions imposed upon the six components M_{ij}. Also, we can see that there exist only three independent components of the seismic tensor M_{ij} from the conditions $M_0 = M_{ii} = 0$, $M_{ij}s_js_i = 0$ (or $M_{ij}a_ia_j = 0$) and $m_i^2 = 1$. The later equality arises from the covariance condition, which, together with the energy conservation, determines the duration T of the seismic activity in the focus, the volume V of the focal region and the magnitude parameter M of the seismic moment. Equation (6.16) solves completely the inverse problem.

7 Isotropic seismic moment

An isotropic seismic moment $M_{ij} = -M\delta_{ij}$ is an interesting particular case, since it can be associated with seismic events caused by explosions (e.g., Knopoff and Randall 1970; Minson and Dreger 2008; Patton and Taylor 2011). In this case the transverse displacement is vanishing ($\mathbf{u}_t^{n,f} = 0$), $\mathbf{M} = -M\mathbf{n}$, $M_4 = -M$ and $\mathbf{v}_l = (R/c_lT)\mathbf{u}_l^n$ (Eqs. (3.2) and (3.4)); from Eqs. (3.5) and (4.6), we get

$$\mathbf{M} = -4\pi\rho TRc_l^3\mathbf{v}_l \ \ , \ \ E = \frac{4\pi\rho R^2}{T}c_lv_l^2 \tag{7.1}$$

we can see that $\mathbf{v}_l\mathbf{n} > 0$ corresponds to $M > 0$ (explosion), while the case $\mathbf{v}_l\mathbf{n} < 0$ corresponds to an implosion. The focal zone is a sphere with radius of the order l, and the vectors $\mathbf{s}$ and $\mathbf{a}$ are equal ($\mathbf{s} = \mathbf{a}$) and depend on the point on the focal surface; the magnitude of the focal displacement is $u^0 = l$. The considerations made above for the geometry of the focal region lead to the representation

$$M_{ij} = -2V(2\mu + \lambda)\delta_{ij} = -2\rho c_l^2V\delta_{ij} \ \ , \tag{7.2}$$

where $V = Sl$ denotes the focal volume and S is the area of the focal region (we note that t_{ij} changes sign in Eq. (5.2)). Similarly, the energy is $E = W = \frac{1}{2}M$ ($M > 0$), such that, making use of Eq. (7.1), we get $c_lT = \sqrt{2Rv_l}$,

$$M = 2\pi\rho c_l^2(2Rv_l)^{3/2} = 2\rho c_l^2V \ \ , \tag{7.3}$$

and the focal volume $V = \pi(2Rv_l)^{3/2}$. These equations determine the seismic moment and the volume of the focal region from the displacement v_l measured at distance R. A superposition of shear faulting and isotropic focal mechanisms cannot be resolved, because the longitudinal displacement $\mathbf{v}_l$ includes indiscriminately contributions from both mechanisms.

8 Numerical examples: validity and limitations

We include here a numerical example, for the earthquake with magnitude $M_w = 4.6$, which occurred on March 14, 2018, in Vrancea (Romania, latitude 45.67°, longitude 26.58°) at 139-km depth (Institute of Earth's Physics, Magurele, Romania, http://www.infp.ro/). The displacement, estimated from measurementa made at Bucharest, is $v_l = 5 \times 10^{-2}$ cm and $\mathbf{v}_t = (2.7, 2.6, -1) \times 10^{-2}$ cm (this displacement is one of the main sources of experimental errors; we estimate 20% an error). According to the theoretical description presented in this paper, we get $M_w = 4.3$, $V = 2.9 \times 10^{10}$ cm^3 and the tensor of the seismic moment (in units 10^{22} erg)

$$M_{ij} = \begin{pmatrix} -0.35 & 1.52 & -2.04 \\ 1.52 & 1.35 & -0.41 \\ -2.04 & -0.41 & -0.98 \end{pmatrix} \tag{8.1}$$

(for density $\rho = 5.5$ g/cm^3 and velocities $c_l = 3$ km/s, $c_t = 7$ km/s). The coordinates of the intersections of the vectors $\mathbf{s}$ (fault orientation) and $\mathbf{a}$ with Earth's surface are latitude 40.64°, 45.5° and longitude 29.28°, 25.1°, respectively. We can see that the computed magnitude (4.3; 0.2 error) is close to the reported magnitude (4.6). The errors of the input data (displacement and coordinates of the focus) cause often a discrepancy between the direction of the longitudinal displacement (vector $\mathbf{v}_l$) and the direction of the focus (vector $\mathbf{n}$), which reflects itself in deviations from the perpendicularity between the displacement vectors $\mathbf{v}_l$ and $\mathbf{v}_t$. In this case the angle made by these two vectors may differ by up 10° (or more) from 90°.

Another numerical example is the earthquake with magnitude $M_w = 6$ of October 27, 2004, Vrancea (latitude 45.84°, longitude 26.63°), depth 105-km depth. The displacement estimated at Bucharest is $v_l = 0.28$ cm and $\mathbf{v}_t = (-0.39, -0.35, 0.45)$ cm. We get the magnitude $M_w = 5.4$, a seismic tensor of the order 10^{23} erg (except for the component $M_{33} = -6.23 \times 10^{21}$ erg) and a focal volume $V = 1.17 \times 10^{12}$ cm^3. These estimations are affected by errors, which arise mainly from the local variability of the amplitude of the P and S seismic waves and the inherent noise. The solution given here to the seismological inverse problem has only a theoretical character. Nevertheless, it may serve both as an example of a mathematically elegant solution and a rapid way for a rough estimate of the seismic moment and parameters of the seismic source. In addition, it may throw light upon the complex connections between the earthquake effects and the seismic-source parameters.

It is worthwhile emphasizing the validity conditions and the limitations of the procedure presented here for solving the inverse seismological problem. The solution is valid for a seismic focus (seismic source) localized both in space and time. The solution makes use of far-field P and S seismic waves computed for a homogeneous isotropic body with localized tensorial forces and measured at an observation point on Earth's surface. It is assumed that the position of the focal source is known. The known (input) parameters are the vector from the seismic source to the observation point (unit vector $\mathbf{n}$ and radius R), Earth's density ρ, the velocities $c_{l,t}$ of the P and S waves, the amplitude of the longitudinal (P wave) displacement v_l and the amplitude of the transverse (S wave) displacement vector $\mathbf{v}_t$. All these input parameters are, in principle, measurable (can be determined), of course with errors. A statistical analysis of errors is necessary, such that the results are given with their own errors. For instance, it is sufficient, in principle, to measure the wave amplitudes only at one location, but it is advisable to measure them at several locations and compute the mean value and the error of the results. A typical error is related to the measured longitudinal displacement vector $\mathbf{v}_l$, which, often, is not directed along the vector $\mathbf{n}$, or/and is not perpendicular to the measured transverse vector $\mathbf{v}_t$. Although only the magnitude v_l is needed, it is necessary to check these compatibility conditions of the input quantities $\mathbf{v}_{l,t}$ and $\mathbf{n}$. Another source of errors arises from the position of the focal source (especially the depth of the focus); an optimization of the accuracy related to this parameter is possible (it will be given in a forthcoming paper). Finally, although the sensitivity of the measurement of the wave amplitudes is, usually, sufficiently high (three or four decimal digits in cm), appreaciable errors may occur in practice in estimating these amplitudes from a seismogram, caused by the irregular pattern exhibited by the seismic records.

9 Discussion and concluding remarks

We can summarize the results as follows. Making use of the longitudinal displacement $\mathbf{v}_l$ and the transverse displacement $\mathbf{v}_t$, measured at Earth's surface (amplitudes of the P and S waves) , we compute the magnitude parameter M from Eq. (6.10) and the vector $\mathbf{m}$ and the scalar m_4 from Eq. (6.13); then, from Eq. (6.16) we get the seismic moment M_{ij}. The energy released by the earthquake is $E = M/2$ and an estimate of the focal volume is given by $V = M/2\rho c_t^2$ ($M/2\mu$, Eqs. (5.9) and (6.7)). An estimation of the duration T of the seismic activity in the focus is provided by Eq. (6.8). The focal slip is of the order $V^{1/3}$ and the focal strain is of the order $M_{ij}/2M$ (Eq. (5.10)). From the magnitude $\left(M_{ij}{}^2\right)^{1/2} = \sqrt{2}M$ of the seismic moment, we may estimate the magnitude M_w of the earthquake by means of the Hanks-Kanamori relation. A similar procedure holds for an isotropic seismic moment (preceding Section).

Making use of $\mathbf{m}$ and m_4 in Eq. (6.14), we compute the normal $\mathbf{s}$ to the fault plane and the unit slip vector $\mathbf{a}$ in the fault plane; the quadratic form associated to the seismic moment is a degenerate hyperboloid which reduces to a hyperbola in the $(\mathbf{s}, \mathbf{a})$-plane with asymptotes along the vectors $\mathbf{s}$ and $\mathbf{a}$. This hyperbola is tighter (closer to the origin) for higher M.

It is often convenient to have a rough estimation of the order of magnitude of the various quantities introduced in this paper. To this end we use a generic velocity c for the seismic waves and a generic vector $\mathbf{v}$ for the displacement in the far-field seismic waves. Equation (6.8) (which is $m^2 = 1$) gives $cT \simeq \sqrt{2Rv}$, which provides an estimate of the duration T in terms of the displacement measured at distance R. The focal volume can be estimated from Eq. (5.8) as $V \simeq \pi(2Rv)^{3/2} \simeq \pi(cT)^3$, as expected (dimension l of the focal region of the order cT; the rate of the focal slip is $l/T \simeq c$). Also, from Eq. (6.10) we have the energy $E \simeq \mu V \simeq M/2 \simeq 2\rho c^2 V$, where M is related to the magnitude $\left(M_{ij}^2\right)^{1/2} = \sqrt{2}M$ of the seismic moment (and the magnitude of the vector $M_{ij}n_j$). From Eq. (5.10) we get a focal strain of the order unity, as expected. In addition, we can see the relationship $\lg v = M_w + const.$

In conclusion, it is shown in this paper that the displacement in the far-field P and S seismic waves, which includes information about the structure of the focal region, can be employed, in principle, to determine the seismic-moment tensor for a fault slip, localized both in space and time (the inverse problem in Seismology), and all the relevant parameters of the seismic source. The vectorial (Kostrov) representation of the seismic moment (dyadic representation) for a shear faulting is written with four (unknown) parameters; one is the magnitude of the focal displacement, while the other three define the spatial orientation of the seismic tensor (orientation of the fault and the displacement direction). These unknown parameters are determined from the three equations relating the far-field displacement to the seismic tensor and the equation which relates the energy released in the earthquake (and carried by the seismic waves) to the focal displacement (and the fault focal volume), via the mechanical work done in the focal region, together with the covariance condition. The solution of the resulting system of equations makes the graphical representation of the quadratic form associated to the seismic-moment tensor, which is a hyperbola (hyperboloid), to offer a (three-dimensional) image of the focal region. The asymptotes of the hyperbola give the direction of the focal displacement and the orientation of the fault (seismic hyperbola). The solution presented here provides also a reasonable characterization of a localized fault slip, the geometry of the focal region (which leads to Kostrov representation) and the displacement in the far-field seismic waves provides reasonable estimations of the fault focal volume, focal strain, duration of the seismic activity and the energy of the earthquake and magnitude of the seismic moment. Also, the special case of an isotropic seismic moment is presented. More complex situations, like a superposition of point-like faults, or a combination of point-like faults and isotropic and dipole components imply more than four unknowns in the seismic tensor; since we have only four equations, the inverse problem in such cases is undetermined, within the present procedure. The procedure presented in this work makes use of manifestly covariant expressions of the data for determining the seismic moment.

Finally, we note that a similar deduction of the seismic-moment tensor can be done by using the (quasi)-static displacement at Earth's surface (Apostol 2017b); since it implies a specific treatment, its presentation is deferred to a forthcoming publication.

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