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Elastic Equilibrium of the Half-Space Revisited. Mindlin and Boussinesq Problems

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Abstract The displacement caused in an isotropic elastic half-space by a point force localized on or beneath its surface is calculated here by a new method. These classical problems are known as Boussinesq and, respectively, Mindlin problems. The motivation for the present work resides in the fact that the original solutions involve some particular procedures, required by the complexity of the boundary conditions, which may limit their general application. The solutions presented here are obtained by including in a generalized Poisson equation the values of the function and its derivatives on the boundary, and by using in-plane Fourier transforms. This method is general and can be extended to other, similar problems.

Keywords Elastic half-space · Mindlin problem · Boussinesq problem · Generalized Poisson equation

Mathematics Subject Classification 74 B05 · 74 G70 · 74 G05 · 86 A15 · 74-01

1 Introduction

As it is well known, the elastic displacement caused in an infinite body by a localized (point) force was calculated as early as 1848 by Kelvin [1–3]. The displacement caused by a force localized in a point on the plane surface of an elastic half-space is known as the Boussinesq problem [4–6]. Various generalizations of such problems have been worked out [7–11], in order to estimate the effects of concentrated forces in elastic bodies bounded by closed surfaces with various boundary conditions. An important recent generalization concerns the derivation of the Green functions for incremental deformations in infinite incompressible solids with homogeneous pre-stress and concentrated load (non-linear elasticity) [12, 13].

The displacement of an isotropic elastic half-space caused by a force localized beneath its surface was tackled as early as 1936 by Mindlin [14, 15], and reworked by him in 1953

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[16]; it is referred to as Mindlin problem. The displacement caused in an isotropic elastic body by a force localized on or beneath its surface is calculated here by a new method. The motivation of undertaking the present research is derived from some particular devices used in the original solutions, which may limit their application to other, similar problems. The original derivation of the solutions involves the use of the Green theorem and the identification of convenient combinations of functions (including Green functions) which satisfy the boundary conditions (usually, free surface). This method depends on the particularities of the problem, and it cannot be straightforwardly extended to other, similar problems, like, for instance, a half-space with fixed surface, or thick plates (where two surfaces are involved), or problems with cylindrical or spherical geometry. The usual, standard methods of treating these classical problems are described in a series of books and research articles [17–24]. In particular, a superposition of the Boussinesq problem and an image force is discussed in Ref. [23] for the Mindlin problem; a review of the application of analytical methods to boundary-value problems in elasticity is included in Ref. [24]. The Boussinesq and Mindlin problems, as well as other related problems (Flamant, Cerruti, Kelvin and Melan problems) are solved in Ref. [22] by an original, heuristic method, which consists in guessing the solution by using the underlying symmetries. We derive here the solution by using Fourier transforms with respect to the coordinates parallel to the plane surface of the half-space, which allow a convenient inclusion of the values of the functions and their derivatives on the surface in a generalized Poisson equation. This may be viewed as a general method, suitable for other, similar problems. It is worth noting that several applications and generalizations of these problems have been discussed recently, like the extension of the Boussinesq problem to soils with inhomogeneities [25], contact problems with friction [26, 27], various surface effects [28–34], etc.

2 General Form of the Solution

Consider the equilibrium equation

$$\Delta \mathbf{u} + \frac{1}{1-2\sigma}\operatorname{grad}\operatorname{div}\mathbf{u} = -\frac{2(1+\sigma)}{E}\mathbf{f} \tag{1}$$

for the displacement $\mathbf{u}$ in an isotropic elastic body with Poisson ratio σ and Young modulus E, subjected to a body force with density $\mathbf{f}$. The derivatives are taken with respect to the coordinates x, y, z and the components of the displacement vector $\mathbf{u}$ are u_x, u_y, u_z. Similarly, the components of the vector $\mathbf{f}$ will be denoted by f_x, f_y, f_z; the same notation will be used everywhere in this paper for vector components. As it is well known, the solution can be written with Helmholtz potentials φ and $\mathbf{h}$ as

$$\mathbf{u} = \operatorname{grad}\varphi + \operatorname{curl}\mathbf{h} \tag{2}$$

with $\operatorname{div}\mathbf{h} = 0$; inserting this solution in (1), we get

$$\Delta \mathbf{b} = -\frac{2(1+\sigma)}{E}\mathbf{f}, \tag{3}$$

where

$$\mathbf{b} = \frac{2(1-\sigma)}{1-2\sigma}\operatorname{grad}\varphi + \operatorname{curl}\mathbf{h}. \tag{4}$$

Taking the div in (4), we are led to

$$\operatorname{div}\mathbf{b} = \frac{2(1-\sigma)}{1-2\sigma}\Delta\varphi. \tag{5}$$

The general solution of (5) is

$$\varphi = \frac{1-2\sigma}{4(1-\sigma)}(\mathbf{r}\cdot\mathbf{b}+\beta), \tag{6}$$

where $\mathbf{r}$ is the position vector of the point with coordinates (x, y, z) and

$$\Delta\beta = \frac{2(1+\sigma)}{E}\mathbf{r}\cdot\mathbf{f}. \tag{7}$$

It follows that we can represent the solution as

$$\mathbf{u} = \mathbf{b} - \frac{1}{4(1-\sigma)}\operatorname{grad}(\mathbf{r}\cdot\mathbf{b}+\beta), \tag{8}$$

in terms of two functions $\mathbf{b}$ and β which satisfy Poisson equations (3) and (7); these functions are sometimes called Grodskii functions or Neuber–Papkovitch potentials [35–37]. For a force density $\mathbf{f} = \mathbf{f}_0\delta(\mathbf{r})$ localized at the origin in an infinite body, by solving (3) and (7) and using (8), we get the well-known Kelvin solution

$$\mathbf{u} = \frac{1+\sigma}{8\pi E(1-\sigma)}\left[\frac{(3-4\sigma)\mathbf{f}_0}{r} + \frac{(\mathbf{r}\cdot\mathbf{f}_0)\mathbf{r}}{r^3}\right], \tag{9}$$

where $\mathbf{f}_0$ is the force and r is the magnitude of $\mathbf{r}$.

3 The Problem and the Solving Method

The problem is to solve the equilibrium equation (1) for an elastic half-space which occupies the spatial region $z < 0$, bounded by a plane surface $z = 0$, and a force localized on or beneath its surface. We consider first a force localized beneath the surface (Mindlin problem). We use the notation $\mathbf{r}$ for the position vector of a point having the coordinates (x, y, z) and $\boldsymbol{\rho}$ for the projection of the vector $\mathbf{r}$ on the plane $z = 0$, corresponding to the coordinates (x, y). We denote by $\mathbf{f}_0$ a force with the components $(f_{0x}, f_{0y,}, f_{0z})$ localized at the point with the position vector $\mathbf{r}_0$, with the coordinates $(0, 0, z_0)$, $z_0 < 0$; the density of this force is given by $\mathbf{f} = \mathbf{f}_0\delta(\mathbf{r}-\mathbf{r}_0)$, where $\delta(\mathbf{r}-\mathbf{r}_0)$ is the Dirac δ-function. The surface $z = 0$ is free; consequently, the force (per unit area) with the components $p_i = -n_j\sigma_{ij}$ on the surface $z = 0$, where $\mathbf{n}$ is the unit vector normal to the surface $z = 0$ (with components $(0, 0, 1)$) and σ_{ij} is the stress tensor, is vanishing: $\sigma_{iz} = 0$ for $z = 0$ (summation over repeated indices is implied in this notation). As it is well-known [7], the stress tensor is $\sigma_{ij} = \frac{E}{1+\sigma}[u_{ij} + \frac{\sigma}{1-2\sigma}u_{kk}\delta_{ij}]$, where $u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the strain tensor; the boundary conditions read

$$u_{xz} = u_{yz} = 0, \qquad (1-\sigma)u_{zz} + \sigma(u_{xx}+u_{yy}) = 0, \qquad z = 0. \tag{10}$$

The usual method of tackling this problem is to solve the Poisson equations (3) and (7) for the functions $\mathbf{b}$ and β with the boundary conditions given by (10) and to use (8) for obtaining the displacement. The Poisson equations are solved by using the Green function for the Laplacian and the Green theorem. Since the boundary conditions (10) are not simply Dirichlet or Neumann boundary conditions for the potentials, their inclusion in the Green theorem requires special elaborations. We give here a different method, which can lead more directly to solution.

Consider the Poisson equation

$$\Delta v = g \tag{11}$$

in the domain D bounded by the closed surface S, where v is the unknown function (the solution of the Poisson equation) and g denotes the inhomogeneous term. We introduce the function $w = vE_D$, where E_D is the characteristic function of the domain D; it is easy to see, by direct calculations, that

$$\Delta w = \Delta v \cdot E_D - \left.\frac{\partial v}{\partial n}\right|_S \delta(n - n_s) - v\big|_S \frac{\partial}{\partial n}\delta(n - n_S), \tag{12}$$

where n is the coordinate along the normal $\mathbf{n}$ to the surface and n_S is the value of n on the surface. Making use of (11) we get

$$\Delta w = g - \left.\frac{\partial v}{\partial n}\right|_S \delta(n - n_s) - v\big|_S \frac{\partial}{\partial n}\delta(n - n_S) \tag{13}$$

in the closed domain D. Equation (13) provides a generalized form of the original Poisson equation (11); the function w, which is the restriction of the function v to the domain D, is the solution v of the original Poisson equation [38]. Using the Green function G, $\Delta G = -4\pi\delta(\mathbf{r} - \mathbf{r}')$, we recover the Green theorem

$$\begin{aligned} v(\mathbf{r}) = &-\frac{1}{4\pi}\int_D dV' G(\mathbf{r} - \mathbf{r}') g(\mathbf{r}') \\ &+ \frac{1}{4\pi}\int_S dS' \left[G(\mathbf{r} - \mathbf{r}')\frac{\partial v(\mathbf{r}')}{\partial n'} - v(\mathbf{r}')\frac{\partial G(\mathbf{r} - \mathbf{r}')}{\partial n'} \right] \end{aligned} \tag{14}$$

from (13), for the restriction of the function v to the domain D (dV' and dS' denote the volume and, respectively, area elements).

We apply this method to the Poisson equation (11) for the half-space $z < 0$ with the characteristic function $\theta(-z)$ and force density $f = f_0\delta(\mathbf{r} - \mathbf{r}_0)$, where $\mathbf{r}_0$ is the position vector of the point with coordinates $(0, 0, z_0)$, $z_0 < 0$, where the force is localized; the step function is $\theta(z) = 1$ for $z > 0$ and $\theta(z) = 0$ for $z < 0$. It is convenient to use in-plane Fourier transforms of the type

$$v(\boldsymbol{\rho}, z) = \frac{1}{(2\pi)^2}\int dk_x dk_y \cdot \widetilde{v}(\mathbf{k}, z) e^{i\mathbf{k}\cdot\boldsymbol{\rho}}, \tag{15}$$

where the integration is extended to the whole plane of $\mathbf{k}$-vectors ((k_x, k_y) are the components of the vector $\mathbf{k}$). This is a decomposition in plane waves, where $\mathbf{k}$ plays the role of a wavevector; the wavevector $\mathbf{k}$ is the argument of the Fourier transform $\widetilde{v}(\mathbf{k}, z)$, and k denotes its magnitude. These partial (or mixed) Fourier transformations are performed only with respect to the in-plane coordinates (x, y) (associated with the vector $\boldsymbol{\rho}$), while the perpendicular-to-the surface coordinate z is not affected. As it is well known, the inverse Fourier transform is

$$\widetilde{v}(\mathbf{k}, z) = \int dxdy \cdot v(\boldsymbol{\rho}, z) e^{-i\mathbf{k}\cdot\boldsymbol{\rho}}, \tag{16}$$

where the integration extends to the whole (x, y)-plane ((x, y) are the coordinates of the position vector $\boldsymbol{\rho}$). Henceforth, throughout this paper, symbols endowed with a tilde are Fourier transforms of the type given by (15) and (16). The Poisson equation becomes

$$\Delta v = f_0\delta(\mathbf{r}-\mathbf{r}_0) - v^{(1)}\delta(z) - v^{(0)}\delta'(z), \tag{17}$$

where $v^{(0)} = v|_{z=0}$, $v^{(1)} = \frac{\partial v}{\partial z}|_{z=0}$; the superscripts (0) and (1) will be used throughout this paper for the values of the functions and, respectively, their derivative with respect to z at $z = 0$. If we apply the Laplacian to the Fourier transformation given by (15), the derivatives with respect to x and y will give the contribution $-k^2\widetilde{v}$ for the corresponding Fourier transform; taking also the Fourier transform on the right in (17), this equation becomes

$$\frac{d^2\widetilde{v}}{dz^2} - k^2\widetilde{v} = f_0\delta(z-z_0) - \widetilde{v}^{(1)}\delta(z) - \widetilde{v}^{(0)}\delta'(z) \tag{18}$$

in Fourier transforms, where $\widetilde{v}^{(0)} = \widetilde{v}|_{z=0}$, $\widetilde{v}^{(1)} = \frac{\partial\widetilde{v}}{\partial z}|_{z=0}$; for the sake of the simplicity we may omit the arguments $(\boldsymbol{\rho}, z)$ or $(\mathbf{k}, z)$, as they can be easily read from the context of the equations. It is known that the Green function G_1 for the Helmholtz equation $d^2G_1/dz^2 - k^2G_1 = \delta(z)$ in one dimension is $G_1 = -(1/2k)e^{-k|z|}$, such that the solution of (18) reads

$$\widetilde{v} = -\frac{1}{2k}f_0e^{-k|z-z_0|} + \frac{1}{2k}\widetilde{v}^{(1)}e^{-k|z|} + \frac{1}{2}\widetilde{v}^{(0)}e^{-k|z|} \tag{19}$$

for $z < 0$; we eliminate $\widetilde{v}^{(1)}$ from this equation and get

$$\begin{aligned} \widetilde{v} &= -\frac{1}{2k}f_0\left(e^{-k|z-z_0|} - e^{-k|z+z_0|}\right) + \widetilde{v}^{(0)}e^{-k|z|}, \\ \widetilde{v}^{(1)} &= f_0e^{-k|z_0|} + k\widetilde{v}^{(0)}; \end{aligned} \tag{20}$$

we recognize here (in the brackets) the Green function for the Helmholtz equation in one dimension vanishing on the surface $z = 0$.

We use the representation given by (20) for the solutions $\mathbf{b}$ and β of the Poisson equations (3) and (7) and derive the functions and their derivatives (of the form $\widetilde{v}^{(0)}$ and $\widetilde{v}^{(1)}$ in (20)) from the boundary conditions; in addition, we note that the second-order derivative on the surface, denoted by $v^{(2)}$, $v^{(2)} = \frac{\partial^2 v}{\partial z^2}|_{z=0}$, has the Fourier transform given by is $\widetilde{v}^{(2)} = k^2\widetilde{v}^{(0)}$, as it follows immediately from (18).

Using the representation given by (20) for the solutions of (3) and (7) we get

$$\begin{aligned} \widetilde{\mathbf{b}} &= \frac{(1+\sigma)}{E}\frac{\mathbf{f}_0}{k}\left(e^{-k|z-z_0|} - e^{-k|z+z_0|}\right) + \widetilde{\mathbf{b}}^{(0)}e^{-k|z|}, \\ \widetilde{\beta} &= \frac{(1+\sigma)}{E}\frac{|z_0|f_{0z}}{k}\left(e^{-k|z-z_0|} - e^{-k|z+z_0|}\right) + \widetilde{\beta}^{(0)}e^{-k|z|}; \end{aligned} \tag{21}$$

in addition, we have the relations

$$\begin{aligned} \widetilde{\mathbf{b}}^{(1)} &= -\frac{2(1+\sigma)}{E}\mathbf{f}_0e^{-k|z_0|} + k\widetilde{\mathbf{b}}^{(0)}, \qquad \widetilde{\mathbf{b}}^{(2)} = k^2\widetilde{\mathbf{b}}^{(0)}, \\ \widetilde{\beta}^{(1)} &= -\frac{2(1+\sigma)}{E}|z_0|f_{0z}e^{-k|z_0|} + k\widetilde{\beta}^{(0)}, \qquad \widetilde{\beta}^{(2)} = k^2\widetilde{\beta}^{(0)}. \end{aligned} \tag{22}$$

4 Force Perpendicular to the Surface

Now we specialize to the case of a force perpendicular to the surface, *i.e.,* we take $f_{0x} = f_{0y} = 0$ and $f_{0z} = f_0$; due to the symmetry of the problem we may also take $b_x = b_y = 0$. Using the Fourier transforms, the boundary conditions from (10) are given by

$$(1-2\sigma)\widetilde{b}_z^{(0)} - \widetilde{\beta}^{(1)} = 0, \qquad 2(1-\sigma)\widetilde{b}_z^{(1)} - k^2\widetilde{\beta}^{(0)} = 0, \tag{23}$$

whence, by using relations (22), we get

$$\begin{aligned} \widetilde{b}_z^{(0)} &= \frac{2(1+\sigma)f_0}{E}\left[\frac{2(1-\sigma)}{k} - z_0\right]e^{-k|z_0|}, \\ \widetilde{\beta}^{(0)} &= \frac{4(1-\sigma)(1+\sigma)f_0}{E}\left(\frac{1-2\sigma}{k^2} - \frac{z_0}{k}\right)e^{-k|z_0|}. \end{aligned} \tag{24}$$

Making use of (21), (24) and the Sommerfeld integral [39]

$$\frac{1}{2\pi}\int dk_x dk_y \frac{e^{i\mathbf{k}\cdot\boldsymbol{\rho}}}{k}e^{-k|z|} = \frac{1}{(\rho^2+z^2)^{1/2}}, \tag{25}$$

where $\boldsymbol{\rho}$ is the projection of the position vector $\mathbf{r}$ on the plane $z = 0$, corresponding to the coordinates (x, y), we get, by inverse Fourier transformation,

$$b_z = \frac{(1+\sigma)f_0}{2\pi E}\left[\frac{1}{r_1} + \frac{3-4\sigma}{r_2} + \frac{2z_0(z+z_0)}{r_2^3}\right], \tag{26}$$

where

$$r_1 = \left[\rho^2 + (z-z_0)^2\right]^{1/2}, \qquad r_2 = \left[\rho^2 + (z+z_0)^2\right]^{1/2} \tag{27}$$

and ρ is the magnitude of the vector $\boldsymbol{\rho}$; we can see in (27) the contribution of the "image" solution corresponding to $z_0 \to -z_0$. Similarly, we get from (21) and (24)

$$\beta = \frac{(1+\sigma)f_0}{2\pi E}\left[\frac{|z_0|}{r_1} + \frac{(3-4\sigma)|z_0|}{r_2} + 4(1-\sigma)(1-2\sigma)I\right], \tag{28}$$

where we define the integral I as

$$I = \frac{1}{2\pi}\int dk_x dk_y \frac{1}{k^2}e^{i\mathbf{k}\cdot\boldsymbol{\rho}}e^{-k|z+z_0|}. \tag{29}$$

In the original solution [16] the function I is replaced by $\ln(r_2 + |z+z_0|)$, which can be obtained by integration of the derivative $\partial I/\partial|z+z_0|$ (a minus sign should be included for the half-space $z < 0$ in comparison with the half-space $z > 0$). For the displacement functions u_x, u_y, u_z given by (8) we need $\operatorname{grad}\beta$ and, therefore, $\operatorname{grad} I$. The derivatives of the function I can be calculated directly from (29) by using Bessel functions. For example, it is easy to get

$$\frac{\partial I}{\partial\boldsymbol{\rho}} = -\left(1 - \frac{|z+z_0|}{r_2}\right)\frac{\boldsymbol{\rho}}{\rho^2} = -\frac{\boldsymbol{\rho}}{r_2(r_2+|z+z_0|)}. \tag{30}$$

We give here the displacement of the surface $z=0$, calculated from (8) by using b_z given by (26) and the derivatives of the function β and function I; we get

$$
\begin{aligned}
u_\rho &= \frac{(1+\sigma)f_0}{2\pi E}\left(\frac{|z_0|}{r_0^2}+\frac{1-2\sigma}{r_0+|z_0|}\right)\frac{\rho}{r_0},\\
u_z &= \frac{(1+\sigma)f_0}{2\pi E}\left[2(1-\sigma)+\frac{z_0^2}{r_0^2}\right]\frac{1}{r_0},
\end{aligned}
\tag{31}
$$

where u_ρ is the radial component of the displacement (along the vector $\boldsymbol{\rho}$) and $r_0=[\rho^2+z_0^2]^{1/2}$. We can see that the radial component of the displacement u_ρ has a maximum of the order $\simeq f_0/E|z_0|$ for distances of the order $\rho\simeq|z_0|$, while the z-component u_z of the displacement attains its maximum value $\simeq f_0/E|z_0|$ for $\rho=0$.

5 Force Parallel to the Surface

We consider now a force parallel to the x-axis $f_{0x}=f_0$, $f_{0y}=f_{0z}=0$; due to the symmetry of the problem we take also $b_y=0$. We introduce the function $C=xb_x+\beta$ and the boundary conditions (10) become

$$
\begin{aligned}
&2(1-\sigma)\widetilde{b}_x^{(1)}+i(1-2\sigma)k_x\widetilde{b}_z^{(0)}-ik_x\widetilde{C}^{(1)}=0,\\
&(1-2\sigma)\widetilde{b}_z^{(0)}-\widetilde{C}^{(1)}=0,\\
&2(1-\sigma)(1-2\sigma)\widetilde{b}_z^{(1)}-(1-\sigma)\widetilde{C}^{(2)}+4i\sigma(1-\sigma)k_x\widetilde{b}_x^{(0)}+\sigma k^2\widetilde{C}^{(0)}=0.
\end{aligned}
\tag{32}
$$

Making use of the Fourier transform, $\widetilde{C}=i\partial\widetilde{b}_x/\partial k_x+\widetilde{\beta}$ and (22), we get the solutions of the system of (32). The solutions of the boundary conditions (10) are

$$
\begin{aligned}
\widetilde{b}_x^{(0)} &= \frac{2(1+\sigma)f_0}{E}\frac{1}{k}e^{-k|z_0|},\\
\widetilde{b}_z^{(0)} &= \frac{2(1+\sigma)f_0}{E}\big(1-2\sigma-k|z_0|\big)\frac{ik_x}{k^2}e^{-k|z_0|},\\
\widetilde{\beta}^{(0)} &= \frac{2(1+\sigma)(1-2\sigma)f_0}{E}\big(1-2\sigma-k|z_0|\big)\frac{ik_x}{k^3}e^{-k|z_0|}.
\end{aligned}
\tag{33}
$$

Hence, by using (21), we get

$$
\begin{aligned}
\widetilde{b}_x &= \frac{(1+\sigma)f_0}{E}\frac{1}{k}\big(e^{-k|z-z_0|}+e^{-k|z+z_0|}\big),\\
\widetilde{b}_z &= \frac{2(1+\sigma)f_0}{E}\big(1-2\sigma-k|z_0|\big)\frac{ik_x}{k^2}e^{-k|z+z_0|},\\
\widetilde{\beta} &= \frac{2(1+\sigma)(1-2\sigma)f_0}{E}\big(1-2\sigma-k|z_0|\big)\frac{ik_x}{k^3}e^{-k|z+z_0|}
\end{aligned}
\tag{34}
$$

and the inverse Fourier transforms

$$b_x = \frac{(1+\sigma)f_0}{2\pi E}\left(\frac{1}{r_1}+\frac{1}{r_2}\right),$$
$$b_z = \frac{(1+\sigma)f_0}{\pi E}\left(\frac{|z_0|}{r_2^2}-\frac{1-2\sigma}{r_2+|z+z_0|}\right)\frac{x}{r_2}, \tag{35}$$
$$\beta = \frac{(1+\sigma)(1-2\sigma)f_0}{\pi E}\left[\frac{|z_0|}{r_2}-(1-2\sigma)\right]\frac{x}{r_2+|z+z_0|};$$

in (35) the function

$$H = \frac{1}{2\pi}\int dk_x dk_y \frac{ik_x}{k^3} e^{i\mathbf{k}\cdot\boldsymbol{\rho}} e^{-k|z+z_0|} \tag{36}$$

has been calculated by integrating the derivative $\partial H/\partial|z+z_0|$ ($H = -x/(r_2+|z+z_0|)$). The results given in (35) coincide with the original Mindlin's results [16], (except for the sign of b_z).

Having known the functions b_x, b_z and β, we can calculate the displacement by using (8). We give here the displacement u_z on the surface $z=0$,

$$u_z = \frac{(1+\sigma)f_0}{2\pi E}\left(\frac{|z_0|}{r_0^2}-\frac{1-2\sigma}{r_0+|z_0|}\right)\frac{x}{r_0} \tag{37}$$

and the asymptotic behaviour of u_x, u_y on the surface

$$u_x \simeq \frac{(1+\sigma)(3-2\sigma)f_0}{4\pi E}\frac{1}{|z_0|}, \quad \rho \ll |z_0|,$$
$$u_y \simeq \frac{(1+\sigma)(3+2\sigma)f_0}{8\pi E}\frac{xy}{|z_0|^3}, \quad \rho \ll |z_0|, \tag{38}$$
$$u_x \simeq \frac{(1+\sigma)(1-\sigma)f_0}{\pi E}\frac{1}{\rho}, \qquad u_y \simeq \frac{\sigma(1+\sigma)f_0}{\pi E}\frac{xy}{\rho^3}, \quad \rho \gg |z_0|;$$

for $y=0$ and $|x| \gg |z_0|$ we get $u_x \simeq [(3-4\sigma)(1+\sigma)f_0/4\pi E(1-\sigma)]/|x|$. We can see that u_x has a maximum value of the order $\simeq f_0/E|z_0|$ for $\rho \to 0$, while $u_y \simeq xy/|z_0|^3$, $u_z \simeq x/z_0^2$ for $\rho \to 0$ and attain a maximum value $\simeq f_0/E|z_0|$ for ρ of the order $|z_0|$. It is worth noting that u_z has a zero for a distance ρ of the order $|z_0|$.

6 Force Acting on the Surface

We consider now a force $\mathbf{f}_0$ with components $(0, 0, f_0)$ localized at the origin on the surface $z=0$ (Boussinesq problem); its density is given by $\mathbf{f} = \mathbf{f}_0\delta(\boldsymbol{\rho})$. Equations (21) and (22) for the Grodskii functions are now free of body force, but the surface force appears in the boundary conditions, which read, in Fourier transforms,

$$(1-2\sigma)\widetilde{b}_z^{(0)} - \widetilde{\beta}^{(1)} = 0,$$
$$2(1-\sigma)\widetilde{b}_z^{(1)} - k^2\widetilde{\beta}^{(0)} = -\frac{4(1+\sigma)(1-\sigma)f_0}{E}. \tag{39}$$

Making use of relations (21) and (22), this system of equations is solved immediately, leading to

$$\begin{aligned} b_z &= -\frac{2(1-\sigma^2)f_0}{\pi E}\frac{1}{r}, \\ \beta &= -\frac{2(1-\sigma^2)(1-2\sigma)f_0}{\pi E}I_0, \end{aligned} \tag{40}$$

where $r=(\rho^2+z^2)^{1/2}$ and I_0 is the function given by (29) for $z_0=0$. From (8) we get the well-known displacement for the Boussinesq problem [5, 6]

$$\begin{aligned} u_\rho &= -\frac{(1+\sigma)f_0}{2\pi E}\left(\frac{z}{r^2}+\frac{1-2\sigma}{r+|z|}\right)\frac{\rho}{r}, \\ u_z &= -\frac{(1+\sigma)f_0}{2\pi E}\left[\frac{z^2}{r^2}+2(1-\sigma)\right]\frac{1}{r}. \end{aligned} \tag{41}$$

The case of a force parallel to the surface (Cerruti problem) can be treated in the same manner.

7 Conclusion

A new method has been introduced in this paper for evaluating the displacement of an isotropic elastic half-space as caused by a force localized on or beneath the surface. The original solutions of these problems, known as Boussinesq and, respectively, Mindlin problems [4, 5, 14–16] include some particular devices, conditioned by the complexity of the boundary conditions. Other known methods of solving these problems make use of heuristic guesses, based on the symmetries of the problems. The solution given here is obtained by using in-plane Fourier transforms and by including the values of the functions and their derivatives on the boundary in a generalized Poisson equation. This method can be extended to other boundary-value problems, like a half-space with fixed surface [40, 41], or elastic (thick) plates, or elastic bodies with cylindrical or spherical geometry, or to the related Flamant, Cerruti, or Melan problems [22].

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