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# Elastic Displacement in a Half-Space Under the Action of a Tensor Force. General Solution for the Half-Space with Point Forces

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**Abstract** The elastic displacement in an isotropic elastic half-space with free surface is calculated for a point tensor force which may arise from the seismic moment of seismic sources concentrated at an inner point of the half-space. The starting point of the calculation is the decomposition of the displacement by means of the Helmholtz potentials and a simplified version of the Grodskii-Neuber-Papkovitch procedure. The calculations are carried out by using generalized Poisson equations and in-plane Fourier transforms, which are convenient for treating boundary conditions. As a general result we compute the displacement in the isotropic elastic half-space with free surface caused by point forces with arbitrary structure and orientation, localized either beneath the surface (generalized Mindlin problem) or on the surface (generalized Boussinesq-Cerruti problems). The inverse Fourier transforms are carried out by means of Sommerfeld-type integrals. For forces buried in the half-space explicit results are given for the surface displacement, which may exhibit finite values at the origin, or at distances on the surface of the order of the depth of the source. The problem presented here may be viewed as an addition to the well-known static problems of elastic equilibrium of a half-space under the action of concentrated loads. The application of the method to similar problems and another approach to the starting point of the general solution are discussed.

**Keywords** Elastic half-space · Tensor force · Mindlin, Boussinesq-Cerruti problems · Generalized Poisson equation · Fourier transforms

**Mathematics Subject Classification** 74B05 · 74G70 · 74G05 · 86A15

## 1 Introduction. Tensor Force

The elastic equilibrium of a half-space under the action of concentrated loads is the subject of a well-known family of classical problems in Elasticity; also, these problems have

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a fundamental relevance in Geotechnics. The displacement vector in an infinite isotropic elastic body under the action of a concentrated force (point force) has been calculated as early as 1848 by Kelvin [1, 2]. In the second half of the 19th century forces localized on the surface of an isotropic elastic half-space have been studied. In the Boussinesq problem [3–6] the point force acts perpendicular to the surface, in the Cerruti problem [7] the point force is tangential to the surface, while in the Flamant problem [8] the force perpendicular to the surface is localized along a straight line. The displacement vector in an isotropic elastic half-space caused by a point force concentrated at an inner point has been calculated by Mindlin between 1936 and 1953 [9–13], while the two dimensional version of the Mindlin problem, known as the Melan problem [14], was solved in 1932. In all these problems the displacement is calculated by solving the Navier-Cauchy equation of elastic equilibrium with suitable boundary conditions. The particular approaches vary from a direct application of the Green theorem to using Kelvin approach to Grodskii-Neuber-Papkovitch [15–17], or Helmholtz, potentials. Various accounts of these problems, at various levels of complexity, can be found in the classical treatises given in Refs. [18–25]. A very interesting, original, heuristic method of solving these problems is described in Ref. [26], where the method consists in guessing the solution by using the underlying symmetries. The Mindlin and Boussinesq problems have been recently revisited [27], by using generalized Poisson equations and in-plane Fourier transforms, which are convenient tools for treating boundary conditions.

In Seismology we need concentrated load distributions which have a total vanishing force and angular momentum. In a simplified model, the Earth may be viewed as an isotropic elastic half-space bounded by a plane surface, the seismic sources being localized beneath the surface. For sufficiently long distances the spatially localized seismic sources may be represented as point sources. The "double-couple" representation of point seismic sources by means of the seismic moment tensor emerged gradually in the first half of the 20th century [28–40]. Let $\mathbf{f}(\mathbf{r}) = \mathbf{f}^0 w(\mathbf{r})$ be a force density, where $\mathbf{f}^0$ is the force, $w(\mathbf{r})$ is a distribution function and $\mathbf{r}$ is the position vector of a point with coordinates $(x_1, x_2, x_3)$; a point couple along the $i$-th direction ($i = 1, 2, 3$) can be represented as

$$f_i^0 w(x_1 + h_1, x_2 + h_2, x_3 + h_3) - f_i^0 w(x_1, x_2, x_3) \simeq f_i^0 h_j \partial_j w(x_1, x_2, x_3), \tag{1}$$

where $f_i^0$, $i = 1, 2, 3$, are the components of the force, $h_j$, $j = 1, 2, 3$, are the components of an infinitesimal displacement $\mathbf{h}$, $\partial_j$ denotes the derivative with respect to the coordinate $x_j$ and summation over repeated labels is assumed. The moment $f_i^0 h_j$ is generalized to a symmetric tensor $M_{ij}$, which in Seismology is called the seismic moment (see [39], 2nd edition, p. 60, Exercise 3.6). The tensor of the seismic moment is related to the area of rupture of the seismic fault; indeed, the force $\mathbf{f}^0$ is the area multiplied by the Lame elastic modulus $\mu$ [39]. In addition, the distribution $w(\mathbf{r})$ is replaced by $\delta(\mathbf{r} - \mathbf{r}_0)$, where $\delta$ denotes the Dirac function localized at the point with the position vector $\mathbf{r}_0$. Thus, we get a tensor force density with the components

$$f_i = M_{ij} \partial_j \delta(\mathbf{r} - \mathbf{r}_0); \tag{2}$$

we can check immediately that the total force is vanishing and so is the total angular momentum (due to the symmetry of the tensor $M_{ij}$).

Our aim in this paper is to solve the equation of elastic equilibrium for an isotropic elastic half-space occupying the region $x_3 = z < 0$, bounded by a flat surface $x_3 = z = 0$, with free boundary conditions, with the force given by (2), placed at the position vector $\mathbf{r}_0$ with coordinates $(0, 0, z_0)$, where $z_0 < 0$. Beside the notation $(x_1, x_2, x_3)$ for coordinates we use also the notation $x = x_1$, $y = x_2$, $z = x_3$. We note that the solution can be obtained by the indirect method of taking the moment of forces as given by (1) in the solution of the Mindlin problem. Beside the particular character of this method, which requires calculations for each component of the force moment, it may be impracticable, due to the calculation complexity. This is why we prefer to give here a direct method, which leads to an elegant and compact form of solution. The method is based on a simplified version of the Grodskii-Neuber-Papkovitch potentials, generalized Poisson equations and in-plane Fourier transforms (*i.e.*, Fourier transforms with respect to the coordinates parallel with the half-space surface). In particular, generalized Mindlin and Boussinesq-Cerruti problems are solved here for point forces of arbitrary orientation and structure. The tensor forces are an instance of such forces.

## 2 First Part of Solution. Helmholtz Potentials

The equation of elastic equilibrium with the force density $\bar{\mathbf{f}}$ is [41]

$$\Delta\mathbf{u} + \frac{1}{1-2\sigma}\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} = -\frac{2(1+\sigma)}{E}\bar{\mathbf{f}}, \tag{3}$$

where $\mathbf{u}$ is the displacement vector (with components $u_i$, $i = 1, 2, 3$), $E$ is the Young modulus and $\sigma$ is the Poisson ratio. In order to simplify the calculations (and notations) it is convenient to absorb the factor $-2(1+\sigma)/E$ in the force density $\bar{\mathbf{f}}$ and re-write (3) as

$$\Delta\mathbf{u} + \frac{1}{1-2\sigma}\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} = \mathbf{f}, \tag{4}$$

where $\mathbf{f} = -2(1+\sigma)\bar{\mathbf{f}}/E$ is a reduced force density. We represent the solution as a sum of Helmholtz potentials $\Phi$ and $\mathbf{a}$,

$$\mathbf{u} = \mathrm{grad}\,\Phi + \mathrm{curl}\,\mathbf{a}, \quad \mathrm{div}\,\mathbf{a} = 0, \tag{5}$$

and introduce the vector

$$\mathbf{b} = \frac{2(1-\sigma)}{1-2\sigma}\,\mathrm{grad}\,\Phi + \mathrm{curl}\,\mathbf{a} \tag{6}$$

which satisfies the equation

$$\Delta\mathbf{b} = \mathbf{f}, \tag{7}$$

derived from (4). In addition, by taking the div in (6), we get

$$\Delta\Phi = \frac{1-2\sigma}{2(1-\sigma)}\,\mathrm{div}\,\mathbf{b}. \tag{8}$$

Eliminating $\mathrm{curl}\,\mathbf{a}$ from (5) and (6), the solution is represented as

$$\mathbf{u} = \mathbf{b} - \frac{1}{1-2\sigma}\,\mathrm{grad}\,\Phi. \tag{9}$$

Basically, this is the starting point of the Grodskii-Neuber-Papkovitch procedure [15–17], which continues with using the representation

$$\Phi = \frac{1-2\sigma}{4(1-\sigma)}(\mathbf{r}\cdot\mathbf{b} + \varphi) \tag{10}$$

of the solution of (8), where the potential $\varphi$ satisfies the equation

$$\Delta\varphi = -\mathbf{r} \cdot \Delta\mathbf{b} = -\,\mathbf{r} \cdot \mathbf{f}. \tag{11}$$

Since the formation $\mathbf{r} \cdot \mathbf{f}$ is not convenient for our use of the Fourier transforms, we prefer to preserve the potential $\Phi$ given by (8). This is what we call a simplified version of the Grodskii-Neuber-Papkovitch procedure. Our strategy is to solve (7) for the vector potential $\mathbf{b}$, then use $\mathbf{b}$ to solve (8) for the Helmholtz scalar potential $\Phi$ and, finally, obtain the solution $\mathbf{u}$ from (9). In the infinite space, for a point force distribution, (7) and (8) lead immediately to the Kelvin solution.

In order to prepare ourselves for tackling the boundary conditions, it is convenient to extend (7) to its generalized form [42], by introducing the vector function $\bar{\mathbf{b}} = \mathbf{b}\theta(-z)$, where $\theta(z) = 1$ for $z > 0$ and $\theta(z) = 0$ for $z < 0$ is the step function. The vector $\bar{\mathbf{b}}$, which is the restriction of $\mathbf{b}$ to the domain $z < 0$, is the solution $\mathbf{b}$ of the original Poisson equation. It is easy to see, by direct calculations, that (7) becomes

$$\Delta\mathbf{b} = \mathbf{f} - \mathbf{b}^{(1)}\delta(z) - \mathbf{b}^{(0)}\delta'(z) \tag{12}$$

where $\mathbf{b}^{(0)} = \mathbf{b}|_{z=0}$, $\mathbf{b}^{(1)} = \frac{\partial\mathbf{b}}{\partial z}|_{z=0}$; the superscripts (0) and (1) will be used throughout this paper for the values of the functions and, respectively, their derivative with respect to $z$ at $z = 0$. The prime over the $\delta$-function in (12) means the derivative with respect to $z$. We can see that the Green theorem is recovered from (12) for the restriction of the function $\mathbf{b}$ to the domain $z < 0$.

It is also convenient to use the projection $\boldsymbol{\rho}$ of the position vector $\mathbf{r}$ on the plane $z = 0$, corresponding to the coordinates $(x_1, x_2)$, and to introduce the in-plane Fourier transforms of the type

$$\mathbf{b}(\boldsymbol{\rho}, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \cdot \tilde{\mathbf{b}}(\mathbf{k}, z) e^{i\mathbf{k}\cdot\boldsymbol{\rho}}, \tag{13}$$

where the integration is extended to the whole plane of $\mathbf{k}$-vectors; $(k_1, k_2)$ are the components of the vector $\mathbf{k}$. This is a decomposition in plane waves, where $\mathbf{k}$ plays the role of a wavevector; the wavevector $\mathbf{k}$ is the argument of the Fourier transform $\tilde{\mathbf{b}}(\mathbf{k}, z)$, and $k$ denotes the magnitude of the vector $\mathbf{k}$. These partial (or mixed) Fourier transformations are performed only with respect to the in-plane coordinates $(x_1, x_2)$ (associated with the vector $\boldsymbol{\rho}$), while the perpendicular-to-surface coordinate $x_3 = z$ is not affected. As it is well known, the inverse Fourier transform is

$$\tilde{\mathbf{b}}(\mathbf{k}, z) = \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 \cdot \mathbf{b}(\boldsymbol{\rho}, z) e^{-i\mathbf{k}\cdot\boldsymbol{\rho}}, \tag{14}$$

where the integration extends to the whole $(x_1, x_2)$-plane (the coordinates of the position vector $\boldsymbol{\rho}$). Such type of Fourier transforms are used throughout this paper for various other functions; symbols endowed with a tilde are Fourier transforms of the type given by (13) and (14).

The in-plane Fourier transform of (12) leads to

$$\frac{d^2\tilde{\mathbf{b}}}{dz^2} - k^2\tilde{\mathbf{b}} = \tilde{\mathbf{f}} - \tilde{\mathbf{b}}^{(1)}\delta(z) - \tilde{\mathbf{b}}^{(0)}\delta'(z), \tag{15}$$

where $\tilde{\mathbf{b}}^{(0)} = \tilde{\mathbf{b}}|_{z=0}$, $\tilde{\mathbf{b}}^{(1)} = \frac{\partial\tilde{\mathbf{b}}}{\partial z}|_{z=0}$; for the sake of simplicity we may omit the arguments $(\boldsymbol{\rho}, z)$ or $(\mathbf{k}, z)$, as they can be easily read from the context of the equations. Beside Roman labels $i, j, l, \ldots = 1, 2, 3$ for coordinates and vector and tensor components, we use also throughout the paper Greek suffixes $\alpha, \beta, \gamma, \ldots = 1, 2$ for the coordinates and components labels 1 and 2, summation over such repeated labels being implicit. With regard to the Fourier transformations given above, the derivatives $\partial_\alpha$ (with respect to the coordinates $x_\alpha$, $\alpha = 1, 2$) applied to the $\delta$-function

$$\delta(\boldsymbol{\rho}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \cdot e^{i\mathbf{k}\cdot\boldsymbol{\rho}} \tag{16}$$

(or, in general, to Fourier transforms) yield factors $ik_\alpha$, while the Laplacian $\partial_\alpha^2 = \partial_\alpha \partial_\alpha$ generates a factor $-k^2$ in the Fourier transform; for example, $\partial_\alpha^2 = \partial_\alpha \partial_\alpha$ applied to $\mathbf{b}$ given by (13) generates $-k^2 \tilde{\mathbf{b}}$.

It is well known that the Green function of the one-dimensional Helmholtz operator on the left of (15) is $-(1/2k)e^{-k|z|}$. Making use of this Green function, we get the solution

$$\tilde{\mathbf{b}} = -\frac{1}{2k}\tilde{\mathbf{c}} + \tilde{\mathbf{b}}^{(0)} e^{-k|z|}, \tag{17}$$

where

$$\tilde{\mathbf{c}} = \int_{-\infty}^{0} dz' \tilde{\mathbf{f}}(z') \left[ e^{-k|z-z'|} - e^{-k|z+z'|} \right] \tag{18}$$

and

$$\tilde{\mathbf{b}}^{(1)} = k\mathbf{b}^{(0)} + \tilde{\mathbf{g}}, \quad \tilde{\mathbf{g}} = \int_{-\infty}^{0} dz' \tilde{\mathbf{f}}(z') e^{-k|z'|}; \tag{19}$$

in addition we have $\tilde{\mathbf{b}}^{(2)} = \frac{\partial^2 \tilde{\mathbf{b}}}{\partial z^2}|_{z=0} = k^2 \tilde{\mathbf{b}}^{(0)} + \tilde{\mathbf{f}}^{(0)}$ directly from (15), where $\tilde{\mathbf{b}}^{(2)}$ denotes the second-order derivative of $\tilde{\mathbf{b}}$ with respect to $z$ for $z = 0$. We can see the occurrence of the image Green function $-(1/2k)e^{-k|z+z'|}$, beside the direct Green function $-(1/2k)e^{-k|z-z'|}$, on the right of (18). We note also the useful relations $\tilde{\mathbf{c}}^{(0)} = 0$, $\tilde{\mathbf{c}}^{(1)} = -2k\mathbf{g}$ and $\tilde{\mathbf{c}}^{(2)} = -2k\tilde{\mathbf{f}}^{(0)}$, which can be obtained directly from (18). In these equations $z'$ is an integration variable, according to the definition of the Green function $-(1/2k)e^{-k|z-z'|}$.

We turn now to solving (8). It is not necessary to extend this equation to its generalized form, by introducing the boundary values $\Phi^{(0)}$and $\Phi^{(1)}$, since $\mathbf{b}^{(0)}$ with its three components (which play the role of "constants of integration") suffices to satisfy the boundary conditions; indeed, the boundary conditions consist of three equations, as many as the number of the functions $b_i^{(0)}$, $i = 1, 2, 3$; the system of equations generated by the boundary conditions is determined for the unknowns $b_i^{(0)}$; introducing $\Phi^{(0)}$ and $\Phi^{(1)}$ would be superfluous. By Fourier transforming (8), with a technique similar with that used above for (7) and making use of $\tilde{\mathbf{b}}$ given by (17), we get

$$\begin{aligned}
\tilde{\Phi} = {} & \frac{(1-2\sigma)ik_\alpha}{8(1-\sigma)k^2} \int_{-\infty}^{0} dz' \tilde{c}_\alpha(z') e^{-k|z-z'|} \\
& - \frac{1-2\sigma}{8(1-\sigma)k} \int_{-\infty}^{0} dz' \operatorname{sgn}(z-z') \tilde{c}_3(z') e^{-k|z-z'|} \\
& - \frac{1-2\sigma}{8(1-\sigma)k^2} \left( ik_\alpha \tilde{b}_\alpha^{(0)} + \tilde{b}_3^{(0)} \right)(1-2kz) e^{-k|z|}
\end{aligned} \tag{20}$$

(by Fourier transforming, div $\mathbf{b}$ yields $ik_\alpha \tilde{b}_\alpha + \tilde{b}_3'$, where the prime means the derivative with respect to $z$). In deriving (20) we use some particular forms of the integrals

$$J_+ = \int_{-\infty}^{0} dz' e^{-k|z-z'|} e^{-k|z'+z_0|} = \left(\frac{1}{2k} - z\right) e^{-k|z+z_0|},$$
$$J_- = \int_{-\infty}^{0} dz' e^{-k|z-z'|} e^{-k|z'-z_0|} = \frac{1}{k} e^{-k|z-z_0|} \tag{21}$$
$$+ |z-z_0| \frac{1}{k} e^{-k|z-z_0|} - \frac{1}{2k} e^{-k|z+z_0|}$$

and

$$J_-^s = \int_{-\infty}^{0} dz' \operatorname{sgn}(z'-z_0) e^{-k|z-z'|} e^{-k|z'-z_0|}$$
$$= (z-z_0) e^{-k|z-z_0|} - \frac{1}{2k} e^{-k|z+z_0|}, \tag{22}$$
$$J_0^s = \int_{-\infty}^{0} dz' \operatorname{sgn}(z-z') e^{-k|z-z'|} e^{-k|z'|} = \left(\frac{1}{2k} + z\right) e^{-k|z|},$$

valid for $z \le 0$ and the parameter $z_0 \le 0$.

## 3 Second Part of Solution. The Boundary Conditions

We consider a free surface $z = 0$. Consequently, the force (per unit area) with the components $\bar{p}_i = -n_j \sigma_{ij}$ on the surface $z = 0$, where $\mathbf{n}$ is the unit vector normal to the surface $z = 0$ (with components $(0, 0, 1)$) and $\sigma_{ij}$ is the stress tensor, is vanishing: $\sigma_{i3} = 0$ for $z = 0$. As it is well-known [41], the stress tensor is $\sigma_{ij} = \frac{E}{1+\sigma}[u_{ij} + \frac{\sigma}{1-2\sigma} u_{kk} \delta_{ij}]$, where $u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the strain tensor; the boundary conditions read

$$u_{\alpha 3} = 0, \qquad (1-\sigma) u_{33} + \sigma u_{\alpha\alpha} = 0, \qquad z = 0. \tag{23}$$

We calculate the strain tensor from the Fourier transform of (9), by making use of $\tilde{\mathbf{b}}$ given by (17) and $\tilde{\Phi}$ given by (20). We give here the boundary values of $\tilde{\Phi}$ and its derivatives on the surface (which enter the expressions of the strain tensor):

$$\tilde{\Phi}^{(0)} = \frac{(1-2\sigma) i k_\alpha \tilde{d}_\alpha}{8(1-\sigma)k^2} - \frac{(1-2\sigma)\tilde{d}_3}{8(1-\sigma)k} - \frac{1-2\sigma}{8(1-\sigma)k^2}\left(i k_\alpha \tilde{b}_\alpha^{(0)} + k \tilde{b}_3^{(0)}\right),$$
$$\tilde{\Phi}^{(1)} = -\frac{(1-2\sigma) i k_\alpha \tilde{d}_\alpha}{8(1-\sigma)k} + \frac{(1-2\sigma)\tilde{d}_3}{8(1-\sigma)} + \frac{1-2\sigma}{8(1-\sigma)k}\left(i k_\alpha \tilde{b}_\alpha^{(0)} + k \tilde{b}_3^{(0)}\right), \tag{24}$$
$$\tilde{\Phi}^{(2)} = \frac{(1-2\sigma) i k_\alpha \tilde{d}_\alpha}{8(1-\sigma)} - \frac{(1-2\sigma) k \tilde{d}_3}{8(1-\sigma)} + \frac{(1-2\sigma)\tilde{g}_3}{2(1-\sigma)} + \frac{3(1-2\sigma)}{8(1-\sigma)}\left(i k_\alpha \tilde{b}_\alpha^{(0)} + k \tilde{b}_3^{(0)}\right),$$

where

$$\tilde{\mathbf{d}} = \int_{-\infty}^{0} dz' \tilde{\mathbf{c}}(z') e^{-k|z'|}. \tag{25}$$

Making use of these expressions the boundary conditions become

$$\begin{aligned} k\tilde{b}_\alpha^{(0)} + \frac{k_\alpha k_\beta \tilde{b}_\beta^{(0)}}{4(1-\sigma)k} + \frac{i(3-4\sigma)k_\alpha \tilde{b}_3^{(0)}}{4(1-\sigma)} &= -\tilde{g}_\alpha + \frac{k_\alpha k_\beta \tilde{d}_\beta}{4(1-\sigma)k} + \frac{ik_\alpha \tilde{d}_3}{4(1-\sigma)}, \\ i(3-4\sigma)k_\alpha \tilde{b}_\alpha^{(0)} - (5-4\sigma)k\tilde{b}_3^{(0)} &= 4(1-\sigma)\tilde{g}_3 - ik_\alpha \tilde{d}_\alpha + k\tilde{d}_3. \end{aligned} \tag{26}$$

The solutions of this algebraic system of equations are given by

$$\begin{aligned} ik_\alpha \tilde{b}_\alpha^{(0)} &= -\frac{i(5-4\sigma)k_\alpha \tilde{g}_\alpha}{4k} - \frac{(3-4\sigma)\tilde{g}_3}{4} + \frac{i}{2}k_\alpha \tilde{d}_\alpha - \frac{1}{2}k\tilde{d}_3, \\ k\tilde{b}_3^{(0)} &= -\frac{i(3-4\sigma)k_\alpha \tilde{g}_\alpha}{4k} - \frac{(5-4\sigma)\tilde{g}_3}{4} + \frac{i}{2}k_\alpha \tilde{d}_\alpha - \frac{1}{2}k\tilde{d}_3, \\ \tilde{b}_\alpha^{(0)} &= -\frac{(1-4\sigma)k_\alpha k_\beta \tilde{g}_\beta}{4k^3} + \frac{i(3-4\sigma)k_\alpha \tilde{g}_3}{4k^2} - \frac{\tilde{g}_\alpha}{k} + \frac{k_\alpha k_\beta \tilde{d}_\beta}{2k^2} + \frac{ik_\alpha \tilde{d}_3}{2k}. \end{aligned} \tag{27}$$

With the functions $\tilde{b}_i^{(0)}$ given above the solution of the problem, given by (9), (17) and (20), is completely determined; it remains to perform the reverse Fourier transforms.

## 4 Surface Displacement. Mindlin General Solution

We limit ourselves to give here the surface displacement

$$\tilde{u}_\alpha^{(0)} = \tilde{b}_\alpha^{(0)} - \frac{ik_\alpha \tilde{\Phi}^{(0)}}{1-2\sigma}, \qquad \tilde{u}_3^{(0)} = \tilde{b}_3^{(0)} - \frac{\tilde{\Phi}^{(1)}}{1-2\sigma}, \tag{28}$$

derived from (9). Making use of $\tilde{b}_i^{(0)}$ from (3) and $\tilde{\Phi}^{(0)}$and $\tilde{\Phi}^{(1)}$ from (25), we get

$$\begin{aligned} \tilde{u}_\alpha^{(0)} &= -\frac{\tilde{g}_\alpha}{k} + \frac{\sigma k_\alpha k_\beta \tilde{g}_\beta}{k^3} + \frac{i(1-2\sigma)k_\alpha \tilde{g}_3}{2k^2} + \frac{k_\alpha k_\beta \tilde{d}_\beta}{2k^2} + \frac{ik_\alpha \tilde{d}_3}{2k}, \\ \tilde{u}_3^{(0)} &= -\frac{(1-\sigma)\tilde{g}_3}{k} - \frac{i(1-2\sigma)k_\alpha \tilde{g}_\alpha}{2k^2} + \frac{ik_\alpha \tilde{d}_\alpha}{2k} - \frac{1}{2}\tilde{d}_3. \end{aligned} \tag{29}$$

For the Mindlin problem we assume a point force of the form

$$\mathbf{f} = \mathbf{f}^{(0)}\delta(\mathbf{r}-\mathbf{r}_0) = \mathbf{f}^{(0)}\delta(\boldsymbol{\rho})\delta(z-z_0) \tag{30}$$

localized at the point with the position vector $\mathbf{r}_0$ of coordinates $(0,0,z_0)$, $z_0 < 0$. The in-plane Fourier transform of this force is

$$\tilde{\mathbf{f}} = \mathbf{f}^{(0)}\delta(z-z_0). \tag{31}$$

From (18), (19) and (25) we get

$$\begin{aligned} \tilde{\mathbf{g}} = \mathbf{f}^{(0)}e^{-k|z_0|}, \qquad \tilde{\mathbf{c}} &= \mathbf{f}^{(0)}(e^{-k|z-z_0|} - e^{-k|z+z_0|}), \\ \tilde{\mathbf{d}} &= -z_0\mathbf{f}^{(0)}e^{-k|z_0|}; \end{aligned} \tag{32}$$

the surface displacement given by (29) becomes

$$\tilde{u}_\alpha^{(0)} = \left[ -\frac{f_\alpha^{(0)}}{k} + \frac{\sigma k_\alpha k_\beta f_\beta^{(0)}}{k^3} + \frac{i(1-2\sigma)k_\alpha f_3^{(0)}}{2k^2} - \frac{z_0 k_\alpha k_\beta f_\beta^{(0)}}{2k^2} - \frac{i z_0 k_\alpha f_3^{(0)}}{2k} \right] e^{-k|z_0|},$$
$$\tilde{u}_3^{(0)} = \left[ -\frac{(1-\sigma) f_3^{(0)}}{k} - \frac{i(1-2\sigma)k_\alpha f_\alpha^{(0)}}{2k^2} - -\frac{i z_0 k_\alpha f_\alpha^{(0)}}{2k} + \frac{1}{2} z_0 f_3^{(0)} \right] e^{-k|z_0|}. \tag{33}$$

Since the multiplication by $k_\alpha$ of the Fourier transform is associated with the operator $\partial_\alpha$, it is easy to see that the reverse Fourier transforms of (33) are given by

$$2\pi \cdot u_\alpha^{(0)} = -f_\alpha^{(0)} I^{(1)} - \frac{1}{2} f_\beta^{(0)} \partial_\beta \left[ 2\sigma I_\alpha^{(3)} - z_0 I_\alpha^{(2)} \right] + \frac{1}{2} f_3^{(0)} \left[ (1-2\sigma) I_\alpha^{(2)} - z_0 I_\alpha^{(1)} \right],$$
$$4\pi \cdot u_3^{(0)} = -f_3^{(0)} \left[ 2(1-\sigma) I^{(1)} - z_0 I^{(0)} \right] - f_\alpha^{(0)} \left[ (1-2\sigma) I_\alpha^{(2)} + z_0 I_\alpha^{(1)} \right], \tag{34}$$

where

$$I_\alpha^{(n)} = \partial_\alpha I^{(n)}, \quad I^{(n)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \frac{e^{i\mathbf{k}\cdot\boldsymbol{\rho}}}{k^n} e^{-k|z_0|}, \; n = 0, 1, 2, 3. \tag{35}$$

The integral $I^{(1)} = 1/r_1$ is the Sommerfeld integral [43], where $r_1 = (\rho^2 + z_0^2)^{1/2}$ ($\rho$ being the magnitude of the vector $\boldsymbol{\rho}$). By differentiating $I^{(1)}$ with respect to $z_0$ we get $I^{(0)} = -z_0/r_1^3$. The integral $I^{(2)}$ is singular; in [13] the function $-\ln(r_1 + |z_0|)$ is used for it. However, we need $I_\alpha^{(2)}$, which is finite and can be computed by means of the Bessel function $J_0(k\rho)$; we get

$$I_\alpha^{(2)} = -\frac{x_\alpha}{r_1(r_1 + |z_0|)}. \tag{36}$$

Similarly, the integral $I^{(3)}$ is singular, but $I_\alpha^{(3)}$ is finite and can be calculated by means of the Bessel function $J_1(k\rho)$; indeed, from (35) we have

$$I_\alpha^{(3)} = \frac{1}{2\pi} \partial_\alpha \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \frac{e^{i\mathbf{k}\cdot\boldsymbol{\rho}}}{k^3} e^{-k|z_0|} = \partial_\alpha \int_0^\infty dk \frac{J_0(k\rho)}{k^2} e^{-k|z_0|}$$
$$= -\frac{x_\alpha}{\rho} \int_0^\infty dk \frac{J_1(k\rho)}{k} e^{-k|z_0|} = -\frac{x_\alpha}{r_1 + |z_0|}, \tag{37}$$

the last integral being given in [44]. This completes the solution of the generalized Mindlin problem given by (34) for the surface displacement. It is easy to recover from (34) the two particular cases usually presented in literature [13, 27]. Indeed, for $f_3^{(0)} \neq 0$, $f_\alpha^{(0)} = 0$ we get from (34)

$$u_\rho^{(0)} = -\frac{f_3^{(0)}}{4\pi} \left[ \frac{|z_0|}{r_1^2} + \frac{1-2\sigma}{r_1 + |z_0|} \right] \frac{\rho}{r_1}, \qquad u_3^{(0)} = -\frac{f_3^{(0)}}{4\pi} \left[ 2(1-\sigma) + \frac{z_0^2}{r_1^2} \right] \frac{1}{r_1}, \tag{38}$$

where $u_\rho^{(0)}$ is the radial component of the displacement (along the vector $\boldsymbol{\rho}$); similarly, for $f_1^{(0)} \neq 0$, $f_2^{(0)} = f_3^{(0)} = 0$ we get from (34)

$$u_3^{(0)} = -\frac{f_1^{(0)}}{4\pi} \left[ \frac{|z_0|}{r_1^2} - \frac{1-2\sigma}{r_1 + |z_0|} \right] \frac{x}{r_1} \tag{39}$$

(it is usual to give only the $u_3^{(0)}$ component, because it has a simple expression). The results (38) and (39) coincide with those given in[13, 27]. We can see in (34) the separate contributions of the in-plane components $f_\alpha^{(0)}$ and the perpendicular-to-surface component $f_3^{(0)}$. For a full comparison we note that the force $\mathbf{f}^{(0)}$ in the equations given above includes the factor $-2(1+\sigma)/E$.

## 5 Tensor Force

The (reduced) tensor force given by (2), placed at the point with the position vector $\mathbf{r}_0$ of coordinates $(0, 0, z_0)$, $z_0 < 0$, has the components

$$f_j = M_{jl}\partial_l\delta(\mathbf{r}-\mathbf{r}_0) = M_{j\alpha}\partial_\alpha\delta(\boldsymbol{\rho})\delta(z-z_0) + M_{j3}\delta(\boldsymbol{\rho})\delta'(z-z_0); \tag{40}$$

their in-plane Fourier transform are

$$\tilde{f}_j = iM_{j\alpha}k_\alpha\delta(z-z_0) + M_{j3}\delta'(z-z_0). \tag{41}$$

From (18), (19) and (25) we get

$$\begin{gathered}\tilde{g}_j = (iM_{j\alpha}k_\alpha - M_{j3}k)e^{-k|z_0|},\\ \tilde{c}_j = iM_{j\alpha}k_\alpha\left(e^{-k|z-z_0|} - e^{-k|z+z_0|}\right) - M_{j3}k\left[sgn(z-z_0)e^{-k|z-z_0|} - e^{-k|z+z_0|}\right],\\ \tilde{d}_j = \left[-iz_0M_{j\alpha}k_\alpha + M_{j3}(1+kz_0)\right]e^{-k|z_0|}.\end{gathered} \tag{42}$$

Inserting these quantities in (29) we get

$$\begin{aligned}\tilde{u}_\alpha^{(0)} &= \left[M_{\alpha3} - \frac{iM_{\alpha\beta}k_\beta}{k} + \frac{iM_{\beta\gamma}k_\alpha k_\beta k_\gamma}{2k^3}(2\sigma - kz_0)\right.\\ &\quad\left. + \frac{z_0M_{3\beta}k_\alpha k_\beta}{k} + \frac{iM_{33}k_\alpha}{2k}(2\sigma + kz_0)\right]e^{-k|z_0|},\\ \tilde{u}_3^{(0)} &= \left[\frac{M_{\alpha\beta}k_\alpha k_\beta}{2k^2}(1-2\sigma+kz_0) + iz_0M_{3\alpha}k_\alpha\right.\\ &\quad\left. + \frac{1}{2}M_{33}(1-2\sigma-kz_0)\right]e^{-k|z_0|}\end{aligned} \tag{43}$$

and, with the reverse Fourier transforms,

$$\begin{aligned}2\pi\cdot u_\alpha^{(0)} &= -M_{\alpha\beta}I_\beta^{(1)} + M_{\alpha3}I^{(0)} - \frac{1}{2}M_{\beta\gamma}\partial_\beta\partial_\gamma\left[2\sigma I_\alpha^{(3)} - z_0I_\alpha^{(2)}\right]\\ &\quad - z_0M_{3\beta}\partial_\beta I_\alpha^{(1)} + \frac{1}{2}M_{33}\left[2\sigma I_\alpha^{(1)} + z_0I_\alpha^{(0)}\right],\\ 2\pi\cdot u_3^{(0)} &= -\frac{1}{2}M_{\alpha\beta}\partial_\beta\left[(1-2\sigma)I_\alpha^{(2)} + z_0I_\alpha^{(1)}\right]\\ &\quad + z_0M_{3\alpha}I_\alpha^{(0)} + \frac{1}{2}M_{33}\left[(1-2\sigma)I^{(0)} - z_0\frac{\partial}{\partial z_0}I^{(0)}\right].\end{aligned} \tag{44}$$

Making use of the integrals $I_\alpha^{(n)}$ and $I^{(0)}$ given above, we get from (44) the components $u_\alpha^{(0)}$ and $u_3^{(0)}$ of the surface displacement caused by a point tensor force localized beneath the surface of an isotropic elastic half-space. We can see from (44) that $u_\alpha^{(0)}$ is vanishing for $\rho \to 0$ and goes like $1/\rho^2$ for $\rho \to \infty$; it attains a maximum value for distances $\rho$ of the order of $|z_0|$. The component $u_3^{(0)}$ goes like $1/z_0^2$ for $\rho \to 0$ and like $1/\rho^2$ for $\rho \to \infty$. A simplified version of (44) is obtained for $M_{ij} = M\delta_{ij}$; noting that $\partial_\beta^2 I_\alpha^{(n)} = -I_\alpha^{(n-2)}$, we get

$$\begin{aligned} u_\alpha^{(0)} &= -\frac{1-2\sigma}{2\pi} M I_\alpha^{(1)} = \frac{1-2\sigma}{2\pi} M \frac{x_\alpha}{r_1^3}, \\ u_3^{(0)} &= \frac{1-2\sigma}{2\pi} M I^{(0)} = \frac{1-2\sigma}{2\pi} M \frac{|z_0|}{r_1^3} \end{aligned} \tag{45}$$

(in deriving (45) $\partial_\alpha^2 I^{(1)}$ occurs, which may be written as $-I^{(-1)} = -\partial I^{(0)}/\partial z_0$ by extending the definition of the integrals $I^{(n)}$ to $n = -1$; the same integral appears also in (44)).

## 6 Force on the Surface

If the force density $\bar{\mathbf{p}}$ is applied on the surface $z = 0$, external loads appear in the boundary conditions, which read

$$u_{\alpha 3} = -\frac{1+\sigma}{E}\bar{p}_\alpha, \qquad (1-\sigma)u_{33} + \sigma u_{\alpha\alpha} = -\frac{(1+\sigma)(1-2\sigma)}{E}\bar{p}_3, \qquad z = 0 \tag{46}$$

(the volume force is set equal to zero); it is convenient to absorb the factor $(1+\sigma)/E$ in the force density $\bar{\mathbf{p}}$, by introducing the reduced force density $\mathbf{p} = (1+\sigma)\bar{\mathbf{p}}/E$. The quantities $\tilde{\mathbf{g}}$, $\tilde{\mathbf{c}}$ and $\tilde{\mathbf{d}}$ introduced above (in (18), (19) and (25)) are vanishing, and the expressions for $\tilde{\mathbf{b}}$ and $\tilde{\Phi}$ (given by (17) and, respectively, (20)) are simplified. For a general point force density on the surface given by $\mathbf{p} = \mathbf{p}^{(0)}\delta(\boldsymbol{\rho})$, where $\mathbf{p}^{(0)}$ is the (reduced) force, the boundary conditions (26) become

$$\begin{aligned} k\tilde{b}_\alpha^{(0)} + \frac{k_\alpha k_\beta \tilde{b}_\beta^{(0)}}{4(1-\sigma)k} + \frac{i(3-4\sigma)k_\alpha \tilde{b}_3^{(0)}}{4(1-\sigma)} &= -p_\alpha^{(0)}, \\ i(3-4\sigma)k_\alpha \tilde{b}_\alpha^{(0)} - (5-4\sigma)k\tilde{b}_3^{(0)} &= 8(1-\sigma)p_3^{(0)}, \end{aligned} \tag{47}$$

where $p_\alpha^{(0)}$, $\alpha = 1, 2$, $p_3^{(0)}$ are the components of the (reduced) force $\mathbf{p}^{(0)}$. The solutions of the system of equations (47) are

$$\begin{aligned} ik_\alpha \tilde{b}_\alpha^{(0)} &= -\frac{i(5-4\sigma)k_\alpha p_\alpha^{(0)}}{4k} - \frac{(3-4\sigma)p_3^{(0)}}{2}, \\ k\tilde{b}_3^{(0)} &= -\frac{i(3-4\sigma)k_\alpha p_\alpha^{(0)}}{4k} - \frac{(5-4\sigma)p_3^{(0)}}{2}, \\ \tilde{b}_\alpha^{(0)} &= -\frac{p_\alpha^{(0)}}{k} - \frac{(1-4\sigma)k_\alpha k_\beta p_\beta^{(0)}}{4k^3} + \frac{i(3-4\sigma)k_\alpha p_3^{(0)}}{2k^2}. \end{aligned} \tag{48}$$

Making use of $\tilde{\mathbf{b}}$ and $\tilde{\Phi}$ given by (17) and, respectively, (20), we obtain the Fourier transform of the displacement

$$\begin{aligned}\tilde{u}_\alpha &= \left[-\frac{p_\alpha^{(0)}}{k} + \frac{\sigma k_\alpha k_\beta p_\beta^{(0)}}{k^3} + \frac{i(1-2\sigma)k_\alpha p_3^{(0)}}{k^2} + \frac{izk_\alpha}{2k^2}\left(ik_\beta p_\beta^{(0)} + 2kp_3^{(0)}\right)\right]e^{-k|z|},\\ \tilde{u}_3 &= \left[-\frac{i(1-2\sigma)k_\alpha p_\alpha^{(0)}}{2k^2} - \frac{2(1-\sigma)p_3^{(0)}}{k} + \frac{z}{2k}\left(ik_\alpha p_\alpha^{(0)} + 2kp_3^{(0)}\right)\right]e^{-k|z|}\end{aligned} \tag{49}$$

and, by reverse Fourier transforming,

$$\begin{aligned}2\pi\cdot u_\alpha &= -p_\alpha^{(0)}\bar{I}^{(1)} - \frac{1}{2}p_\beta^{(0)}\partial_\beta\left[2\sigma\bar{I}_\alpha^{(3)} - z\bar{I}_\alpha^{(2)}\right] + p_3^{(0)}\left[(1-2\sigma)\bar{I}_\alpha^{(2)} + z\bar{I}_\alpha^{(1)}\right],\\ 2\pi\cdot u_3 &= -\frac{1}{2}p_\alpha^{(0)}\left[(1-2\sigma)\bar{I}_\alpha^{(2)} - z\bar{I}_\alpha^{(1)}\right] - p_3^{(0)}\left[2(1-\sigma)\bar{I}^{(1)} - z\bar{I}^{(0)}\right],\end{aligned} \tag{50}$$

where the integrals $\bar{I}^{(n)}$, $\bar{I}_\alpha^{(n)}$ are the corresponding integrals $I^{(n)}$, $I_\alpha^{(n)}$ given by (35) with $z_0$ replaced by $z$. This is the general solution of the Boussinesq-Cerruti problem for a point force of arbitrary orientation acting on the surface of the half-space. For the particular case $p_\alpha^{(0)} = 0$ (force perpendicular to the surface, Boussinesq problem) we obtain from (50)

$$\begin{aligned}u_\alpha &= -\frac{p_3^{(0)}}{2\pi}\left(\frac{1-2\sigma}{r+|z|} + \frac{z}{r^2}\right)\frac{x_\alpha}{r},\\ u_3 &= -\frac{p_3^{(0)}}{2\pi}\left[2(1-\sigma) + \frac{z^2}{r^2}\right]\frac{1}{r},\end{aligned} \tag{51}$$

which coincide with the well-known results given in [19, 27, 45]; $r = (\rho^2 + z^2)^{1/2}$ is the magnitude of the position vector $\mathbf{r}$. The solution for the Cerruti problem is obtained from (50) for $p_3^{(0)} = p_2^{(0)} = 0$ and $p_1^{(0)} \neq 0$.

## 7 Discussion and Final Remarks

The displacement vector in an isotropic elastic half-space is calculated in this paper for a point tensor force concentrated at an inner point. The tensor force may correspond to the "double-couple" representation of the forces generated by localized seismic sources and governed by the tensor of the seismic moment. Explicit results are given for the surface displacement. This problem may be viewed as being in the same class of static problems of elastic equilibrium of the half-space under the action of concentrated loads known as Mindlin, Melan, Boussinesq, Cerruti or Flamant problems. The problem is solved by a special method which implies the generalized Poisson equation and the use of in-plane Fourier transforms (*i.e.*, Fourier transforms with respect to the coordinates parallel with the surface). The starting point is the decomposition of the displacement by means of the Helmholtz potentials and the use of a simplified version of the Grodskii-Neuber-Papkovitch procedure. By this approach, we obtain also the solution of the generalized Mindlin and Boussinesq-Cerruti problems for point forces of arbitrary structure and orientation, acting beneath and, respectively, on the surface of the half-space. The method used here is particularly convenient for accounting for the boundary conditions. For inhomogeneous boundary conditions, when the (volume) force is absent in the generalized Poisson equation, and only the values of the functions and their normal derivatives at the surface as "force" terms are included, the external loads appear in the boundary conditions. In Fourier transforms, the boundary conditions generate a system of algebraic equations which can be solved for the values of the functions at the surface, thus providing a completely determined solution. The method can be extended to other geometries, or other similar problems (like a thick plate, for instance). The solution to the problems regarding forces concentrated along infinite lines (Melan and Flamant problems) can be obtained immediately by a direct integration of the solutions of the corresponding point forces.

Finally we comment upon another starting point for the general solution of equilibrium of the isotropic elastic half-space. In (4) we introduce the notation

$$D = \operatorname{div} \mathbf{u}, \tag{52}$$

such that (4) becomes

$$\Delta \mathbf{u} = \mathbf{f} - \frac{1}{1-2\sigma} \operatorname{grad} D. \tag{53}$$

These equations may be considered an alternate starting point with respect to the simplified version of the Grodskii-Neuber-Papkovitch procedure used here. In order to satisfy the boundary conditions it suffices to generalize (53) to

$$\Delta \mathbf{u} = \mathbf{f} - \frac{1}{1-2\sigma} \operatorname{grad} D - \mathbf{u}^{(1)}\delta(z) - \mathbf{u}^{(0)}\delta'(z), \tag{54}$$

while maintaining (52); we solve (54) for $\mathbf{u}$ as a function of $D$ and use this solution to compute $D$ from (52); this provides a self-consistency relation which gives $D$ and, finally, $\mathbf{u}$. The parameters $\mathbf{u}^{(0)}$ ($\mathbf{u}^{(1)}$ are given in terms of $\mathbf{u}^{(0)}$ by the solution of (54)) are sufficient to satisfy the boundary conditions, which become an algebraic system of equations with unknowns $\mathbf{u}^{(0)}$. This $D$-procedure is entirely equivalent with the ($\mathbf{b}$, $\Phi$)-procedure used here; it can be used for both inner or surface point forces in the half-space.

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