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B. F. Apostol

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B. F. Apostol

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Abstract The problem of vibrations generated in a homogeneous and isotropic elastic half-space by spatially concentrated forces, known in Seismology as (part of) the Lamb problem, is formulated here in terms of Helmholtz potentials of the elastic displacement. The method is based on time Fourier transforms, spatial Fourier transforms with respect to the coordinates parallel to the surface (in-plane Fourier transforms) and generalized wave equations, which include the surface values of the functions and their derivatives. This formulation provides a formal general solution to the problem of forced elastic vibrations in the homogeneous and isotropic half-space. Explicit results are given for forces derived from a gradient, localized at an inner point in the half-space, which correspond to a scalar seismic moment of the seismic sources. Similarly, explicit results are given for a surface force perpendicular to the surface and localized at a point on the surface. Both harmonic time dependence and time δ-pulses are considered (where δ stands for the Dirac delta function). It is shown that a δ-like time dependence of the forces generates transient perturbations which are vanishing in time, such that they cannot be viewed properly as vibrations. The particularities of the generation and the propagation of the seismic waves and the effects of the inclusion of the boundary conditions are discussed, as well as the role played by the eigenmodes of the homogeneous and isotropic elastic half-space. Similarly, the distinction is highlighted between the transient regime of wave propagation prior to the establishment of the elastic vibrations and the stationary-wave regime.

Keywords Lamb problem · Half-space · Vibrations · Eigenmodes

Mathematics Subject Classification 35L05 · 35L67 · 74J05 · 74J15 · 74J70 · 86

1 Introduction

The generation and the propagation of the seismic waves is a basic problem in mathematical Seismology. It gives information about the processes occurring in an earthquake focus, about the inner structure of the Earth and the effects the seismic waves have on the Earth's surface. In addressing this problem, it is convenient to approximate the Earth by a half-space bounded by a plane surface; by another useful simplification, the Earth is viewed as a homogeneous and isotropic elastic solid, though the problem may be more complex, involving, for instance, anisotropic stratified structures, or functionally graded media. Similarly, the seismic source may exhibit the complex structure of a moving dislocation, a crack or even multiple cracks. Usually, "elementary" earthquakes are considered, produced by sources localized beneath the Earth's surface, such that, for long distances, we may consider point seismic sources, i.e. sources represented by spatial δ-functions, or derivatives of the δ-functions (where δ stands for the Dirac delta function). A similar representation holds for

B. F. Apostol (✉)
Department of Seismology, Institute of Earth's Physics, PO Box MG-35, Magurele-Bucharest MG-6, Romania
E-mail: afelix@theory.nipne.ro

the time dependence of such "elementary" seismic sources. Within such circumstances, the generation and the propagation of the seismic waves, as well as the vibrations of the elastic half-space, are known as defining the so-called Lamb problem [1–3]. The recent interest in this old problem is related to the soil–structure interaction and the effects the seismic waves and, in general, elastic waves, may have upon surface [4–6].

In the complex of physical phenomena involved in the Lamb problem (generation and propagation of seismic waves, vibrations generated by seismic sources, eigenmodes) a certain particularity exists, related to the requirement of satisfying the boundary conditions at the Earth's surface, usually considered a free plane surface. In general, we may consider elastic waves propagating in the infinite space, or waves guided along surfaces and satisfying boundary conditions, or vibrations of finite domains. The problem of vibrations of an elastic sphere was solved as early as 1882 [7–9], and surface waves in an elastic half-space were derived in 1885 [10]. We show here that the problem of elastic vibrations in a homogeneous and isotropic half-space can be formulated and solved in a general form. We give here a formal solution to this problem, as well as a few explicit results for some particular cases.

The method presented here has a few convenient points. First, we use the decomposition of the displacement field and the force term in Helmholtz potentials, which obey standard wave equations. This is an important simplification, because the elementary solutions of these equations are readily available. In order to include the boundary conditions, we use the Fourier transforms with respect to the coordinates parallel with the surface (in-plane Fourier transforms). The boundary conditions are most conveniently treated by including in the wave equations the values of the potentials and their derivatives on the surface. The solution is then determined by solving a system of algebraic equations. We avoid using the Fourier transforms along the direction perpendicular to the surface (or Laplace transforms), which are not directly appropriate for a half-space.

The problem is particularly relevant for seismic waves, where the standard approach is the Cagniard–de Hoop method [11–14]. Indeed, the particularities related to the Lamb problem are highlighted by the interpretation of the seismic records. The general structure of any seismic record exhibits a preliminary feeble tremor, consisting of the primary P- and S-waves, followed by a main shock with a long tail [15,16]. It is agreed that the main shock and its long tail are related to the surface waves, though the S-wave arrivals bring an important contribution [17]. The surface waves are guided waves, i.e. solutions of the homogeneous elastic waves equation with an undetermined amplitude, satisfying boundary conditions. More generally, the solution is often represented as a superposition of incident and reflected plane waves (which satisfy the boundary conditions), or a series of various other combinations currently made in this context, related to various types of waves, like head (or " lateral") waves, cylindrical or conical waves, leaking waves, inhomogeneous, damped waves, etc. [18–31].

The seismic waves suffer multiple reflections on the Earth's surface (or on the interfaces of the internal Earth's layers), such that a stationary regime of oscillations may set in a finite time interval. The relevant magnitude of this amount of time is of the order R/c, where R is the radius of the Earth and c is the wave velocity. For $R = 6370$ km and a mean velocity $c = 5$ km/s of the elastic waves, we get $R/c \simeq 1274$ s; this time interval is much longer than the time taken by the seismic waves to propagate from the source to the Earth's surface. The effects of the seismic waves on the Earth's surface are produced in a time much shorter than the time needed for attaining a stationary regime of vibrations. It follows that, as regards the effect of the earthquakes, we are interested primarily in the transient regime of the seismic waves, where the boundary conditions are practically radiation conditions. An "elementary" earthquake is produced by sources localized both in space and time. In these circumstances, in a first approximation, the solution consists of primary P- and S-spherical waves generated by temporal and spatial δ-pulses from the seismic source (or derivatives of the δ-function). For sources with a finite temporal or spatial extension (or for multiple sources), a structure factor of the focal region is necessary. The interaction of the primary waves with the surface generates additional wave sources placed on the surface, which give rise to secondary waves; they are responsible for the main shock and the long tail recorded in seismograms [32].

2 The problem and its general solution

The equation of the elastic waves in an isotropic body reads [33]

$$\ddot{\mathbf{u}} - c_2^2 \Delta \mathbf{u} - \left(c_1^2 - c_2^2\right) \operatorname{grad} \operatorname{div} \mathbf{u} = \mathbf{f}, \quad (1)$$

where $\mathbf{u}$ is the displacement, $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$, $c_2 = \sqrt{\mu/\rho}$ are the wave velocities, λ, μ are the Lame elastic moduli, ρ is the density and $\mathbf{f}$ is the force (per unit mass). We consider this equation in the half-space occupying the region $z < 0$ and bounded by the plane surface $z = 0$; the force $\mathbf{f}$ is placed inside the half-space.

As it is well known, in the absence of the force ($\mathbf{f} = 0$) the homogeneous equation (1) extended to the whole space exhibits two types of ("free") waves: longitudinal waves, propagating with velocity c_1, and transverse waves, propagating with velocity c_2 [34–36]. In the half-space, the free equation (1) exhibits a combination of incident and reflected longitudinal and transverse waves which satisfy the boundary conditions. Similarly, surface waves (Rayleigh waves) [10,33] may propagate along the surface, being damped along the direction perpendicular to the surface. They are solutions of the homogeneous equation (1), satisfy the boundary conditions, have a free (undetermined) amplitude and their dependence on the coordinate perpendicular to the surface is separated from the dependence of the other two combined coordinates and time. A similar character may be attributed to other waves which propagate at interfaces, like the well-known Love waves, or Stonely waves [15,18,37,38]. In addition, we shall show below that other particular solutions are present in the homogeneous and isotropic elastic half-space, represented by plane waves which propagate along directions parallel with the surface $z = 0$; we may call them lateral waves, though the term "lateral" used here has a different meaning than the same term used in the current seismological literature (see, for instance, Ref. [31]).

We consider now the solutions determined by the force $\mathbf{f}$ in the inhomogeneous equation (1) for the half-space with boundary conditions; these solutions are "forced" vibrations. The force $\mathbf{p}$ on the surface, with the components p_i is given by $\sigma_{iz} \mid_{z=0} = -p_i$, where $\sigma_{ij} = \rho\left[2c_2^2 u_{ij} + (c_1^2 - 2c_2^2)\mathrm{div}\,\mathbf{u}\delta_{ij}\right]$ is the stress tensor and u_{ij} is the strain tensor [33]. We use labels $i, j, k \ldots = 1, 2, 3$ for the coordinates $x = x_1$, $y = x_2$, $z = x_3$, as well as labels $\alpha, \beta, \gamma \ldots = 1, 2$ for the coordinates $x = x_1$, $y = x_2$. The boundary conditions read

$$\partial_\alpha u_3 + \partial_3 u_\alpha \mid_{z=0} = -\frac{p_\alpha}{\rho c_2^2}, \quad 2\partial_3 u_3 + \frac{c_1^2 - 2c_2^2}{c_2^2}\mathrm{div}\,\mathbf{u} \mid_{z=0} = -\frac{p_3}{\rho c_2^2}; \tag{2}$$

in order to simplify the notations, we replace p_i by $\rho c_2^2 p_i$.

We use the Helmholtz decomposition for the displacement $\mathbf{u}$ and the force $\mathbf{f}$, by introducing

$$\mathbf{u} = \mathrm{grad}\,\Phi + \mathrm{curl}\,\mathbf{a}, \quad \mathrm{div}\,\mathbf{a} = 0,$$
$$\mathbf{f} = \mathrm{grad}\,\varphi + \mathrm{curl}\,\mathbf{h}, \quad \mathrm{div}\,\mathbf{h} = 0. \tag{3}$$

Equation (1) is transformed in two standard wave equations

$$\ddot{\Phi} - c_1^2\Delta\Phi = \varphi, \quad \ddot{\mathbf{a}} - c_2^2\Delta\mathbf{a} = \mathbf{h}, \tag{4}$$

where the potentials φ and $\mathbf{h}$ are given by $\Delta\varphi = \mathrm{div}\,\mathbf{f}$, $\Delta\mathbf{h} = -\mathrm{curl}\,\mathbf{f}$; we consider these potentials as known quantities. Since the time is uniform and the in-plane coordinates (x_1, x_2) are also uniform, it is convenient to use time and in-plane Fourier transforms of the form

$$\mathbf{u}(\mathbf{r}, z, t) = \frac{1}{2\pi}\int d\omega e^{-i\omega t}\frac{1}{(2\pi)^2}\int d\mathbf{k}\mathbf{u}(\mathbf{k}, z, \omega)e^{i\mathbf{kr}}, \tag{5}$$

where $\mathbf{r} = (x_1, x_2)$ is the in-plane position vector. For simplicity, we use the same symbol for the mutual Fourier transforms, without any risk of confusion; similarly, sometimes we drop the arguments, which can easily be read from the context. Equation (4) becomes

$$\Phi'' + \kappa_1^2\Phi = -\varphi/c_1^2, \quad \mathbf{a}'' + \kappa_2^2\mathbf{a} = -\mathbf{h}/c_2^2, \tag{6}$$

where

$$\kappa_{1,2}^2 = \omega^2/c_{1,2}^2 - k^2; \tag{7}$$

the prime means the derivation with respect to z; we note that $\kappa_{1,2}$ may be either real or imaginary, with either sign. It is our basic assumption that the Fourier transforms $\varphi(\boldsymbol{k}, z, \omega)$ and $\boldsymbol{h}(\boldsymbol{k}, z, \omega)$ of the potentials $\varphi(\boldsymbol{r}, z, t)$ and $\boldsymbol{h}(\boldsymbol{r}, z, t)$ exist.

In order to include in a convenient manner the boundary conditions (2), we introduce the surface values of the functions and their derivatives in equations, i.e. we write

$$\Phi'' + \kappa_1^2\Phi = -\varphi/c_1^2 - \Phi^1\delta(z) - \Phi^0\delta'(z),$$
$$\mathbf{a}'' + \kappa_2^2\mathbf{a} = -\mathbf{h}/c_2^2 - \mathbf{a}^1\delta(z) - \mathbf{a}^0\delta'(z), \tag{8}$$

where $\mathbf{\Phi}^0 = \mathbf{\Phi}\mid_{z=0}$, $\mathbf{a}^0 = \mathbf{a}\mid_{z=0}$, $\Phi^1 = \mathrm{d}\Phi/\mathrm{d}z\mid_{z=0}$ and $\mathbf{a}^1 = \mathrm{d}\mathbf{a}/\mathrm{d}z\mid_{z=0}$; integrating these equations along a perpendicular of infinitesimal length across the surface $z = 0$, we check immediately the terms $\Phi^1\delta(z)$, $\mathbf{a}^1\delta(z)$; multiplying the equations by z and repeating the procedure, we check the terms $\Phi^0\delta'(z)$, $\mathbf{a}^0\delta'(z)$. More formally, we can justify the presence of these singular terms by using the generalized functions (distributions) $\Phi\theta(-z)$ and $\mathbf{a}\theta(-z)$, where $\theta(z) = 1$ for $z > 0$, $\theta(z) = 0$ for $z < 0$ is the step function [39]. The parameters Φ^1, $\mathbf{a}^1$ are not independent of parameters Φ^0, $\mathbf{a}^0$.

Making use of the Green function $G = \mathrm{e}^{i\kappa|z|}/2i\kappa$ of the one-dimensional Helmholtz equation $G'' + \kappa^2 G = \delta(z)$, we can write immediately the solutions of Eq. (8):

$$\Phi = -\frac{1}{2i\kappa_1 c_1^2}\int_{-\infty}^{0}\mathrm{d}z'\varphi(z')\left[\mathrm{e}^{i\kappa_1|z-z'|} - \mathrm{e}^{i\kappa_1|z+z'|}\right] + \Phi^0\mathrm{e}^{i\kappa_1|z|},$$
$$\mathbf{a} = -\frac{1}{2i\kappa_2 c_2^2}\int_{-\infty}^{0}\mathrm{d}z'\mathbf{h}(z')\left[\mathrm{e}^{i\kappa_2|z-z'|} - \mathrm{e}^{i\kappa_2|z+z'|}\right] + \mathbf{a}^0\mathrm{e}^{i\kappa_2|z|}. \tag{9}$$

We note the occurrence of the " reflected" ("image") Green function $\mathrm{e}^{i\kappa_{1,2}|z+z'|}/2i\kappa_{1,2}$ in these formulae. We can check the transversality condition div $\mathbf{a} = 0$ in Eq. (9) (due to div $\mathbf{h} = 0$), which in Fourier transforms reads $ik_\alpha a_\alpha + a_3' = 0$; we assume $ik_\alpha a_\alpha^0 + a_3^1 = 0$. It is also worth noting in Eq. (9) that $\kappa_{1,2}$ may have either sign. The derivatives on the surface of these functions are given by

$$\Phi^1 = -\frac{1}{c_1^2}\int_{-\infty}^{0}\mathrm{d}z'\varphi(z')\mathrm{e}^{i\kappa_1|z'|} - i\kappa_1\Phi^0,$$
$$\mathbf{a}^1 = -\frac{1}{c_2^2}\int_{-\infty}^{0}\mathrm{d}z'\mathbf{h}(z')\mathrm{e}^{i\kappa_2|z'|} - i\kappa_2\mathbf{a}^0. \tag{10}$$

We write now the boundary conditions given by Eq. (2) by using the Fourier transforms; to this end, we need the second derivative $\Phi^{(2)} = \mathrm{d}^2\Phi/\mathrm{d}z^2\mid_{z=0}$ on the surface, which can be derived immediately from Eq. (8): $\Phi^{(2)} = -\kappa_1^2\Phi^0 - \varphi^0/c_1^2$, where $\varphi^0 = \varphi\mid_{z=0}$ (a similar notation will be used for $\mathbf{h}$). The boundary conditions can now be written as

$$2\kappa_1 k_1\Phi^0 + 2k_1k_2a_1^0 + \left(\kappa_2^2 + k_2^2 - k_1^2\right)a_2^0 = q_1,$$
$$2\kappa_1 k_2\Phi^0 - \left(\kappa_2^2 + k_1^2 - k_2^2\right)a_1^0 - 2k_1k_2a_2^0 = q_2,$$
$$\left(k^2 - \kappa_2^2\right)\Phi^0 - 2\kappa_2 k_2 a_1^0 + 2\kappa_2 k_1 a_2^0 = q_3, \tag{11}$$

where

$$q_1 = -p_1 - h_2^0/c_2^2 + \frac{2ik_1}{c_1^2}\int_{-\infty}^{0}\mathrm{d}z'\varphi(z')\mathrm{e}^{i\kappa_1|z'|},$$
$$q_2 = -p_2 + h_1^0/c_2^2 + \frac{2ik_2}{c_1^2}\int_{-\infty}^{0}\mathrm{d}z'\varphi(z')\mathrm{e}^{i\kappa_1|z'|},$$
$$q_3 = -p_3 + \varphi^0/c_2^2 + \frac{2i}{c_2^2}\int_{-\infty}^{0}\mathrm{d}z'\left[k_1h_2(z') - k_2h_1(z')\right]\mathrm{e}^{i\kappa_2|z'|}; \tag{12}$$

in these equations $k_{1,2}$ are the components of the vector $\mathbf{k} = (k_1, k_2)$. Equation (11) represents a system of three equations with the unknowns Φ^0, a_1^0 and a_2^0; a_3^0 is eliminated by the transversality condition div $\mathbf{a} = 0$ ($ik_\alpha a_\alpha + a_3' = 0$); from Eq. (10), it is given by

$$\kappa_2 a_3^0 = k_\alpha a_\alpha^0 + \frac{i}{c_2^2}\int_{-\infty}^{0}\mathrm{d}z' h_3(z')\mathrm{e}^{i\kappa_2|z'|}. \tag{13}$$

The solution of the system of Eq. (11) is

$$\begin{aligned}
\Phi^0 &= \frac{1}{\Delta}\left[-2\kappa_2\left(\kappa_2^2+k^2\right)(k_1q_1+k_2q_2)+\left(\kappa_2^4-k^4\right)q_3\right],\\
a_1^0 &= \frac{1}{\Delta}\left\{-2k_1k_2\left(k^2-\kappa_2^2+2\kappa_1\kappa_2\right)q_1+\left[4\kappa_1\kappa_2k_1^2\right.\right.\\
&\quad\left.\left.-\left(k^2-\kappa_2^2\right)\left(\kappa_2^2+k_2^2-k_1^2\right)\right]q_2+2\kappa_1k_2\left(\kappa_2^2+k^2\right)q_3\right\},\\
a_2^0 &= \frac{1}{\Delta}\left\{\left[-4\kappa_1\kappa_2k_2^2+\left(k^2-\kappa_2^2\right)\left(\kappa_2^2+k_1^2-k_2^2\right)\right]q_1\right.\\
&\quad\left.+2k_1k_2\left(k^2-\kappa_2^2+2\kappa_1\kappa_2\right)q_2-2\kappa_1k_1\left(\kappa_2^2+k^2\right)q_3\right\},
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
\Delta &= -\left(\kappa_2^2+k^2\right)\left[\left(\kappa_2^2-k^2\right)^2+4\kappa_1\kappa_2k^2\right]\\
&= -\frac{\omega^2}{c_2^2}\left[\left(\kappa_2^2-k^2\right)^2+4\kappa_1\kappa_2k^2\right].
\end{aligned} \tag{15}$$

Having known the parameters Φ^0 and $\mathbf{a}^0$, the potentials Φ and $\mathbf{a}$ given by Eq. (9) and the displacement $\mathbf{u} = \operatorname{grad}\Phi + \operatorname{curl}\mathbf{a}$ are completely determined; it remains to perform the inverse time and spatial Fourier transforms. This is the formal general solution to our problem.

3 General time dependence

The displacement $\mathbf{u}$ computed from the above formulae includes terms of the general form

$$\begin{aligned}
&f(\omega,\mathbf{k})F(\omega^2,\mathbf{k}),\quad \frac{f(\omega,\mathbf{k})F(\omega^2,\mathbf{k})}{\kappa_{1,2}},\\
&\frac{f(\omega,\mathbf{k})F(\omega^2,\mathbf{k})}{\Delta},\quad \frac{f(\omega,\mathbf{k})F(\omega^2,\mathbf{k})}{\kappa_2\Delta},
\end{aligned} \tag{16}$$

where the functions f come from both the volume force (via φ and $\mathbf{h}$) and the surface force and the functions F arise from the structure of the wave equations. The κ_1 in the denominator arises from the potential Φ, the κ_2 arises from a_3^0 [Eq. (9)], and Δ originates in the parameters Φ^0, $\mathbf{a}^0$ [Eq. (14)].

A force which generates vibrations includes time harmonic oscillations with a general form

$$f = f_1\delta(\omega-\omega_0) + f_1^*\delta(\omega+\omega_0), \tag{17}$$

for the function f, where ω_0 is the frequency of the force. It is easy to see that, by multiplying the functions in Eq. (16) by $\mathrm{e}^{-i\omega t}$ and integrating over ω, in order to get the time inverse Fourier transform, we get a time dependence of the form $\sim \cos\omega_0 t$, as expected; harmonic time oscillations of the force generate harmonic vibrations.

From a technical standpoint, a δ-impulse time dependence of the force is worth considering; in this case, f in Eq. (16) does not depend on ω. The denominator Δ in Eq. (16) brought a double pole $\omega = 0$; this pole does not contribute to the Fourier transform, since the functions F, F/κ_2 are even functions of ω. It is convenient to change $\kappa_{1,2} \rightarrow i\kappa_{1,2}$ in Eq. (16); then we can see that $\Delta = 0$ implies

$$\left(\kappa_2^2+k^2\right)^2 - 4\kappa_1\kappa_2k^2 = 0, \tag{18}$$

which is the well-known dispersion relation of the damped surface waves (Rayleigh waves) [33]; the solution of this equation is $\omega_0 = c_2\xi k$, where ξ $(0 < \xi < 1)$ is the solution of the Rayleigh equation

$$\xi^6 - 8\xi^4 + 8\left(3-2c_2^2/c_1^2\right)\xi^2 - 16\left(1-c_2^2/c_1^2\right) = 0. \tag{19}$$

Indeed, the Rayleigh waves are eigenmodes of the homogeneous isotropic elastic half-space, and their dispersion relation is expected to occur in the denominator of the forced vibrations. As it is well known [33], the ratio c_2/c_1 varies from $1/\sqrt{2}$ to 0, and the root ξ of Eq. (19) varies approximately from 0.87 to 0.95. We expand the determinant Δ in powers of $\omega-\omega_0$ and get

$$\Delta=-\frac{4\omega_0^2}{c_2^3}\Delta_0(\xi)k^3(\omega-\omega_0)+\cdots, \tag{20}$$

where

$$\Delta_0(\xi)=\xi\left[\xi^2-2+\frac{4\left(1-2c_2^2\xi^2/c_1^2+c_2^2/c_1^2\right)}{\sqrt{\left(1-\xi^2\right)\left(1-c_2^2\xi^2/c_1^2\right)}}\right]; \tag{21}$$

we can see that terms with Δ in the denominator in Eq. (16) have a pole $\omega=\omega_0$. The inverse time Fourier transforms of such terms gives contributions of the form

$$g(\mathbf{k})\mathrm{e}^{-ic_2\xi kt}+\text{c.c.}, \tag{22}$$

where g is a function of the wavevector $\mathbf{k}$ (we note that there exists another pole at $\omega=-\omega_0$); these contributions are harmonic oscillations of the form $\cos c_2\xi kt$, $\sin c_2\xi kt$, with frequencies depending on k.

The spatial dependence is more difficult to be computed, in general. On effecting integrals of the form

$$u=\int \mathrm{d}\mathbf{k} f(\mathbf{k},\kappa)\mathrm{e}^{i\mathbf{kr}}\mathrm{e}^{i\kappa|z|}, \tag{23}$$

which appear in the inverse spatial Fourier transform, we should be aware of the presence of κ in the integrand $f(\mathbf{k},\kappa)$, which must obey the symmetry condition $f^*(-\mathbf{k},-\kappa)=\mathbf{f}(\mathbf{k},\kappa)$ in order the function u to be real. It is more convenient to use the formula

$$u=\frac{1}{2}\int \mathrm{d}\mathbf{k}\mathrm{e}^{i\mathbf{kr}}\left[f(\mathbf{k},\kappa)\mathrm{e}^{i\kappa|z|}+f^*(-\mathbf{k},\kappa)\mathrm{e}^{-i\kappa|z|}\right], \tag{24}$$

for such integrals, which does not imply the change of the sign of κ.

A typical spatial dependence is provided by the Sommerfeld–Weyl integral [40]

$$\int \mathrm{d}\mathbf{k}\frac{\mathrm{e}^{i\mathbf{kr}}}{\kappa}\mathrm{e}^{i\kappa|z|}, \tag{25}$$

which, in this context, leads to terms of the form

$$\int \mathrm{d}\mathbf{k}\frac{\mathrm{e}^{i\mathbf{kr}}}{i\alpha k}\mathrm{e}^{-\alpha k|z|}\mathrm{e}^{-ic_2\xi kt}, \tag{26}$$

where $\alpha=\sqrt{1-c_2^2\xi^2/c_1^2}$ for $\kappa=\kappa_1$ and $\alpha=\sqrt{1-\xi^2}$ for $\kappa=\kappa_2$. The analytic continuation of the Sommerfeld–Weyl integral gives

$$\frac{1}{\sqrt{S}}\mathrm{e}^{-i\chi/2},\ S=\left[\left(r^2+\alpha^2z^2-c_2^2\xi^2t^2\right)^2+4\alpha^2z^2c_2^2\xi^2t^2\right]^{1/2},$$
$$\tan\chi=\frac{2\alpha c_2\xi\,|z|\,t}{r^2+\alpha^2z^2-c_2^2\xi^2t^2}, \tag{27}$$

for Eq. (26). We can see that in the limit of long times $S\to\infty$ and $\chi\to 0$, i.e. forces with a pulse-like time dependence do not contribute to vibrations, as expected. It is the transient regime, prior to the establishment of the stationary vibrations regime, which is relevant for time δ-pulses of perturbations. It is worth noting that a similar conclusion is reached by using the Cagniard–de Hoop method for the Green functions of the Lamb problem [3,14].

The vanishing of the denominators $\kappa_{1,2}$ in Eq. (16) means $\omega=c_{1,2}k$; in this case a z-dependence does not exist; the contribution of these terms corresponds to lateral waves, i.e. waves which propagate along directions which are parallel with the surface $z=0$. These modes are particular cases of the incident and reflected plane waves, with dispersion relations $\omega^2=c_{1,2}^2(\kappa_{1,2}^2+k^2)$ [already included in the solution, Eqs. (9) and (11)].

4 A gradient force: harmonic oscillations

The seismic sources are currently associated with the so-called tensorial representation of the forces, given by

$$f_i = m_{ij}(t)\partial_j\delta(\mathbf{r})\delta(z - z_0), \tag{28}$$

where $m_{ij}(t)$ is the seismic moment and the source is placed at $\mathbf{r} = 0$, $z = z_0$, $z_0 < 0$ (see, for instance, Ref. [31], 2nd edition, p. 60, Exercise 3.6). We consider here a particular case, where the tensor of the seismic moment reduces to a scalar $m(t)$; it is easy to see that such an expression for the force may mimic an explosion source (for a time dependence proportional to $\delta(t)$). We consider also a free surface, i.e. we set $p_i = 0$. In this case, it is easy to see that the force derives from a potential, $\mathbf{f} = \mathrm{grad}\varphi$, where the Fourier transform of the potential φ is $\varphi = m\delta(z - z_0)$; the potential $\mathbf{h}$ is zero and

$$q_\alpha = \frac{2ik_\alpha m}{c_1^2}\mathrm{e}^{i\kappa_1|z_0|},\ \ q_3 = 0; \tag{29}$$

similarly, the boundary parameters a_3^0 and a_α^0 are not zero. Making use of Eqs. (9) and (14), we get immediately Φ and Φ^0, a_α^0; it is convenient to limit ourselves to the surface displacement only, given by

$$\begin{aligned} u_\alpha^0 &= -\frac{2m\kappa_2 k_\alpha}{c_1^2\overline{\Delta}}\mathrm{e}^{i\kappa_1|z_0|},\\ u_3^0 &= v_3^0 + w_3^0,\ \ v_3^0 = -\frac{m}{c_1^2}\mathrm{e}^{i\kappa_1|z_0|},\\ w_3^0 &= -\frac{2mk^2\left(\kappa_2^2 - k^2 - 2\kappa_1\kappa_2\right)}{c_1^2\overline{\Delta}}\mathrm{e}^{i\kappa_1|z_0|}, \end{aligned} \tag{30}$$

where

$$\overline{\Delta} = \left(\kappa_2^2 - k^2\right)^2 + 4\kappa_1\kappa_2 k^2. \tag{31}$$

If the source is a harmonic oscillation with frequency ω_0, of the form $m = m_0\cos\omega_0 t$ (as for an isotropic source concentrated at an inner point in the half-space), the surface displacement has the same time dependence $u^0(\mathbf{k}, t) \sim m_0\cos\omega_0 t$, where ω in Eq. (30) is replaced by ω_0. The Fourier transforms of the term v_3^0 can be computed by means of the Sommerfeld–Weyl integral [40]

$$\frac{i}{2\pi}\int \mathrm{d}\mathbf{k}\frac{\mathrm{e}^{i\mathbf{kr}}}{\kappa}\mathrm{e}^{i\kappa|z|} = \frac{\mathrm{e}^{i\omega R/c}}{R}, \tag{32}$$

where $\kappa = \sqrt{\omega^2/c^2 - k^2}$ and $R = \sqrt{r^2 + z^2}$; we get

$$v_3^0(\mathbf{r}, t) = -\frac{m_0}{\pi c_1^2}\frac{\partial}{\partial|z_0|}\frac{\sin\omega_0 R_0/c_1}{R_0}\cos\omega_0 t, \tag{33}$$

where $R_0 = \sqrt{r^2 + z_0^2}$; we recognize in Eq. (33) a spherical-wave vibration.

In order to estimate the spatial dependence of u_α^0 and w_3^0, we note that we are often interested in distances much longer than the wavelengths $c_{1,2}/\omega_0$, such that we may assume $k < k_c \ll \omega_0/c_{1,2}$ in Eq. (30), where k_c is a cutoff wavevector. Within this approximation, we get

$$u_\alpha^0(\mathbf{r}, t) \simeq -\frac{4m_0c_2^3}{(2\pi)^2c_1^2\omega_0^3}\partial_\alpha\int \mathrm{d}\mathbf{k}\mathrm{e}^{i\mathbf{kr}}\sin\frac{\omega_0}{c_1}|z_0|\cos\omega_0 t; \tag{34}$$

the integration over a finite range of $\mathbf{k}$ in Eq. (34) leads to a function localized over a range of the order $(\Delta r)^2 \simeq 1/k_c^2$ (if we extend the integration to infinity, we get the function $\delta(\mathbf{r})$).

Within the same short-wavelength approximation, the term w_3^0 in Eq. (30) is

$$w_3^0(\mathbf{r}, t) \simeq -\frac{m_0c_2^2(c_1 - 2c_2)}{4\pi c_1^3\omega_0^2}k_c^4\cos\frac{\omega_0}{c_1}|z_0|\cos\omega_0 t. \tag{35}$$

We can see that for large distances the main contribution to the displacement is

$$v_3^0(\mathbf{r},t) \simeq -\frac{m_0\omega_0}{\pi c_1^3}\frac{|z_0|}{R_0^2}\cos\frac{\omega_0 R_0}{c_1}\cos\omega_0 t, \tag{36}$$

arising from Eq. (33).

In all the cases presented above, the spatial dependence is separated from the harmonic time dependence, as expected for typical vibrations.

5 A gradient force: δ-pulse time dependence

If the time dependence of the seismic moment is of the form $m(t) = m_0\delta(t)$, the inverse Fourier transform of the term v_3^0 given by Eq. (30) can be calculated by using the integral in Eq. (32); it leads to

$$v_3^0(\mathbf{r},t) = \frac{m_0}{2\pi c_1^2}\frac{\partial}{\partial|z_0|}\frac{\delta(t-R_0/c_1)}{R_0}, \tag{37}$$

which is the derivative of a propagating spherical wave; since the support of this function is zero, its contribution to the boundary conditions is zero.

For u_α^0 and w_3^0 in Eq. (30), the poles associated with the surface waves are active. With $\kappa_{1,2} \to i\kappa_{1,2}$, the denominator $\overline{\Delta}$ in Eq. (30) has poles at $\omega = \pm\omega_0$, where $\omega_0 = c_2\xi k$ is the frequency of the Rayleigh surface waves. The expansion in powers of $\omega \pm \omega_0$ gives

$$\left(\kappa_2^2 + k^2\right)^2 - 4\kappa_1\kappa_2 k^2 = \pm 4\frac{\Delta_0(\xi)}{c_2}k^3(\omega \mp \omega_0) + \ldots, \tag{38}$$

where $\Delta_0(\xi)$ is given by Eq. (21); taking the inverse time Fourier transforms in Eq. (30), we get

$$\begin{aligned} u_\alpha(\mathbf{k},t) &= \frac{m_0 c_2\sqrt{1-\xi^2}}{c_1^2\Delta_0(\xi)}\frac{k_\alpha}{k^2}\mathrm{e}^{-\alpha k|z_0|}\sin c_2\xi k t, \\ w_3^0(\mathbf{k},t) &= \frac{m_0 c_2\left(2-\xi^2-2\alpha\sqrt{1-\xi^2}\right)}{c_1^2\Delta_0(\xi)}\frac{1}{k}\mathrm{e}^{-\alpha k|z_0|}\sin c_2\xi k t, \end{aligned} \tag{39}$$

where $\alpha = \sqrt{1 - c_2^2\xi^2/c_1^2}$ in the exponent and the prefactors (not to be mistaken for the label of the displacement component). The inverse spatial Fourier transform of $w_3^0(\mathbf{k},t)$ in Eq. (39) implies integrals given in Eq. (26); we get $w_3^0(\mathbf{r},t) \sim \frac{1}{\sqrt{S}}\sin\chi/2$, where S and χ are given in Eq. (27) with z replaced by z_0; in the limit of large t the function $w_3^0(\mathbf{r},t)$ vanishes.

The inverse spatial Fourier transform of $u_\alpha^0(\mathbf{k},t)$ given in Eq. (39) can be effected by using the identities

$$J_\alpha = \frac{1}{2\pi}\int \mathrm{d}\mathbf{k}\frac{\mathrm{e}^{i\mathbf{kr}}}{k^2}k_\alpha \mathrm{e}^{-k|z|} = -i\partial_\alpha J, \quad J = \frac{1}{2\pi}\int \mathrm{d}\mathbf{k}\frac{\mathrm{e}^{i\mathbf{kr}}}{k^2}\mathrm{e}^{-k|z|}, \tag{40}$$

and

$$\frac{\partial J}{\partial|z|} = -\frac{1}{2\pi}\int \mathrm{d}\mathbf{k}\frac{\mathrm{e}^{i\mathbf{kr}}}{k}\mathrm{e}^{-k|z|} = -\frac{1}{R}, \tag{41}$$

where $R = \sqrt{r^2 + z^2}$; we get

$$J = -\ln\left(|z| + R\right), \quad J_\alpha = \frac{ix_\alpha}{R(|z|+R)}; \tag{42}$$

making use of $R = \sqrt{r^2 + (\alpha|z_0| \mp ic_2\xi t)^2}$ and replacing $|z|$ by $\sqrt{\alpha^2 z_0^2 + c_2^2\xi^2 t^2}$ in Eq. (42), we can see that $u_\alpha^0(\mathbf{r},t) \to 0$ for $t \to \infty$, as expected. It is worth noting that in this formulation of the problem the surface displacement given by $u_\alpha^0(\mathbf{r},t)$ and $w_3^0(\mathbf{r},t)$ is different from zero immediately after the initial moment $t = 0$, when the $\delta(t)$-perturbation occurs at the point $\mathbf{r} = 0$, $z = z_0$, $z_0 < 0$ (as expected for a vibrations approach);

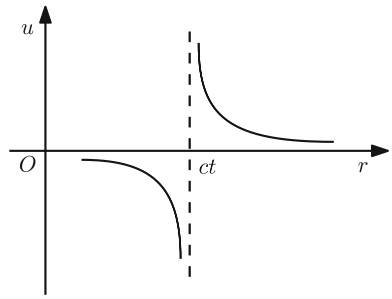

Fig. 1 Qualitative behaviour of the surface displacement $u = w_3^0$, u_r^0 versus surface distance r for a scalar-moment force and a free surface [Eq. (43)], displaying the discontinuity at $r = ct$

this is not so for the spherical wave $v_3^0(\mathbf{r}, t)$; hence, we can see that applying the vibrations approach to δ-like point forces concentrated both in time and space is not an adequate formulation of the problem.

In order to get an insight into the nature of the solution obtained above, we may use the notation $c_2\xi = c$ and the approximation $c \lesssim c_1$ ($\alpha \simeq 0$). The displacement given by Eq. (39) can be represented approximately as

$$w_3^0 \sim \frac{\mathrm{sgn}(r-ct)}{|r^2-c^2t^2|^{1/2}}, \quad u_r^0 \sim \frac{r\cdot\mathrm{sgn}(r-ct)}{|r^2-c^2t^2|^{1/2}[ct+|r^2-c^2t^2|^{1/2}]}, \tag{43}$$

where u_r^0 is the radial component of the surface displacement. These functions are represented qualitatively in Fig. 1, where we can see the sudden variation at $r \simeq ct$, specific of a source localized both in space and time ($\sim \delta(\boldsymbol{r})\delta(t)$), as well as the vanishing of the solution for long times. The singularity at $r = ct$ arises from the approximation $\alpha \longrightarrow 0$. A similar behaviour is exhibited by the component v_3^0 [Eq. (37)]. Algebraic discontinuities as those shown by Eq. (43) have also been obtained by applying the Huygens principle to secondary waves generated by the surface of a half-space [32].

6 Force on the surface

Let us assume that the volume force is zero ($\mathbf{f} = 0$, $\varphi = 0$, $\mathbf{h} = 0$) and only the component $p_3 = p(t)\delta(\mathbf{r})$ of a surface force localized at $\mathbf{r} = 0$ on the surface $z = 0$ is non-vanishing. Making use of Eqs. (9), (12) and (14), we get immediately the components of the displacement

$$\begin{aligned} u_\alpha(\mathbf{k}, \omega) &= \frac{ik_\alpha p}{\overline{\Delta}}\left[\left(\kappa_2^2 - k^2\right) \mathrm{e}^{i\kappa_1|z|} + 2\kappa_1\kappa_2 \mathrm{e}^{i\kappa_2|z|}\right], \\ u_3(\mathbf{k}, \omega) &= -\frac{i\kappa_1 p}{\overline{\Delta}}\left[\left(\kappa_2^2 - k^2\right) \mathrm{e}^{i\kappa_1|z|} - k^2 \mathrm{e}^{i\kappa_2|z|}\right]. \end{aligned} \tag{44}$$

For a harmonic force $p(t) = p_0 \cos\omega_0 t$, within the short-wavelength approximation described above, we get

$$\begin{aligned} u_\alpha(\mathbf{r}, z, t) &= \frac{c_2^2(c_1 + 2c_2)}{(2\pi)^2 c_1 \omega_0^2} p_0 \partial_\alpha \int \mathrm{d}\mathbf{k}\mathrm{e}^{i\mathbf{kr}} \left(\cos\frac{\omega_0}{c_1}z + \cos\frac{\omega_0}{c_2}z\right)\cos\omega_0 t, \\ u_3(\mathbf{r}, z, t) &= \frac{c_2^2 p_0}{2\pi^2 c_1\omega_0}\left[\int \mathrm{d}\mathbf{k}\mathrm{e}^{i\mathbf{kr}} \sin\frac{\omega_0}{c_1}|z| - \frac{2c_2^2}{\omega_0^2}\int \mathrm{d}\mathbf{k}k^2\mathrm{e}^{i\mathbf{kr}}\sin\frac{\omega_0}{c_2}|z|\right]\cos\omega_0 t, \end{aligned} \tag{45}$$

where the integration is performed over a finite range $0 < k < k_c \ll \omega_0/c_{1,2}$. For a time impulse $p(t) = p_0\delta(t)$, the contribution to the inverse time Fourier transform comes from the poles of $\overline{\Delta}$; in this case, we reach the same conclusion as above, *viz.* in the limit $t \to \infty$ the displacement is vanishing.

7 Static limit

It is worth noting that we are not allowed to take the static limit $\omega \to 0$ in the formulation given here for the Lamb problem, as expected. Indeed, both volume and surface static forces determine a deformation of the surface $z = 0$, such that the boundary conditions should be imposed on the deformed surface; it follows that the boundary conditions imposed here on the surface $z = 0$ become inadequate in this case. This can also be seen from the boundary-conditions system of Eq. (11), whose solutions given by Eq. (14) become meaningless in the static limit, since they include terms like $\delta(\omega)/\omega^2$, or $\omega^2\delta(\omega)/\omega^2$, arising from $\Delta \sim \omega^2$, $\kappa_2^2 + k^2 = \omega^2/c_2^2$ and time Fourier transforms of static forces, which are proportional to $\delta(\omega)$.We note that long-wavelength asymptotics for soliton-like Lamb and Love waves have been obtained in anisotropic multi-layered media [41–43].

The static limit exhibits a special problem. We can start, in the present formulation, with Eq. (8) for potentials and Eq. (11) for the boundary conditions in the static limit, i.e. for $\omega = 0$. Then, we see immediately that the boundary-conditions system of Eq. (11) is incompatible (its determinant is vanishing). This is due to the condition $\operatorname{div} \mathbf{a} = 0$ which is too restrictive in this case. This particularity of the static limit is related to the fact that the contributions associated with Δ and grad div in the equation of static equilibrium are entangled in the static limit. If we give up this condition, then the boundary-conditions system of Eq. (11) is compatible, and we may set $a_3^0 = 0$, for instance (or any other convenient relationship between the four unknowns Φ^0 and $\mathbf{a}^0$). Such special features of the static limit can be seen in the well-known Grodskii–Neuber–Papkovitch approach [44–46] (see also Ref. [47]).

8 Concluding remarks

The problem of elastic vibrations in a homogeneous and isotropic half-space, known usually as part of the Lamb problem in Seismology, is formulated here in terms of the Helmholtz potentials of the elastic displacement. The formulation is based on time Fourier transforms, spatial Fourier transforms with respect to the coordinates parallel to the surface of the half-space and wave equations for generalized functions (distributions), which include the surface values of the functions and their derivatives. This formulation allows a formal general solution for the vibrations in the homogeneous and isotropic half-space. Explicit results are given for forced vibrations generated in the half-space by forces derived from a spatial gradient and concentrated at an inner point in the half-space; these forces correspond to a scalar seismic moment of the seismic sources. Similarly, explicit results are given for forces concentrated at a point on the surface of the half-space. Both harmonic oscillations and δ-like time pulses are considered for these forces. It is shown that time pulses of the δ-type generate transient perturbations which are vanishing in time; consequently, such perturbations cannot be properly regarded as vibrations, as expected. For harmonic oscillations, the vibrations of the half-space are driven by forces, while for a δ-like time dependence of the forces the results are governed by surface (Rayleigh) and lateral eigenmodes. It is emphasized that the vibrations formulation of the problem is meaningful only for long times, such that the waves have sufficient time to reach the surface and establish the stationary vibrations regime.

It is well known that forces concentrated both in time and space (like δ-functions, or derivatives of δ-functions) generate elastic spherical waves; the boundary conditions are irrelevant for such propagating waves, due to their vanishing support. Their interaction with the surface generates additional wave sources, which produce secondary waves; the boundary conditions are included in constructing the secondary-waves sources [32]. It is this transient regime of propagating waves which is relevant in the "elementary" earthquakes, i.e. earthquakes which are produced by forces concentrated both in space and time. The original spherical waves are the primary waves associated with the " preliminary feeble tremor" [15,16]; the secondary waves generate the main shock and the long tail, documented by the seismic records. The surface waves, or the lateral waves, as eigenmodes, have no direct bearing on the vibrations, other than contributing through their dispersion relations to the perturbations produced by time δ-pulses. As regards the propagating regime of these transient perturbations, all the eigenmodes of the homogeneous and isotropic elastic half-space have no relevance.

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