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The effect of the inhomogeneities on the propagation of elastic waves in isotropic bodies

B.F. Apostol

Institute of Earth's Physics, Magurele-Bucharest MG-6, P.O. Box MG-35, Romania

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ABSTRACT

A new method is introduced for estimating the effects of the inhomogeneities on the propagation of the elastic waves in isotropic bodies. The method is based on the Kirchhoff electromagnetic potentials. It is applied here for estimating the effect of a static density inhomogeneity, either extended or localized, on the elastic waves propagating in an infinite, or a semi-infinite (half-space) body. For a semi-infinite body the method leads to coupled integral equations, which are solved. It is shown that such a density inhomogeneity may renormalize the waves velocity, or may even produce dispersive waves, depending on the geometry of the body and the spatial extension of the inhomogeneity. The method can be extended to other types of geometries or inhomogeneities, as, for instance, those occurring in the elastic constants.

The effect of the inhomogeneities on the propagation of the elastic waves in structures with special, restricted geometries has always enjoyed a great deal of interest (Baker and Copson, 1950; Ewing et al., 1957; Henneke, 1972; Achenbach, 1973; Richards and Frasier, 1976; Amstrong, 1980; Rokhlin et al., 1986; Wu, 1989; Mandal, 1991; Jackson and Ivakin, 1998; Cai et al., 2000; Rathore et al., 2003). Apart from its practical importance in engineering, the problem is particularly relevant for the effect the seismic waves may have on the Earth's surface (Bullen, 1976; Aki and Richards, 1980; Ben-Menahem and Singh, 1981; Albuquerque and Mauriz, 2003; van Manen et al., 2005; Sepehrinia et al., 2008). The propagation of elastic waves in bodies with finite, special geometries, like, for instance, a semi-infinite space, poses certain technical problems. We present herein a new method of dealing with elastic waves in isotropic media, borrowed from electromagnetism. The method is based on Kirchhoff retarded potentials for the wave equation. In the present paper we analyze the change produced in the eigenfrequencies of the elastic modes by static density inhomogeneities of a certain spatial extent, distributed in an infinite, or a semi-infinite (half-space) isotropic body.

As it is well known (Landau and Lifshitz, 2005), the propagation of the elastic waves in an isotropic body is governed by the equation of motion

$$\rho\ddot{\mathbf{u}} = \mu\Delta\mathbf{u} + (\lambda+\mu)grad\cdot div\mathbf{u}, \tag{1}$$

where ρ is the body density, $\mathbf{u}$ is the field displacement and λ and μ are the Lame coefficients. We leave aside the external forces and write this equation in the form

$$\frac{1}{v_t^2}\ddot{\mathbf{u}} - \Delta\mathbf{u} = q\cdot grad\cdot div\mathbf{u}, \tag{2}$$

where $v_t = \sqrt{\mu/\rho}$ is the velocity of the transverse waves, $q = v_l^2/v_t^2 - 1$ and $v_l = \sqrt{(\lambda+2\mu)/\rho}$ is the velocity of the longitudinal waves. As it is well-known, for reasons of stability, the inequality $q > 1/3$ (actually $q > 1$ for real bodies) holds. A particular solution of Eq. (2) is given by the well-known Kirchhoff potential (Born and Wolf, 1959, Chapter II, p. 73)

$$\mathbf{u}(\mathbf{R},t) = \frac{q}{4\pi}\int d\mathbf{R}'\frac{grad\cdot div\mathbf{u}(\mathbf{R}', t-|\mathbf{R}-\mathbf{R}'|/v_t)}{|\mathbf{R}-\mathbf{R}'|}. \tag{3}$$

Indeed, making use of Fourier transforms and using also the well-known integral

$$\int d\mathbf{R}\frac{e^{i\mathbf{KR}+i\omega R/v_t}}{R} = -\frac{4\pi v_t^2}{\omega^2 - v_t^2K^2}, \tag{4}$$

E-mail address: afelix@theory.nipne.ro.

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we get the eigenvalue equation

$$(-\rho\omega^2+\mu K^2)\bar{\mathbf{u}} = -(\lambda+\mu)(\mathbf{K}\bar{u})\mathbf{K}, \tag{5}$$

where ω denotes the frequency, $\mathbf{K}$ is the wavevector and $\bar{\mathbf{u}}(\mathbf{K},\omega)$ is the Fourier transform of $\mathbf{u}(\mathbf{R},t)$. One can check immediately that Eq. (5) gives the well-known transverse and longitudinal elastic waves propagating in an infinite isotropic body.

For a semi-infinite body extending over the region $z>0$, with a free surface in the (x,y)-plane $z=0$, we use

$$\mathbf{u}\rightarrow\mathbf{u}\theta(z) = (\mathbf{v},u_3)\theta(z) \tag{6}$$

for the displacement field, where $\theta(z)=1$ for $z>0$, $\theta(z)=0$ for $z<0$ is the step function, $\mathbf{v}$ is the (x,y) in-plane component and u_3 is the normal-to-surface component of the displacement (directed along the z-coordinate). We note that we employ in fact distributions (in the sense of generalized functions) like $\theta(z)$ or $\delta(z)$, etc., instead of usual functions. For the function $\mathbf{u}$ in $\mathbf{u}\theta(z)$ (defined over the entire space) we use Fourier transforms of the type

$$\mathbf{u}(\mathbf{r},z;t) = \sum_{\mathbf{k}}\int d\omega\tilde{\mathbf{u}}(\mathbf{k}\omega;z)e^{i\mathbf{kr}}e^{-i\omega t}, \tag{7}$$

where $\mathbf{R}=(\mathbf{r},z)$ and $\tilde{\mathbf{u}}(\mathbf{k}\omega;z)$ is the (partial) Fourier transform of $\mathbf{u}(\mathbf{r},z;t)$ with respect to $\mathbf{r}$ and t. The divergence occurring in Eq. (3) can then be written as

$$div\mathbf{u} = \left(div\mathbf{v}+\frac{\partial u_3}{\partial z}\right)\theta(z)+u_3(0)\delta(z), \tag{8}$$

where we can see the occurrence of the surface term $u_3(0)=u_3(z=0)$. The gradient can be computed similarly, by using the Fourier transform given by Eq. (7).

We assume a certain region in the body, whose shape and extension is described by a function $g(\mathbf{r},z)$, where the density of the body is modified according to

$$\rho\rightarrow\rho+\rho g(\mathbf{r},z). \tag{9}$$

We employ Eq. (9) for describing an inhomogeneity in the body. It is easy to see that this change in density introduces a new source term in Eq. (2), which can be written as

$$-\frac{1}{v_t^2}g(\mathbf{r},z)\ddot{\mathbf{u}}(\mathbf{r},z;t) = \sum_{\mathbf{k}}\int d\omega\frac{\omega^2}{v_t^2}\tilde{\mathbf{h}}(\mathbf{k}\omega;z)e^{i\mathbf{kr}}e^{-i\omega t}, \tag{10}$$

where

$$\tilde{\mathbf{h}}(\mathbf{k}\omega;z) = \sum_{\mathbf{k}_1}\tilde{g}(\mathbf{k}-\mathbf{k}_1,z)\tilde{\mathbf{u}}(\mathbf{k}_1\omega;z). \tag{11}$$

Consequently, Eq. (3) becomes

$$\mathbf{u}(\mathbf{R},t) = \frac{q}{4\pi}\int d\mathbf{R}'\frac{grad\cdot div\mathbf{u}(\mathbf{R}',t-|\mathbf{R}-\mathbf{R}'|/v_t)}{|\mathbf{R}-\mathbf{R}'|} - \frac{1}{4\pi v_t^2}\int d\mathbf{R}'\frac{g(\mathbf{r}',z')\ddot{\mathbf{u}}(\mathbf{R}',t-|\mathbf{R}-\mathbf{R}'|/v_t)}{|\mathbf{R}-\mathbf{R}'|}. \tag{12}$$

Making use of the representations given above, and after performing conveniently a few integrations by parts, Eq. (12) can be simplified appreciably. The intervening integrals can be performed straightforwardly. They reduce to the known integral (Gradshteyn and Ryzhik, 2000)

$$\int_{|z|}^{\infty}dxJ_0(k\sqrt{x^2-z^2})e^{i\omega x/v_t} = \frac{i}{\kappa_0}e^{i\kappa_0|z|}, \tag{13}$$

where J_0 is the Bessel function of the first kind and zeroth order and

$$\kappa_0 = \sqrt{\frac{\omega^2}{v_t^2}-k^2}. \tag{14}$$

We get the system of coupled integral equations

$$\tilde{\mathbf{v}}(\mathbf{k}\omega;z) = -\frac{iq\mathbf{k}}{2\kappa_0}\int_0 dz'\mathbf{k}\tilde{\mathbf{v}}(\mathbf{k}\omega;z')e^{i\kappa_0|z-z'|} - \frac{q\mathbf{k}}{2\kappa_0}\frac{\partial}{\partial z}\int_0 dz'\tilde{u}_3(\mathbf{k}\omega;z')e^{i\kappa_0|z-z'|} + \frac{i\omega^2}{2v_t^2\kappa_0}\int_0 dz'\tilde{\mathbf{h}}_{\parallel}(\mathbf{k}\omega;z')e^{i\kappa_0|z-z'|} \tag{15}$$

and

$$\tilde{u}_3(\mathbf{k}\omega;z) = -\frac{q}{2\kappa_0}\frac{\partial}{\partial z}\int_0 dz'\mathbf{k}\tilde{\mathbf{v}}(\mathbf{k}\omega;z')e^{i\kappa|z-z'|} + \frac{iq}{2\kappa_0}\frac{\partial^2}{\partial z^2}\int_0 dz'\tilde{u}_3(\mathbf{k}\omega;z')e^{i\kappa|z-z'|} + \frac{i\omega^2}{2v_t^2\kappa_0}\int_0 dz'\tilde{h}_3(\mathbf{k}\omega;z')e^{i\kappa_0|z-z'|}, \tag{16}$$

where $\tilde{\mathbf{h}}_{\parallel}$ is the in-plane component of the vector $\tilde{\mathbf{h}}$ defined by Eq. (11) and $\tilde{h}_3$ is its component along the z-direction. The details for deriving these equations are given in Appendix A.

It is convenient to introduce the notations $\tilde{v}_1=\mathbf{k}\tilde{\mathbf{v}}/k$, $\tilde{v}_2=\mathbf{k}_{\perp}\tilde{\mathbf{v}}/k$, and similar ones for the vector $\tilde{\mathbf{h}}$, where $\mathbf{k}_{\perp}$ is a vector perpendicular to $\mathbf{k}$, $\mathbf{kk}_{\perp}=0$, and of the same magnitude k. Under these conditions Eq. (15) for $\tilde{v}_2$ reduces to

$$\tilde{v}_2(\mathbf{k}\omega;z) = \frac{i\omega^2}{2v_t^2\kappa_0}\int_0 dz'\tilde{h}_2(\mathbf{k}\omega;z')e^{i\kappa_0|z-z'|}. \tag{17}$$

This equation corresponds to the transverse wave polarized perpendicular to the plane of propagation (it is known in electromagnetism as the s-wave, from the German "senkrecht" which means "perpendicular"). Taking the second derivative with respect to z in this equation we get

$$\frac{\partial^2\tilde{v}_2}{\partial z^2} = -\kappa_0^2\tilde{v}_2-\frac{\omega^2}{v_t^2}\tilde{h}_2. \tag{18}$$

Here, it is worth noting the non-invertibility of the (second) derivative and the integral in Eq. (17), as a result of the discontinuity in the derivative of the function $e^{i\kappa_0|z-z'|}$ for $z=z'$. In Eq. (18) we perform a Fourier transform with respect to the coordinate z. Introducing the wavevectors $\mathbf{K}=(\mathbf{k},\kappa)$ and $\mathbf{K}_1=(\mathbf{k},\kappa_1)$ and making use of Eq., (14), Eq. (18) becomes

$$\left(\frac{\omega^2}{v_t^2}-K^2\right)\bar{v}_2(\mathbf{K}\omega) = -\frac{\omega^2}{v_t^2}\sum_{\mathbf{K}_1}\bar{g}(\mathbf{K}-\mathbf{K}_1)\bar{v}_2(\mathbf{K}_1\omega). \tag{19}$$

We assume first that function $g(\mathbf{R})$ is a constant, $g(\mathbf{R})=g$. Then, $\bar{g}(\mathbf{K})=g\delta_{\mathbf{K},0}$ and Eq. (19) gives the frequency

$$\omega = \frac{v_t}{\sqrt{1+g}}K, \tag{20}$$

an expected result, which shows that the wave velocity is renormalized as a consequence of the change in density, as described by the parameter g. Second, we assume that the function $g(\mathbf{R})$ is localized at some position $\mathbf{R}_0$ in the body over a small spatial range of linear extension a. Then, its Fourier transform can be taken almost constant, $\bar{g}(\mathbf{K})\simeq ga^3/V$, over a range $\sim 1/a$, where V is the volume of the Fourier integration and $g=g(\mathbf{R}_0)$. Under these conditions we get from Eq. (19) the dispersion relation

$$1 = -\frac{\omega^2ga^3}{v_t^2V}\sum_{\mathbf{K}}\frac{1}{\omega^2/v_t^2-K^2}. \tag{21}$$

For small values of g the solutions of this equation are given by

$$\omega^2/v_t^2 = K^2 - \frac{g\omega^2}{6\pi^2 v_t^2} = K^2 - \frac{g}{6\pi^2}K^2 + \cdots, \tag{22}$$

whence, in the first approximation, we get another renormalization of the wave velocity

$$v_t \to v_t\left(1 - \frac{g}{12\pi^2}\right). \tag{23}$$

More specifically, we may take for the localized function $g(\mathbf{R})$ a Gaussian normal distribution of the form $g(\mathbf{R}) = g\exp(-|\mathbf{R}-\mathbf{R}_0|/2a^2)$, centered at $\mathbf{R}_0$ and of standard deviation a. As it is well known, its Fourier transform is $\bar{g}(\mathbf{K}) = (g/V)(2\pi a^2)^{3/2}\exp(-i\mathbf{K}\mathbf{R}_0)\exp(-K^2a^2/2)$, which is, essentially, another Gaussian distribution, centered at $\mathbf{K}=0$ and of standard deviation $1/a$. In this case, the correction term $(g/6\pi^2)K^2$ in Eq. (22) acquires an additional factor $(2\pi)^{3/2}$. However, we emphasize here that all these numerical results are only qualitative estimations.

We note that the renormalization given by Eq. (23) does not depend on the spatial extension of the function $g(\mathbf{R})$. We also note that these results are the same for an infinite body. For a general function $g(\mathbf{R})$ we may obtain a renormalization of the velocity comprised between the two limiting cases given above by Eqs. (20) and (23). As one can see, there is a qualitative resemblance between these two results (for instance, Eq. (20) can also be written as $v_t \to v_t(1-g/2)$). But we must keep in mind that all these are only approximate, qualitative estimations. The exact solution would imply solving the integral equation (19) (a homogeneous Fredholm equation of the second kind), which, for a general kernel $\bar{g}(\mathbf{K}-\mathbf{K}_1)$, is a difficult problem. Generally speaking, it implies finding out the eigenfunctions and eigenvalues of the kernel. Under certain conditions, we may try an iterative technique, which may offer an insight into the qualitative behaviour of the solution: the dispersion relation $\omega(\mathbf{K})$ will exhibit both dispersion and anisotropy, and the waves will get anisotropic, dispersive group velocities. They may, more appropriately, be then viewed as dispersive, anisotropic wave packets.

It is also interesting the case where the localized inhomogeneities are randomly distributed in the whole body, i.e. the function $g(\mathbf{R})$ is given by

$$g(\mathbf{R}) = \sum_{i=1}^{N} g_i(\mathbf{R}-\mathbf{R}_{0i}), \tag{24}$$

where $g_i(\mathbf{R}-\mathbf{R}_{0i})$ is a function of strength g_i localized over the volume a_i^3 centered at $\mathbf{R}_{0i}$ and N denotes the number of these inhomogeneities. The Fourier transform is given then approximately by $g_i(\mathbf{K}) \simeq g_i a_i^3/V$, which extends over a volume $\sim 1/a_i^3$. Repeating the above calculations for Eq. (19) we get a renormalization of the velocity given by

$$v_t \to v_t\left(1 - \frac{1}{12\pi^2}\sum_i g_i\right) = v_t\left(1 - \frac{N\bar{g}}{12\pi^2}\right), \tag{25}$$

where $\bar{g}$ is the mean strength, as expected. If the inhomogeneities are distributed in a regular, periodic array, then the problem becomes more complicated, because the Fourier transforms will then be peaked at all the reciprocal vectors of the array. The integral equation (19) is then replaced by another integral equation, implying summation over all the reciprocal vectors, but with similar (common) kernels in all the terms of the summation. The qualitative behaviour of the solutions of such equations are known from the theory of energy bands in solids (or the propagation of light in periodic media (Brillouin and Parodi, 1956)): due to the multiple reflections, the waves may form stationary waves and the frequencies ω will be distributed in bands, separated by frequency gaps. However, such subjects will take the present discussion too far.

We can also consider a layer of thickness a, i.e. take $g(\mathbf{R}) = g(z-z_0)$, where $g(z-z_0)$ is a function localized over the thickness a around z_0. Its Fourier transform is $\bar{g}(\mathbf{k},\kappa) \simeq (ga/L)\delta_{\mathbf{k},0}$, where L is the length of the Fourier integration along the z-direction and $\bar{g}(\mathbf{k},\kappa)$ extends over a range $\sim 1/a$ as a function of κ. We note that function $g(\mathbf{R}) = g(z-z_0)$ does not depend on $\mathbf{r}$. Of course, the definition of such a (full) Fourier transform is

$$\mathbf{u}(\mathbf{r},z;t) = \mathbf{u}(\mathbf{R},t) = \sum_{\mathbf{k}\kappa}\int d\omega\bar{\mathbf{u}}(\mathbf{k},\kappa,\omega)e^{i\mathbf{k}\mathbf{r}}e^{i\kappa z}$$
$$= \sum_{\mathbf{K}}\int d\omega\bar{\mathbf{u}}(\mathbf{K},\omega)e^{i\mathbf{K}\mathbf{R}}, \tag{26}$$

(compare with Eq. (7)), where the summations (integrations) over $\mathbf{k}$, κ and ω extend over the entire space. The velocity is then renormalized according to

$$v_t \to v_t\left(1 - \frac{g}{4\pi}\right). \tag{27}$$

We turn now to Eq. (15) written for $\tilde{v}_1$ and Eq. (16) for $\tilde{u}_3$. We leave aside arguments $\mathbf{k}$, ω for simplicity, and preserve explicitly only the argument z. It is easy to see that these two equations imply

$$\tilde{u}_3(z) = -\frac{i}{k}\frac{\partial\tilde{v}_1}{\partial z} - \frac{\omega^2}{2v_t^2\kappa_0 k}\frac{\partial\tilde{H}_1}{\partial z} + \frac{i\omega^2}{2v_t^2\kappa_0}\tilde{H}_3(z), \tag{28}$$

where

$$\tilde{H}_{1,3}(z) = \int_0 dz'\tilde{h}_{1,3}(z')e^{i\kappa_0|z-z'|}. \tag{29}$$

We introduce $\tilde{u}_3(z)$ as given by Eq. (28) in Eq. (15) for $\tilde{v}_1(z)$ and take the second derivative in the resulting equation. We get

$$\frac{\partial^2\tilde{v}_1}{\partial z^2} + \kappa_0'^2\tilde{v}_1 = \frac{i\omega^2}{2v_t^2\kappa_0}\left(\frac{\partial^2\tilde{H}_1}{\partial z^2} + \frac{\kappa_0^2 v_t^2}{v_l^2}\tilde{H}_1\right) + \frac{qk\omega^2}{2v_l^2\kappa_0}\frac{\partial\tilde{H}_3}{\partial z}, \tag{30}$$

where

$$\kappa_0' = \sqrt{\frac{\omega^2}{v_l^2} - k^2}. \tag{31}$$

We introduce Fourier transforms with respect to the z-coordinate both in Eqs. (28) and (30). The Fourier transforms of the functions $\tilde{H}_{1,3}(z)$ are

$$\bar{H}_{1,3}(\kappa) = -\frac{2i\kappa_0}{\kappa^2-\kappa_0^2}\bar{h}_{1,3}(\kappa) \tag{32}$$

for $\kappa \neq \kappa_0$. Restoring the arguments, $\bar{h}_1(\kappa)$ is written, by Eq. (11), as

$$\bar{h}_1(\mathbf{K}) = \sum_{\mathbf{K}_1}\bar{g}(\mathbf{K}-\mathbf{K}_1)\bar{v}_1(\mathbf{K}_1); \tag{33}$$

a similar expression holds for $\bar{h}_3$. Doing so, we get two coupled equations

$$\bar{u}_3(\mathbf{K}) - \frac{\kappa}{k}\bar{v}_1(\mathbf{K}) + \frac{\omega^2}{\omega^2 - v_t^2K^2}\sum_{\mathbf{K}_1}\bar{g}(\mathbf{K}-\mathbf{K}_1)\left[\bar{u}_3(\mathbf{K}_1) - \frac{\kappa}{k}\bar{v}_1(\mathbf{K}_1)\right]$$
$$= 0 \tag{34}$$

and

$$(\omega^2 - v_t^2K^2)(\omega^2 - v_l^2K^2)\bar{v}_1(\mathbf{K}) + \omega^2(\omega^2 - v_l^2\kappa^2 - v_t^2k^2)\sum_{\mathbf{K}_1}\bar{g}(\mathbf{K}-\mathbf{K}_1)\bar{v}_1(\mathbf{K}_1) + qv_t^2\kappa k\omega^2\sum_{\mathbf{K}_1}\bar{g}(\mathbf{K}-\mathbf{K}_1)\bar{u}_3(\mathbf{K}_1) = 0. \quad (35)$$

In analyzing these equations we proceed as before. For a constant function $g(\mathbf{R}) = g$, whose Fourier transform is $\bar{g}(\mathbf{K}) = g\delta_{\mathbf{K},0}$, Eqs. (34) and (35) give two types of waves. For the longitudinal wave, $\bar{u}_3 = \kappa\bar{v}_1/k$, Eq. (34) is satisfied identically, while from Eq. (35) we get a renormalization of the velocity v_l which is the same as that given above by Eq. (20). For the transverse wave $\bar{u}_3 = -k\bar{v}_1/\kappa$ (p-wave, whose polarization lies in the plane of propagation) we get from Eqs. (34) and (35) the same renormalization of the velocity v_t as that given by Eq. (20).

We assume now a function $g(\mathbf{R})$ localized at some position $\mathbf{R}_0$ within the body and extending over a range $\sim a$. Its Fourier transform can be taken as $\bar{g}(\mathbf{K}) \simeq ga^3/V$ for $\mathbf{K}$ extending over a range $\sim 1/a$ and $g = g(\mathbf{R}_0)$. It is easy to see that, according to Eqs. (34) and (35), the velocity v_t is not renormalized in the first order of the (small) parameter g, but the velocity v_l acquires a renormalization given by

$$v_l \to v_l\left(1 - \frac{g}{36\pi^2}\right). \quad (36)$$

Similarly, for a layer of thickness a the velocity v_t is not renormalized in the first order of the parameter g but the frequency of the longitudinal waves becomes

$$\omega = v_lK\left(1 - \frac{gak}{4}\right); \quad (37)$$

we can see that the longitudinal waves become dispersive in this case.

For comparison we give here the results for a density inhomogeneity in an infinite elastic body. By using Fourier transforms, Eq. (12) leads to

$$\bar{\mathbf{u}}(\mathbf{K}\omega) = \frac{qv_t^2}{\omega^2 - v_t^2K^2}(\mathbf{K}\bar{\mathbf{u}})\mathbf{K} - \frac{\omega^2}{\omega^2 - v_t^2K^2}\bar{\mathbf{h}}(\mathbf{K}\omega), \quad (38)$$

where

$$\bar{\mathbf{h}}(\mathbf{K}\omega) = \sum_{\mathbf{K}_1}\bar{g}(\mathbf{K}-\mathbf{K}_1)\bar{\mathbf{u}}(\mathbf{K}_1\omega) \quad (39)$$

and we have used the integral given by Eq. (4). Eq. (38) reduces to

$$\bar{u}_{1,2}(\mathbf{K}\omega) + \frac{\omega^2}{\omega^2 - v_{l,t}^2K^2}\sum_{\mathbf{K}_1}\bar{g}(\mathbf{K}-\mathbf{K}_1)\bar{u}_{1,2}(\mathbf{K}_1\omega) = 0 \quad (40)$$

for the longitudinal waves $\bar{u}_1 = \bar{\mathbf{u}}\mathbf{K}/K$ (velocity v_l) and, respectively, transverse waves $\bar{u}_2 = \bar{\mathbf{u}}\mathbf{K}_\perp/K$ (velocity v_t), where $\mathbf{K}_\perp$ is a vector perpendicular to the wavevector $\mathbf{K}$, $\mathbf{K}\mathbf{K}_\perp = 0$, and of the same magnitude K. Both Eqs. (40) lead to a dispersion equation of the same form as the one corresponding to the s-wave (Eq. (19)). For an extended inhomogeneity both $v_{t,l}$ are renormalized according to Eq. (20), for a localized inhomogeneity both velocities are renormalized according to Eq. (23). This is different than the semi-infinite body (compare with Eq. (36)).

In conclusion we may say that we have introduced herein a new method, based on the Kirchhoff electromagnetic potentials, to estimate the effects of density inhomogeneities on the propagation of the elastic waves in isotropic bodies. We have applied this method both to an infinite body and a semi-infinite (half-space) body. For an infinite body a density inhomogeneity renormalizes the velocity of the transverse and longitudinal waves. We have estimated this effect both for an extended and a localized inhomogeneity, or for a layer, assuming that the strength of the inhomogeneity is small (parameter g). For a semi-infinite body the present method leads to coupled integral equations which we have solved. The transverse s-wave is affected in the same manner as in an infinite body, and this holds also for all the waves for an extended inhomogeneity, as expected. For a localized inhomogeneity the transverse p-wave is affected in the second-order of the parameter g, while the longitudinal wave undergoes a renormalization of velocity (different than in an infinite body). In addition, for a layer inhomogeneity, the longitudinal waves become dispersive.

The method presented here can be extended to other types of inhomogeneities, as, for instance, those produced in the elastic properties of the body (the Lame coefficients). This problem is left for a forthcoming investigation.

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Appendix A. The derivation of the Eqs. (15) and (16)

Let us denote by

$$\mathbf{F}(\mathbf{r}, z; t) = -\frac{1}{4\pi v_t^2}\int d\mathbf{R}'\frac{g(\mathbf{r}', z')\ddot{\mathbf{u}}(\mathbf{R}', t - |\mathbf{R}-\mathbf{R}'|/v_t)}{|\mathbf{R}-\mathbf{R}'|} \quad (41)$$

the second term in the *rhs* of Eq. (12), where $\mathbf{R} = (\mathbf{r}, z)$ and $\mathbf{R}' = (\mathbf{r}', z')$. We note that the integration here extends over the whole space (according to the definition of the Kirchhoff potentials). First, we replace $\mathbf{u}$ by $\mathbf{u}\theta(z)$, which will restrict the integration with respect to z' to $0 < z' < \infty$. Second, we perform the Fourier transform with respect to the time, according to Eq. (7), which will bring a factor $-\omega^2$. Then, we introduce the spatial Fourier transforms (according to the same Eq. (7)) and get

$$\tilde{\mathbf{F}}(\mathbf{r}, z; \omega) = \frac{\omega^2}{4\pi v_t^2}\sum_{\mathbf{k}_1\mathbf{k}_2}\int_0^\infty dz'\int d\mathbf{r}'\frac{\tilde{g}(\mathbf{k}_2, z')\tilde{\mathbf{u}}(\mathbf{k}_1\omega; z')}{\sqrt{(\mathbf{r}-\mathbf{r}')^2 + (z-z')^2}} \times e^{i(\omega/v_t)\sqrt{(\mathbf{r}-\mathbf{r}')^2+(z-z')^2}}e^{i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}'}. \quad (42)$$

In this equation we introduce the new variable $\mathbf{r}_1 = \mathbf{r}' - \mathbf{r}$ and put $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}$. We get immediately the Fourier transform

$$\tilde{\mathbf{F}}(\mathbf{k}\omega; z) = \frac{\omega^2}{4\pi v_t^2}\sum_{\mathbf{k}_1}\int_0^\infty dz'\int d\mathbf{r}_1\frac{\tilde{g}(\mathbf{k}-\mathbf{k}_1, z')\tilde{\mathbf{u}}(\mathbf{k}_1\omega; z')}{\sqrt{r_1^2 + (z-z')^2}} \times e^{i(\omega/v_t)\sqrt{r_1^2+(z-z')^2}}e^{i\mathbf{k}\mathbf{r}_1}. \quad (43)$$

Now, by successive integrations, we have

$$\int d\mathbf{r}_1\frac{e^{i(\omega/v_t)\sqrt{r_1^2+z^2}}}{\sqrt{r_1^2+z^2}}e^{i\mathbf{k}\mathbf{r}_1} = 2\pi\int_0^\infty dr_1 r_1\frac{e^{i(\omega/v_t)\sqrt{r_1^2+z^2}}}{\sqrt{r_1^2+z^2}}J_0(kr_1) = 2\pi\int_{|z|}^\infty dx J_0(k\sqrt{x^2-z^2})e^{i(\omega/v_t)x} = \frac{2\pi i}{\kappa_0}e^{i\kappa_0|z|}, \quad (44)$$

according to Eq. (13), where $\kappa_0 = \sqrt{\omega^2/v_t^2 - k^2}$ (Eq. (14)). In Eq. (44) J_0 is the Bessel function of the first kind and zeroth order and we made the change of variable $r_1^2 + z^2 = x^2$. This result will be used in all the subsequent calculations. We may replace now the integral with respect to $\mathbf{r}_1$ in Eq. (43) by this result and get

$$\tilde{\mathbf{F}}(\mathbf{k}\omega; z) = \frac{i\omega^2}{2v_t^2\kappa_0}\sum_{\mathbf{k}_1}\int_0^\infty dz'\tilde{g}(\mathbf{k}-\mathbf{k}_1, z')\tilde{\mathbf{u}}(\mathbf{k}_1\omega; z')e^{i\kappa_0|z-z'|}, \quad (45)$$

or

$$\tilde{\mathbf{F}}(\mathbf{k}\omega; z) = \frac{i\omega^2}{2v_t^2\kappa_0}\int_0^\infty dz'\tilde{\mathrm{h}}(\mathbf{k}\omega; z')e^{i\kappa_0|z-z'|}, \quad (46)$$

according to definition (11). We can already recognize the last term in the *rhs* of Eqs. (15) and (16).

Now, we pass to the first term in the *rhs* of Eq. (12). First, we replace here $\mathbf{u}$ by $\mathbf{u}\theta(z)$. Second, we note that this term is computed for $\mathbf{u}(\mathbf{R}', t')$, where the time t' is then replaced by the retarded time $t - |\mathbf{R} - \mathbf{R}'|/v_t$ (according to the definition of the retarded Kirchhoff potentials). Making use of Eqs. (6) and (8), and introducing the Fourier transform with respect to $\mathbf{r}$, we get

$$div\mathbf{u} = \sum_{\mathbf{k}}\left[\left(i\mathbf{k}\mathbf{v} + \frac{\partial u_3}{\partial z}\right)\theta(z) + u_3(0)\delta(z)\right]e^{i\mathbf{k}\mathbf{r}} \quad (47)$$

and

$$(grad \cdot div\mathbf{u})_\parallel = \sum_{\mathbf{k}}\left[i\mathbf{k}\left(i\mathbf{k}\mathbf{v} + \frac{\partial u_3}{\partial z}\right)\theta(z) + i\mathbf{K}u_3(0)\delta(z)\right]e^{i\mathbf{k}\mathbf{r}} \quad (48)$$

for the in-plane component of the gradient and

$$(grad \cdot div\mathbf{u})_3 = \sum_{\mathbf{k}}\left[\left(ik\frac{\partial v}{\partial z} + \frac{\partial^2 u_3}{\partial z^2}\right)\theta(z) + \left(i\mathbf{k}\mathbf{v} + \frac{\partial u_3}{\partial z}\right)\delta(z)\right]e^{i\mathbf{k}\mathbf{r}} + \sum_{\mathbf{k}}u_3(0)\delta'(z)e^{i\mathbf{k}\mathbf{r}} \quad (49)$$

for its component normal to the surface. The symbol $\delta'(z)$ denotes here the derivative of the δ-function with respect to the coordinate z. Making the Fourier transform with respect to the time, the contribution of the in-plane component of the gradient (Eq. (48)) to Eq. (12) becomes

$$\frac{q}{4\pi}\sum_{\mathbf{k}}\int_0^\infty dz'\int d\mathbf{r}'\frac{i\mathbf{k}\left(i\mathbf{k}\mathbf{v}+\frac{\partial u_3}{\partial z'}\right)}{\sqrt{(\mathbf{r}-\mathbf{r}')^2+(z-z')^2}}e^{i(\omega/v_t)\sqrt{(\mathbf{r}-\mathbf{r}')^2+(z-z')^2}}e^{i\mathbf{k}\mathbf{r}'} + \frac{q}{4\pi}\sum_{\mathbf{k}}\int d\mathbf{r}'\frac{i\mathbf{k}u_3(0)}{\sqrt{(\mathbf{r}-\mathbf{r}')^2+z^2}}e^{i(\omega/v_t)\sqrt{(\mathbf{r}-\mathbf{r}')^2+z^2}}e^{i\mathbf{k}\mathbf{r}'}. \quad (50)$$

Here, we introduce again the variable $\mathbf{r}_1 = \mathbf{r}' - \mathbf{r}$ and use the result given by Eq. (44). Now, we can write the Fourier transform of $\mathbf{v}$ as given by Eq. (12) (including the contribution given by Eq. (46)) as

$$\tilde{\mathbf{v}}(\mathbf{k}\omega; z) = -\frac{iq\mathbf{k}}{2\kappa_0}\int_0^\infty dz'\mathbf{k}\tilde{\mathbf{v}}e^{i\kappa_0|z-z'|} - \frac{q\mathbf{k}}{2\kappa_0}\int_0^\infty dz'\frac{\partial\tilde{u}_3}{\partial z'}e^{i\kappa_0|z-z'|} - \frac{q\mathbf{k}}{2\kappa_0}\tilde{u}_3(0)e^{i\kappa_0 z} + \tilde{\mathbf{F}}_\parallel(\mathbf{k}\omega; z). \quad (51)$$

In the second integral in this equation we make an integration by parts and pass from $\partial/\partial z'$ to $-\partial/\partial z$ in the derivatives of function $e^{i\kappa_0|z-z'|}$. We get immediately the Eq. (15) given in the main text.

The gradient component normal to the surface (Eq. (49)) is treated in the same way. We introduce the Fourier transform with respect to the time, then use Eq. (44) for the integration over $\mathbf{r}'$ and get the partial Fourier transform of the u_3. Thereafter, we perform an integration by parts in the first bracket in Eq. (49) which cancels out the contribution of the second bracket in this equation. Finally, we make another integration by parts for the term containing $\partial\tilde{u}_3/\partial z'$ which cancels out the contribution of the δ'-term. We give here the contribution of this δ'-term, which is perhaps a bit more difficult to compute. We have successively

$$\int_{-\infty}^{+\infty} dz'\delta'(z')e^{i\kappa_0|z-z'|} = -\int_{-\infty}^{+\infty} dz'\delta(z')\frac{\partial}{\partial z'}e^{i\kappa_0|z-z'|} = \frac{\partial}{\partial z}\int_{-\infty}^{+\infty} dz'\delta(z')e^{i\kappa_0|z-z'|} = i\kappa_0 e^{i\kappa_0 z}. \quad (52)$$

This completes the proof of the derivation of the Eqs. (15) and (16) given in the main text.

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