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ELECTROMAGNETIC EIGENMODES IN MATTER. VAN DER WAALS-LONDON AND CASIMIR FORCES

M. Apostol and G. Vaman

Department of Theoretical Physics
Institute of Atomic Physics
Magurele-Bucharest MG-6, P. O. Box MG-35, Romania

Abstract—We derive van der Waals-London and Casimir forces by calculating the eigenmodes of the electromagnetic field interacting with two semi-infinite bodies (two halves of space) with parallel surfaces separated by distance d. We adopt simple models for metals and dielectrics, well-known in the elementary theory of dispersion. In the non-retarded (Coulomb) limit we get a d^{-3}-force (van der Waals-London force), arising from the zero-point energy (vacuum fluctuations) of the surface plasmon modes. When retardation is included we obtain a d^{-4}-(Casimir) force, arising from the zero-point energy of the surface plasmon-polariton modes (evanescent modes) for metals, and from propagating (polaritonic) modes for identical dielectrics. The same Casimir force is also obtained for "fixed surfaces" boundary conditions, irrespective of the pair of bodies. The approach is based on the equation of motion of the polarization and the electromagnetic potentials, which lead to coupled integral equations. These equations are solved, and their relevant eigenfrequencies branches are identified.

1. INTRODUCTION

The Casimir force was originally derived by estimating the zero-point energy (vacuum fluctuations) of the electromagnetic field comprised in-between two ideal, perfectly reflecting, semi-infinite metals (two halves of space) separated by distance d [1]. As it is well-known, it goes like d^{-4} for distances greater than the characteristic electromagnetic wavelengths of the bodies (plasmon "wavelengths"). Further on, the calculations have been cast in a different form, by resorting to the fluctuations theory [2, 3], and a d^{-3}-force has been obtained for the

Corresponding author: M. Apostol (apoma@theory.nipne.ro).

non-retarded (Coulomb) interaction, which corresponds to the van der Waals-London force. The matter polarization is usually represented in this case by a dielectric function. Recently, there is a renewed interest in this subject, motivated, on one hand, by the role played by plasmons, polaritons and other surface effects arising from the interaction between the electromagnetic field and matter and, on the other hand, by the querries related to the applicability of a dielectric function for discontinuous bodies [4–20]. We report here on a different investigation of these forces, based on the calculation of the eigenfrequencies of the electromagnetic field interacting with matter.

We assume a simple model of matter, consisting of mobile particles with charge $-e$ and mass m, moving in a rigid neutralizing background, and subjected to certain forces. Such a model is reminiscent of the well-known jellium model of electron plasma, though it is generalized here to some extent. In the presence of the electromagnetic field matter polarizes. We leave aside the magnetization (we consider only non-magnetic matter) and relativistic effects. We represent the small disturbance in the density of the mobile charges as $\delta n = -n\mathrm{div}\mathbf{u}$, where n is the (constant) concentration of the charges and $\mathbf{u}$ is a displacement field in the positions of these charges. The charge disturbance is therefore $\rho = en\mathrm{div}\mathbf{u}$. This representation is valid for $\mathbf{K}\mathbf{u}(\mathbf{K}) \ll 1$, where $\mathbf{K}$ is the wavevector and $\mathbf{u}(\mathbf{K})$ is the Fourier transform of the displacement field.

For homogeneous and isotropic matter the displacement field obeys an equation of motion which can be taken of the form

$$m\ddot{\mathbf{u}} = -e\mathbf{E} - e\mathbf{E}_0 - m\omega_0^2\mathbf{u} - m\gamma\dot{\mathbf{u}}, \tag{1}$$

where $\mathbf{E}$ is the (internal) electric field, $\mathbf{E}_0$ is an external electric field, ω_0 is a frequency parameter corresponding to an elastic force and γ is a dissipation parameter. Making use of the temporal Fourier transform we get

$$\mathbf{u}(\omega) = \frac{e}{m}\frac{1}{\omega^2 - \omega_0^2 + i\omega\gamma}(\mathbf{E} + \mathbf{E}_0) \tag{2}$$

(where we dropped out the argument ω of the electric fields). On the other hand, from Maxwell's equation $\mathrm{div}\mathbf{E} = 4\pi en\mathrm{div}\mathbf{u}$, we get the (internal) electric field $\mathbf{E} = 4\pi ne\mathbf{u}$ (equal to $-4\pi\mathbf{P}$, where $\mathbf{P}$ is the polarization). Making use of Equation (2) we get the dielectric function

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 + i\omega\gamma} \tag{3}$$

from its definition $\mathbf{E}_0 = \varepsilon(\mathbf{E} + \mathbf{E}_0)$, where ω_p, given by $\omega_p^2 = 4\pi ne^2/m$, is the plasma frequency. The dielectric function given by Equation (3) is well known in the elementary theory of dispersion [21]. It proves to be a fairly adequate representation for matter polarization in various bodies. We can view ω_p, ω_0 and γ as free parameters, thus being able to simulate various models of matter. For $\omega_0 = \gamma = 0$ we get the well-known dielectric function of an ideal plasma; if $\omega_0 = 0$ we have the dielectric function of the optical properties of simple metals for $\omega \gg \gamma$ (Drude model), and the dielectric function corresponding to the static (or quasi-static) currents in metals for $\omega \ll \gamma$; for $\omega_0 \gg \omega_p$ we have a dielectric function of dielectrics with loss; and so on.

In addition, making use of Equation (2), we can compute also the electric conductivity σ, from its definition $\mathbf{j} = \sigma(\mathbf{E} + \mathbf{E}_0)$, where $\mathbf{j} = -en\dot{\mathbf{u}}$ is the current density. We get the well-known conductivity

$$\sigma(\omega) = \frac{\omega_p^2}{4\pi} \frac{i\omega}{\omega^2 - \omega_0^2 + i\omega\gamma}, \qquad (4)$$

whence, for instance, the static conductivity for metals $\sigma = \omega_p^2/4\pi\gamma$; parameter γ can be viewed as the reciprocal of a damping time τ (or relaxation time, or lifetime), $\gamma = 1/\tau$, and we get the well-known static conductivity $\sigma = ne^2\tau/m\gamma$.

Therefore, the equation of motion (1) turns out to be an adequate starting point for representing the matter polarization. However, we must note that for dielectrics, which may imply oscillations in localized atoms (in our model through the frequency ω_0), the classical dynamics assumed here turns out to be inadequate in the retarded regime, and a quantum treatment is then required.

In the non-retarded limit the electric field $\mathbf{E}$ in Equation (1) is given by the Coulomb law, i.e., $\mathbf{E} = -\mathrm{grad}\Phi$, where Φ is the static Coulomb potential arising in matter. The latter depends on the charge disturbance $\rho = -e\delta n$, therefore on $\mathbf{u}$. Then, it is easy to see that the equation of motion (1) leads to an integral equation for the displacement field $\mathbf{u}$. Its eigenvalues give the plasmon modes. For retarded interaction, the electric field $\mathbf{E}$ in Equation (1) is given by the vector potential $\mathbf{A}$ and the scalar potential Φ trough $\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \mathrm{grad}\Phi$. Making use of the radiation (Kirchhoff) formulae, these potentials can be expressed as integrals containing the displacement field $\mathbf{u}$ (through the charge and current densities), and we get again an integral equation for $\mathbf{u}$. Its eigenvalues give polariton-like modes. The use of integral equations in treating the electromagnetic field interacting with matter was previously indicated in connection with the so-called Ewald-Oseen extinction theorem [22]. We have applied this approach to a semi-infinite (half-space) body, as well as to a slab of finite thickness [23]. In this case, beside the bulk displacement field, there appears a surface displacement field also, and the integral equations couple these degrees of freedom. We have solved these coupled integral equations and computed bulk and surface plasmons and polaritons, dielectric response, reflected, refracted and transmitted fields, and derived generalized Fresnel relations. We employ the same procedure here for two semi-infinite bodies (two halves of the space) separated by distance d, in order to get the electromagnetic eigenfrequencies and to derive van der Waals-London and Casimir forces. We do it in two steps: First, for static Coulomb (non-retarded) interaction (valid for wavelengths much longer than the characteristic size of the bodies) and, second, for retarded interaction.

2. SURFACE PLASMONS. VAN DER WAALS-LONDON FORCES

We consider two semi-infinite bodies (two halves of space) with parallel surfaces in the (x, y)-plane, separated by distance d. The bodies occupy the regions $z < -d/2$ and, respectively, $z > d/2$. We take two displacement fields $\mathbf{u}_{1,2}$, giving rise to two charge disturbances $\delta n_{1,2} = -n_{1,2}\mathrm{div}\mathbf{u}_{1,2}$. We consider first the equation of motion for an ideal plasma. In general, we leave aside the dissipation (parameter γ in Equation (1)), which is irrelevant for our discussion. The equation of motion reads

$$m\ddot{\mathbf{u}}_1 = \mathrm{grad}\int d\mathbf{R}'U\left(\left|\mathbf{R}-\mathbf{R}'\right|\right)\left[n_1\mathrm{div}\mathbf{u}_1\left(\mathbf{R}'\right)+n_2\mathrm{div}\mathbf{u}_2\left(\mathbf{R}'\right)\right], \quad (5)$$

and a similar equation for $\mathbf{u}_2$, which can be obtained from Equation (5) by interchanging the labels 1 and 2 ($1 \longleftrightarrow 2$); $U(R) = e^2/R$ in Equation (5) is the Coulomb interaction. Since we are interested in the eigenmodes, we leave aside the external field $\mathbf{E}_0$. We use $\mathbf{R} = (\mathbf{r}, z)$ for the position vector $\mathbf{R}$, where $\mathbf{r} = (x, y)$, and the representation

$$\mathbf{u}_{1,2} = (\mathbf{v}_{1,2}, w_{1,2})\theta(\pm z - d/2) \quad (6)$$

for the displacement fields, where $\theta(z) = 1$ for $z > 0$ and $\theta(z) = 0$ for $z < 0$ is the step function; the $\pm$ sign is associated with labels 1 and 2, respectively. The divergence in Equation (5) can now be written as

$$\mathrm{div}\mathbf{u}_{1,2}=\left(\mathrm{div}\mathbf{v}_{1,2}+\frac{\partial w_{1,2}}{\partial z}\right)\theta(\pm z-d/2)+w_{1,2}(\pm d/2)\delta(\pm z-d/2), \quad (7)$$

where $w_{1,2}(\pm d/2)$ means $w_{1,2}(\mathbf{r}, z = \pm d/2)$. We notice in Equation (7) the (de)polarization charge arising at the surfaces $z = \pm d/2$. We employ Fourier representations of the form

$$\mathbf{v}_{1,2}(\mathbf{r}, z; t) = \sum_{\mathbf{k}}\int d\omega\mathbf{v}_{1,2}(\mathbf{k}, z; \omega)e^{i\mathbf{kr}}e^{-i\omega t} \quad (8)$$

and similar ones for $w_{1,2}$, and use the Fourier transform

$$\frac{1}{\sqrt{r^2+z^2}}=\sum_{\mathbf{k}}\frac{2\pi}{k}e^{-k|z|}e^{i\mathbf{kr}} \tag{9}$$

for the Coulomb potential. Then, we notice that Equation (5) implies that $\mathbf{v}_{1,2}$ are parallel with the wavevector $\mathbf{k}$ (in-plane "longitudinal" modes), and $i\mathbf{k}w_{1,2}=\frac{\partial \mathbf{v}_{1,2}}{\partial z}$. We use this latter relation to eliminate $w_{1,2}$ from the equations of motion. In addition, we introduce the notation $v_{1,2}=\mathbf{kv}_{1,2}/k$. Then, it is easy to see that Equation (5) yields two coupled integral equations

$$\begin{aligned}\omega^2 v_1 =& \frac{\omega_1^2 k}{2}\int_{d/2}^{\infty}dz' e^{-k|z-z'|}v_1+\frac{\omega_1^2}{2k}\int_{d/2}^{\infty}dz'\frac{\partial}{\partial z'}e^{-k|z-z'|}\frac{\partial v_1}{\partial z'}\\ &+\frac{\omega_2^2 k}{2}\int_{-\infty}^{-d/2}dz' e^{-k(z-z')}v_2+\frac{\omega_2^2}{2}\int_{-\infty}^{-d/2}dz' e^{-k(z-z')}\frac{\partial v_2}{\partial z'}\\ &\qquad z>d/2,\\ \omega^2 v_2 =& \frac{\omega_1^2 k}{2}\int_{d/2}^{\infty}dz' e^{k(z-z')}v_1-\frac{\omega_1^2}{2}\int_{d/2}^{\infty}dz' e^{k(z-z')}\frac{\partial v_1}{\partial z'}\\ &+\frac{\omega_2^2 k}{2}\int_{-\infty}^{-d/2}dz' e^{-k|z-z'|}v_2+\frac{\omega_2^2}{2k}\int_{-\infty}^{-d/2}dz'\frac{\partial}{\partial z'}e^{-k|z-z'|}\frac{\partial v_2}{\partial z'}\\ &\qquad z<-d/2,\end{aligned} \tag{10}$$

where $\omega_{1,2}^2=4\pi n_{1,2}e^2/m$ and we dropped out the arguments ω, $\mathbf{k}$. Integrating by parts in Equation (10) we obtain a system of two algebraic equations

$$\begin{aligned}\left(\omega^2-\omega_1^2\right)v_1 &= -\frac{1}{2}e^{-kz}\left[\omega_1^2 e^{kd/2}v_1(d/2)-\omega_2^2 e^{-kd/2}v_2(-d/2)\right],\\ &\qquad z>d/2,\\ \left(\omega^2-\omega_2^2\right)v_2 &= \frac{1}{2}e^{kz}\left[\omega_1^2 e^{-kd/2}v_1(d/2)-\omega_2^2 e^{kd/2}v_2(-d/2)\right],\\ &\qquad z<-d/2.\end{aligned} \tag{11}$$

We can see that in this non-retarded limit the two bodies are coupled only through their surfaces.

For $v_1(d/2) = v_2(-d/2) = 0$ in Equation (11) we get the bulk plasmons $\omega = \omega_{1,2}$. Making $z = \pm d/2$ in Equation (11) we get the system of equations for the surface modes. The corresponding dispersion equation is given by

$$\left(\omega^2 - \frac{1}{2}\omega_1^2\right)\left(\omega^2 - \frac{1}{2}\omega_2^2\right) - \frac{1}{4}\omega_1^2\omega_2^2 e^{-2kd} = 0. \tag{12}$$

For $d = 0$ we obtain the surface plasmon of a metallic interface given by $\omega^2 = \frac{1}{2}(\omega_1^2 + \omega_2^2)$, while for $d \to \infty$ we get the surface plasmons $\omega = \omega_{1,2}/\sqrt{2}$ for free (uncoupled) surfaces. If the body labelled by 2 for instance is a dielectric, then ω^2 in the second Equation (11) is replaced by $\omega^2 - \omega_0^2$. In the limit $\omega_0 \gg \omega_2$ and for $d = 0$ we get the surface plasmon $\omega = \omega_1/\sqrt{1+\varepsilon_2}$, corresponding to a dielectric-metal interface, where $\varepsilon_2 = 1+\omega_2^2/\omega_0^2$. For two identical metals $\omega_1 = \omega_2 = \omega_p$ we get the surface plasmons given by

$$\omega^2 = \frac{1}{2}\omega_p^2\left(1 \pm e^{-kd}\right). \tag{13}$$

They are identical with the surface plasmons of a plasma slab of thickness d. These are well-known results [24–31].

Let us label by α all the eigenvalues Ω_α of the system of Equation (11). We compute the force acting between the two bodies by using the zero-point energy

$$F = \frac{\partial}{\partial d}\sum_\alpha \frac{1}{2}\hbar\Omega_\alpha. \tag{14}$$

Although it can be included straightforwardly, it is easy to see that the temperature plays no significant role, so we may neglect the temperature effects, as usually.

We introduce the two surface modes $\Omega_{1,2}$ given by Equation (13) (labeled by wavevector $\mathbf{k}$) into Equation (14). We can see that these eigenfrequencies are functions of kd, so the force depends on distance d as $F \sim 1/d^3$. As it is well-known, such a force between two bodies implies an inter-atomic interaction $\sim 1/R^6$, where R is the distance between two atoms. This is the well-known van der Waals-London interaction [32]. Making use of the change of variable $x = kd$, and using $\sum_{\mathbf{k}} = (2\pi)^{-2}\int d\mathbf{k}$, it is a matter of straightforward computation to get a force

$$F = \frac{\hbar\omega_p}{8\pi\sqrt{2}d^3}\int_0^\infty dx \cdot x^2 e^{-x}\left(\frac{1}{\sqrt{1-e^{-x}}} - \frac{1}{\sqrt{1+e^{-x}}}\right) \tag{15}$$

per unit area from Equation (14), acting between two identical metals. The integral in Equation (15) is $\simeq 4$, so we get $F \simeq \hbar\omega_p/2\pi\sqrt{2}d^3$.

In like manner we can compute the force between two (identical) dielectrics, by replacing ω^2 in Equation (13) by $\omega^2-\omega_0^2$ and taking the limit $\omega_0 \gg \omega_p$. The result is a much weaker force $F=\hbar\omega_p^4/128\omega_0^3 d^3$. It can also be written as $F=\hbar\omega_0(\varepsilon-1)^2/128d^3$, where $\varepsilon \simeq 1+\omega_p^2/\omega_0^2$ is the (static) dielectric function in the limit $\omega \ll \omega_0$. The same result is obtained by making use of the formulae given in [32] for non-retarded interaction within the framework of the fluctuations theory (equation 82.3 p. 343 in [32]). Making use of the eigenvalues given by the roots of the dispersion Equation (12), we can compute in the same manner the force acting between two distinct bodies. For instance, we can consider a dielectric-metal pair and get straightforwardly the force $F=\hbar\omega_1\omega_2^2/32\pi\sqrt{2}\omega_0^2 d^3$, where ω_1 belongs to the metal and ω_2, ω_0 represent the dielectric.

It is worth noting that the present calculations are performed for ideal solids (in particular loss-less materials). It was shown recently [33] that the van der Waals-London or Casimir forces may change drastically in complex materials, like poor conductors for instance, both in their numerical coefficients and in their d-dependence. Some of such changes seem to be related to the high-frequency dielectric functions $\varepsilon\,(\omega\to\infty)$, which is not unity anymore, in contrast with our Equation (3). For instance, the van der Waals-London force $\sim 1/d^3$ may be removed entirely by a non-unity high-frequency dielectric function. In order to include such effects in the present theory we need a more elaborate model for such complex materials, which is beyond the aim of the present paper.

3. SURFACE PLASMON-POLARITON MODES. CASIMIR FORCE

We pass now to the retarded interaction. The electric field in Equation (1) is given by $E=-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}-\mathrm{grad}\Phi$, where $\mathbf{A}$ is the vector potential and Φ is the scalar potential. These potentials are given by

$$\mathbf{A}(\mathbf{r},z;t)=\frac{1}{c}\int d\mathbf{r}'\int dz'\frac{\mathbf{j}\,(\mathbf{r}',z';t-R/c)}{R} \tag{16}$$

and

$$\Phi(\mathbf{r},z;t)=\int d\mathbf{r}'\int dz'\frac{\rho\,(\mathbf{r}',z';t-R/c)}{R}, \tag{17}$$

where

$$\mathbf{j}=-en_1\,(\dot{\mathbf{v}}_1,\dot{w}_1)\,\theta(z-d/2)-en_2\,(\dot{\mathbf{v}}_2,\dot{w}_2)\,\theta(-z-d/2) \tag{18}$$

is the current density,

$$\rho = en_1\left(\mathrm{div}\mathbf{v}_1 + \frac{\partial w_1}{\partial z}\right)\theta(z-d/2) + w_1(d/2)\delta(z-d/2)$$
$$+en_2\left(\mathrm{div}\mathbf{v}_2 + \frac{\partial w_2}{\partial z}\right)\theta(-z-d/2) + w_2(-d/2)\delta(z+d/2) \quad (19)$$

is the charge density and $R = \sqrt{(\mathbf{r}-\mathbf{r}')^2+(z-z')^2}$. We use the Fourier representations given by Equation (8) and the Fourier transform [34]

$$\frac{e^{i\frac{\omega}{c}\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}} = \sum_{\mathbf{k}} \frac{2\pi i}{\kappa} e^{i\mathbf{kr}} e^{i\kappa|z|}, \quad (20)$$

where $\kappa = \sqrt{\frac{\omega^2}{c^2} - k^2}$. Then we compute the electric field from the potentials given by Equations (16) and (17) and use Equation (1) for $\omega_0 = 0$, $\gamma = 0$, $\mathbf{E}_0 = 0$ in order to get integral equations for $\mathbf{v}_{1,2}$, $w_{1,2}$. We define the wavevector $\mathbf{k}_\perp$ of magnitude k and perpendicular to the wavevevctor $\mathbf{k}$, and introduce the notations $v_{1,2} = \mathbf{k}\mathbf{v}_{1,2}/k$, $v_{1,2}^\perp = \mathbf{k}_\perp \mathbf{v}_{1,2}/k$. Doing so, we get the first set of integral equations

$$\begin{aligned} v_1^\perp &= -\frac{i\omega_1^2}{2c^2\kappa}\int_{d/2}^{\infty} dz' e^{i\kappa|z-z'|} v_1^\perp(z') \\ &\quad - \frac{i\omega_2^2}{2c^2\kappa}\int_{-\infty}^{-d/2} dz' e^{i\kappa(z-z')} v_2^\perp(z'), \quad z > d/2, \\ v_2^\perp &= -\frac{i\omega_1^2}{2c^2\kappa}\int_{d/2}^{\infty} dz' e^{-i\kappa(z-z')} v_1^\perp(z') \\ &\quad - \frac{i\omega_2^2}{2c^2\kappa}\int_{-\infty}^{-d/2} dz' e^{i\kappa|z-z'|} v_2^\perp(z'), \quad z < -d/2, \end{aligned} \quad (21)$$

where we dropped out the arguments ω, $\mathbf{k}$.

A similar set of coupled integral equations are obtained from Equations (16), (17) and (1) for $v_{1,2}$ and $w_{1,2}$. It is easy to notice from these integral equations the relationship

$$w_{1,2} = \frac{ik}{\kappa^2 - \omega_{1,2}^2/c^2}\frac{\partial v_{1,2}}{\partial z}, \quad (22)$$

which we use to eliminate $w_{1,2}$; so, we are left with the second set of two integral equations for $v_{1,2}$:

for $z > d/2$

$$\frac{c^2\kappa^2(\omega^2-\omega_1^2)}{c^2\kappa^2-\omega_1^2}v_1 = -\frac{i\kappa\omega_1^2(\omega^2-\omega_1^2)}{2(c^2\kappa^2-\omega_1^2)}\int_{d/2}^{\infty}dz' e^{i\kappa|z-z'|}v_1(z')$$
$$-\frac{i\kappa\omega_2^2(\omega^2-\omega_2^2)}{2(c^2\kappa^2-\omega_2^2)}\int_{-\infty}^{-d/2}dz' e^{i\kappa(z-z')}v_2(z')$$
$$+\frac{c^2k^2\omega_1^2}{2(c^2\kappa^2-\omega_1^2)}e^{i\kappa(z-d/2)}v_1(d/2)-\frac{c^2k^2\omega_2^2}{2(c^2\kappa^2-\omega_2^2)}e^{i\kappa(z+d/2)}v_2(-d/2) \quad (23)$$

and

$$\frac{c^2\kappa^2(\omega^2-\omega_2^2)}{c^2\kappa^2-\omega_2^2}v_2 = -\frac{i\kappa\omega_1^2(\omega^2-\omega_1^2)}{2(c^2\kappa^2-\omega_1^2)}\int_{d/2}^{\infty}dz' e^{-i\kappa(z-z')}v_1(z')$$
$$-\frac{i\kappa\omega_2^2(\omega^2-\omega_2^2)}{2(c^2\kappa^2-\omega_2^2)}\int_{-\infty}^{-d/2}dz' e^{i\kappa|z-z'|}v_2(z')$$
$$-\frac{c^2k^2\omega_1^2}{2(c^2\kappa^2-\omega_1^2)}e^{-i\kappa(z-d/2)}v_1(d/2)+\frac{c^2k^2\omega_2^2}{2(c^2\kappa^2-\omega_2^2)}e^{-i\kappa(z+d/2)}v_2(-d/2) \quad (24)$$

for $z < -d/2$. It is worth observing in deriving these equations the non-intervertibility of the derivatives and the integrals, according to the identity

$$\frac{\partial}{\partial z}\int_{d/2}^{\infty}dz' f\left(z'\right)\frac{\partial}{\partial z'}e^{i\kappa\left|z-z'\right|} = \kappa^2\int_{d/2}^{\infty}dz' f\left(z'\right)e^{i\kappa\left|z-z'\right|} - 2i\kappa f(z) \quad (25)$$

for any function $f(z)$, $z > d/2$; a similar identity holds for $z, z' < -d/2$. It is due to the discontinuity in the derivative of the function $e^{i\kappa\left|z-z'\right|}$ for $z = z'$. We can see that these equations become Equation (10) in the non-retarded limit by taking formally the limit $c \to \infty$. However, this is not so for their dispersion equations, as we shall see below. One can also see from Equations (21), (23) and (24) that the coupling between the two bodies is performed through both bulk and surface degrees of freedom, in contrast to the non-retarded situation, where this coupling occurs only through surfaces (Equation (11)).

We turn now to Equation (21). Taking the second derivative with respect to z in these equations we get

$$\frac{\partial^2 v_{1,2}^{\perp}}{\partial z^2} + \left(\kappa^2 - \frac{\omega_{1,2}^2}{c^2}\right)v_{1,2}^{\perp} = 0, \quad (26)$$

which tells that $v_{1,2}^{\perp}$ are a superposition of two waves $e^{\pm i\kappa_{1,2}z}$, where

$$\kappa_{1,2} = \sqrt{\kappa^2 - \frac{\omega_{1,2}^2}{c^2}}. \quad (27)$$

We note that such modes are polaritonic modes, since $\omega^2 = c^2\left(k^2+\kappa^2\right) = c^2\left(k^2+\kappa_{1,2}^2\right)+\omega_{1,2}^2 = c^2K_{1,2}^2+\omega_{1,2}^2$, where $\mathbf{K}_{1,2} = (\mathbf{k},\kappa_{1,2})$, which is the well-kown dispersion relation for the polaritonic modes. It can also be written as $\omega^2\varepsilon_{1,2} = c^2K_{1,2}^2$, where $\varepsilon_{1,2} = 1-\omega_{1,2}^2/\omega^2$ is the dielectric function for metals. This relation is well-known in the so-called thery of "effective medium permittivity". We take $v_{1,2}^{\perp} = A_{1,2}e^{i\kappa_{1,2}z}$, where $A_{1,2}$ are amplitudes to be determined. Then, Equation (21) have non-trivial solutions for frequencies ω given by the roots of the dispersion equation

$$e^{2i\kappa d} = \frac{(\kappa_1+\kappa)(\kappa_2-\kappa)}{(\kappa_1-\kappa)(\kappa_2+\kappa)}. \tag{28}$$

Equation (28) has a branch of roots for the damped regime (evanescent modes) $\kappa_1 = i\alpha_1$, $\kappa_2 = -i\alpha_2$, given by

$$\tan\kappa d = \frac{\kappa\left(\alpha_1+\alpha_2\right)}{\kappa^2-\alpha_1\alpha_2}, \tag{29}$$

where

$$\alpha_{1,2} = \sqrt{\frac{\omega_{1,2}^2}{c^2}-\kappa^2}, \quad \omega_{1,2} > c\kappa, \tag{30}$$

and κ real. Since these modes are damped inside the bodies and propagating in-between the bodies they may be called surface plasmon-polariton modes. It is worth noting the correct choice of the sign of the square root in this case, in order to get the correct behaviour at infinity, $v_1^{\perp} = A_1^{\perp}e^{-\alpha_1 z}$ for $z > d/2$ and $v_2^{\perp} = A_2^{\perp}e^{\alpha_2 z}$ for $z < -d/2$. The roots of Equation (29) can be written as

$$\Omega_1 = c\sqrt{k^2+\frac{\pi^2x_n^2}{d^2}}, \tag{31}$$

where $x_0 = 0$ and $n-1/2 < x_n < n+1/2$, $n = 1,2,3,\ldots$ for $x_n < \min\left(\omega_1,\omega_2\right)d/\pi c$. For identical bodies the roots are given by

$$\Omega = c\sqrt{k^2+\frac{\pi^2n^2}{d^2}} \tag{32}$$

for any integer $n = 0,1,2,\ldots$. They correspond to propagating (polariton) modes ($\kappa_1 = \kappa_2$ and κ all real numbers) and arise from Equation (28) for $e^{2i\kappa d} = 1$. Equation (29) may have another solution in the vicinity of the vertical asymptote of the function in its *rhs*, which, however, is irrelevant for our discussion.

Similarly, $v_{1,2}$ from Equations (23) and (24) obey the same Equation (26). We look again for solutions of the form $v_{1,2} =$

$A_{1,2}e^{i\kappa_{1,2}z}$, where $A_{1,2}$ are amplitudes to be determined. According to Equation (22) these modes are transverse modes, as they should be (for $\kappa_{1,2}$ real). The relevant dispersion equation is given by

$$e^{2i\kappa d} = \frac{(\kappa_1 + \kappa)(\kappa_2 - \kappa)(\kappa\kappa_1 + k^2)(\kappa\kappa_2 - k^2)}{(\kappa_1 - \kappa)(\kappa_2 + \kappa)(\kappa\kappa_1 - k^2)(\kappa\kappa_2 + k^2)}. \quad (33)$$

We note that this dispersion equation does not become the non-retarded dispersion Equation (28) by taking formally the limit $c \to \infty$.

An analysis similar to the one performed above for Equation (28) shows that Equation (33) has a branch of roots

$$\Omega_2 = c\sqrt{k^2 + \frac{\pi^2 y_n^2}{d^2}}, \quad (34)$$

where $y_0 = 0$ and $y_n < \min(\omega_1, \omega_2)\, d/\pi c$. They correspond to surface plasmon-polariton modes $\kappa_1 = i\alpha_1$, $\kappa_2 = -i\alpha_2$ and κ real. We note that y_n may differ from x_n. For identical bodies these roots are those given by Equation (32). Some other isolated roots may appear, as for instance the one corresponding to an overall damping, i.e., $\kappa_1 = i\alpha_1$, $\kappa_2 = -i\alpha_2$, $\kappa = i\alpha$, where $\alpha = \sqrt{k^2 - \omega^2/c^2}$, $\omega < ck$. It is given by

$$\Omega_0 = c\sqrt{k^2 - \frac{\pi^2 z_0^2}{d^2}}, \quad (35)$$

where $\min(\omega_1, \omega_2) < \pi\sqrt{2}cz_0/d < \max(\omega_1, \omega_2)$. Such an isolated mode does not contribute significantly to the energy, so we may neglect it in our subsequent analysis.

We can take the limit $d \to \infty$ in Equation (33). It can be shown that this limit amounts formally to put $e^{2i\kappa d} = 0$ [23]. We get in this case the surface plasmon-polariton modes corresponding to a semi-infinite body, given by $\alpha\alpha_{1,2} = k^2$, i.e.,

$$\omega^2 = \frac{2\omega_{1,2}^2 c^2 k^2}{\omega_{1,2}^2 + 2c^2k^2 + \sqrt{\omega_{1,2}^4 + 4c^4k^4}}, \quad (36)$$

as derived previously [23]. In general, there are problems with taking formally the limits $d \to 0$ or $d \to \infty$ in the above equations, as expected.

It is also worth interesting to look for solutions of the type

$$v_{1,2} = A_{1,2}\left[e^{i\kappa_{1,2}z} - e^{\pm i\kappa_{1,2}(d\mp z)}\right] \quad (37)$$

for Equations (23) and (24), which are vanishing on the surfaces, $v_{1,2}(\pm d/2)$ ("fixed surfaces" boundary conditions). In this case, we get again the resonance modes Ω given by Equation (32), irrespective of the bodies being distinct or identical. In addition, we may get special modes $\omega = \omega_{1,2}$, $\omega^2 = c^2k^2 + \omega_{1,2}^2$ ($\kappa_{1,2} = 0$) or $\omega = ck$ ($\kappa = 0$), which do not depend on distance d. Other boundary conditions can be put on surfaces $z = \pm d/2$, and we can get the corresponding eigenmodes.

We note that the dispersion Equations (28) and (33) appear, though in a disguised form, in various formulations of the fluctuations theory [2, 3, 5, 32]. Within the framework of this theory the dielectric function is included from the beginning. On the contrary, we recover the dielectric function in the final results of the present approach, which shows that our approach is equivalent with the so-called "effective medium permittivity" theory.

We pass now to the zero-point energy corresponding to the $\Omega_{1,2}$-eigenmodes given by Equations (31) and (34), or the Ω-branch given by Equation (32) (for identical bodies or "fixed surfaces", in the limit $\min(\omega_1, \omega_2)\, d/\pi c \gg 1$. These are the only eigenfrequencies which depend on distance d. In the limit $\min(\omega_1, \omega_2)\, d/\pi c \gg 1$ these modes are dense sets, and it is easy to see that their contributions to the zero-point energy are equal (corresponding to the two polarizations), so we can write the total zero-point energy as

$$E = \hbar c \sum_{\mathbf{k}n=0} \sqrt{k^2 + \frac{\pi^2 x_n^2}{d^2}}, \tag{38}$$

where x_n are defined above; for identical bodies (or for "fixed surfaces") $x_n = n$. We follow the standard regularization procedure by removing the ultraviolet divergencies and using the Euler-MacLaurin formula [35]. As it is well-known, the energy thus regularized reads

$$E = \frac{\hbar c}{2\pi} \sum_{k=1} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x_0), \tag{39}$$

where B_{2k} are Bernoulli's numbers and

$$f(x) = \int_0 dk \cdot k \sqrt{k^2 + \frac{\pi^2 x^2}{d^2}} = \frac{1}{2} \int_{\pi^2 x^2/d^2} du \cdot \sqrt{u}. \tag{40}$$

Since $x_0 = 0$ (and $y_0 = 0$), we get the well-known energy $E = -\pi^2 \hbar c B_4/4! d^3 = -\pi^2 \hbar c/720 d^3$ and Casimir force $F = \pi^2 \hbar c/240 d^4$ per unit area. The same result is obtained for the Ω-modes given by Equation (32) with $n = 0, 1, 2, \ldots$, corrresponding to identical bodies or the "fixed surfaces" boundary conditions $v_{1,2}(\pm d/2)$. It is easy to see that for decreasing $\min(\omega_1, \omega_2)\, d/\pi c$ the number of x_n-roots contributing to energy decreases, the numerical coefficient of the Casimir force decreases gradually, and the d^{-4}-dependence deteriorates, untill a cross-over may occur to the non-retarded van der Waals-London d^{-3}-force.

The dispersion Equations (28) and (33) hold also for dielectrics, providing the wavevectors $\kappa_{1,2}$ are changed according to

$$\kappa_{1,2}^2 \to \kappa^2 - \frac{\omega_{1,2}^2}{c^2}\frac{\omega^2}{\omega^2 - \omega_{01,2}^2}. \tag{41}$$

We can get a usual model of dielectric for $\omega_{01,2} \gg \omega_{1,2}$. In this case, the wavevectors $\kappa_{1,2}$ become

$$\kappa_{1,2} = \sqrt{\kappa^2 + \frac{\omega_{1,2}^2}{\omega_{01,2}^2}\frac{\omega^2}{c^2}}, \tag{42}$$

and we cannot have anymore surface plasmon-polariton modes (evanescent modes). In general, under these circumstances, the dispersion Equations (28) and (33) have no solutions, except for identical bodies when we may have the Ω-modes given by Equation (32) ($e^{2i\kappa d} = 1$) for $n = 0, 1, 2, \ldots$. These modes correspond to propagating polaritons and give again the classical result for the Casimir force $F = \pi^2 \hbar c/240 d^4$ per unit area. Similarly, for a dielectric-metal pair there is no force, except for boundary conditions $v_{1,2}(\pm d/2)$ when the resonant Ω-modes given by Equation (32) for $n = 0, 1, 2, \ldots$ are present. The latter result holds for any pair of bodies. It is, however, worth stressing that such results depend on our model of dielectric function for dielectrics, and, in general, it is necessary to have a quantum-mechanical treatment for the internal dynamics of the dielectrics.

4. DISCUSSION AND CONCLUSION

In conclusion, we may say that we have derived here van der Waals-London and Casimir forces acting between two semi-infinite bodies with parallel surfaces by calculating the electromagnetic eigenmodes in matter and estimating their zero-point energy (vacuum fluctuations). We have adopted well-known, simple, usual models for matter polarization in metals and dielectrics and made use of the equation of motion for the polarization in order to get coupled integral equations. The eigenfrequencies of these equations have been identified and used in calculating the zero-point energy. In the non-retarded (Coulomb) limit we get the well-known van der Waals-London d^{-3}-force, arising from the surface plasmons, where d is the distance between the two bodies. The numerical coefficient of this force acquires various values, depending on the nature of the bodies and on their being distinct or identical. When retardation is included we get the Casimir d^{-4}-force arising from surface plasmon-polariton modes (evanescent modes) for a pair of metals. The classical numerical coefficient of this force ($\pi^2/240$) is obtained for distances much larger than the characteristic wavelengths ($\sim c/\omega_{1,2}$, where $\omega_{1,2}$ are the plasmon frequencies) of the bodies, and it diminishes gradually for shorter distances, while the force loses its characteristic d^{-4}-dependence. For a pair of identical dielectrics we get the classical Casimir result arising from propagating polariton modes. The same result holds for any pair of bodies with "fixed surfaces" boundary conditions.

As it is well-known, the fluctuations theory [32] predicts Casimir forces between any pair of bodies, in contrast with our results, which give a vanishing force for two distinct dielectrics, for instance. The difference originates in the circumstance, usually overlooked, that the equivalent of our dispersion Equations (28) and (33) in the fluctuations theory have no solutions in some cases, as, for instance, for distinct dielectrics. The usual theorem of meromorphic functions, applied within the framework of the fluctuations theory [4–6], gives then a finite result, but it does not represent the energy of the eigenmodes. The problem does not appear in the non-retarded regime, where our results coincide with those of the fluctuations theory. On the other hand, we must stress again upon the fact that our model for the dielectric function may not be perfectly adequate for describing the internal polarization of dielectric matter. Again, this is immaterial in the non-retarded regime, and we succeeded to compute a d^{-4}-van der Waals-London force between a classical model of polarizable point-like particle and a semi-infinite body. But our approach fails in this case in the retarded regime, where a quantum mechanical treatment is necessary, as in the original attempt in [36]).

Finally, it is worth noting that the dispersion Equations (28) and (33) can also be obtained by calculating the reflected field in-between the bodies (fields for semi-infinite bodies) [23]. If $r_{1,2}$ are the amplitudes of these fields (for a given polarization), then the dispersion Equations (28) and (33) are obtained from $r_1 = r_2 e^{2i\kappa d}$. We note that $|r_{1,2}|^2$ are the reflection coefficients, and for two perfectly reflecting bodies $|r_1| = |r_2| = 1$. If we neglect the phases of the coefficients $r_{1,2}$, and put $r_1 = r_2 = 1$, we get the Casimir dispersion equation $e^{2i\kappa d} = 1$ (Ω-modes given by Equation (32)). However, it is precisely these phases that give the damped surface plasmon-polariton regime, as we have shown in the present paper, and these phases are not equal in the damped regime, not even for identical bodies. This is related to the correct choice of the sign of the square root in $\kappa_{1,2}$, which, as we have shown here, is $\kappa_1 = i\alpha_1$ and $\kappa_2 = -i\alpha_2$ (Equations (29) and (30)). For the propagating regime (vanishing phases) and identical bodies ($r_1 = r_2$) we get again the Casimir dispersion equation $e^{2i\kappa d} = 1$, as we do for "fixed surfaces" boundary conditions (in the latter case irrespective of the bodies).

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