ELASTIC WAVES INSIDE AND ON THE SURFACE OF A HALF-SPACE

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Summary

The notions of 'elementary' seismic sources and 'elementary' earthquakes are introduced, as being associated with 'elementary' tensorial point forces with a δ-like time dependence (where δ is the Dirac delta function). The tensorial character of these forces, known in Seismology as the dipole (or double-couple) representation, is given by the tensor of the seismic moment. A regular seismic source and a regular earthquake can be represented as a superposition of elementary sources and, respectively, elementary earthquakes, governed by a space-time structure factor of the seismic focal region. All these are new concepts. Elementary seismic sources are considered here for a homogeneous isotropic elastic half-space bounded by a free plane surface, the sources being located at an inner point in the half-space. A transient regime of generation and propagation of seismic waves is identified, as distinct from the stationary regime of elastic vibrations. This is another new concept. It is shown that elementary seismic sources produce (double-shock) spherical-shell waves (in the wave region), which are the well-known P and S waves associated with the feeble tremor in the recorded seismograms. Their mathematical expression, derived here from the tensorial force, differs from known, particular cases. These waves are called here collectively 'primary' waves. It is shown that the primary waves interact with the surface of the half-space, where they give rise to 'secondary' wave sources, placed on the surface. The secondary waves generated by the secondary sources (which may be called 'surface seismic radiation') are estimated here in a simplified model. It is shown that the secondary waves have a delay time in comparison with the primary waves and give rise to a main shock and a long seismic tail, in qualitative agreement with the seismic records. The secondary wave introduced here is a new concept; the main shock and its long tail derived here are elements of novelty. Similarly, the secondary waves generated by an internal discontinuity in the elastic properties of the half-space (an interface parallel with the free surface) are also estimated; it is shown that the discontinuity reduces appreciably the singular main shock on the free surface of the homogeneous half-space.

1. Introduction

It is widely accepted that the main problem in Seismology is the generation and propagation of the seismic waves. It gives information about processes occurring in the earthquake focal region, the nature and structure of the Earth's interior, and the effect of the seismic waves on the Earth's surface. The problem originates with the classical works of Rayleigh, Love and Lamb (it is sometime known as Lamb's problem) (**1**)–(**4**). In a simplified model, the Earth may be viewed as a homogeneous isotropic elastic half-space bounded by a plane surface, the seismic sources being placed at an inner

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point in the half-space. For sufficiently long distances the spatial localisation of the seismic sources may be represented by δ-functions, or their derivatives (point sources), where δ is the Dirac delta function. The dipole (or double-couple) representation of point seismic sources by means of the tensor of the seismic moment emerged gradually in the first half of the 20th century (**5**)–(**8**). The exact mathematical expression of the elementary tensorial point seismic force is derived here, as a basic element for characterising earthquakes' foci.

The standard way of treating the seismic waves starts with a point vector force placed in the seismic source; this is known as the Stokes problem (**9**). Two solutions of the Stokes problem in the infinite space are superposed, as arising from a Stokes point dipole. The point dipole conserves the total force, but it does not conserve the total angular momentum. The angular momentum is conserved by two dipoles suitably arranged; this is the double-couple representation of the seismic sources. A distribution of point dipoles which conserve the total force and angular momentum is not unique. Obviously, the point dipoles indicate a tensorial character of the seismic sources (**10**)–(**12**). We derive here the exact mathematical expression of the tensorial point force governed by the tensor of the seismic moment and solve directly the equation of the elastic waves in the direct space for a homogeneous isotropic medium (in the time-domain) for a δ-like time dependence of the force. We call the point seismic sources (forces) with such a time dependence 'elementary' sources (forces); they generate 'elementary' earthquakes. The importance of the elementary seismic sources introduced here resides in their being associated with Green functions in the time domain which allow direct, explicit calculations. Quasi-static unphysical terms may appear in solution, which should be discarded by a regularisation (calibration) procedure. It is shown that the elementary tensorial forces produce two (double-shock) spherical-shell elastic waves which can be viewed as the well-known P and S seismic waves. They are currently associated with the feeble preliminary tremor recorded in seismograms (**3**), (**13**). We call these waves, collectively, 'primary' waves. (We recall that in the seismological literature the P wave is called primary wave, while the S wave is called secondary wave). The mathematical expression of the P and S waves derived here from the tensorial force differs from known, particular cases. For sources with a finite temporal or spatial extension, or both, or for multiple sources a space-time structure factor of the focal region is introduced here, which may be viewed as an imprint of the focal region in recorded seismograms. Such structure factors are responsible for the 'succession of primitive shocks' (**2**) and the oscillations and the irregular motion exhibited by the seismic records. The concept of structure factor introduced here is new.

Also, a model of isotropic elementary source is introduced here (which may be relevant for explosions) and the seismic waves produced by such a source are derived by solving directly the equation of the elastic motion; it is shown that these waves can be obtained from the general solution corresponding to the tensorial force by using an isotropic (scalar) seismic moment, as expected.

The standard course in the seismological literature continues with the decomposition of the seismic waves in (in-plane) Fourier series for a half-space (or their expansions in normal modes for a spherical model of Earth) and Rayleigh surface waves (**14**), (**15**) are added to these extended waves to satisfy the boundary conditions at the surface (usually a free surface); also, the Laplace transform is used to this end (the well-known Cagniard-de Hoop method (**16**), (**17**)). Thereafter, the recomposition is looked for, in order to get the seismic main shock (which is expected to appear in this context from the surface waves). However, the primary waves are propagating waves concentrated on spherical shells, while the boundary conditions are relevant for surfaces acted continuously, in their entirety, by stationary waves (vibrations). The elastic waves may suffer multiple reflections on the Earth's surface (or on the interfaces of the internal Earth's layers) and a stationary regime of oscillations may set in after a lapse of time. The relevant magnitude of this time interval is of the order R_E/c, where R_E is the radius of the Earth and c is the wave velocity. For $R_E = 6370$ km and a mean velocity $c = 5$ km/s of the elastic waves we get $R_E/c \simeq 1274$ s; this time interval is much longer than the time taken by the seismic waves to propagate from the source to the Earth's surface (that is to the epicentre and the surface zones surrounding the epicenter). We can see that the effects of the seismic waves on the Earth's surface are produced in a time much shorter than the time needed for attaining the stationary regime of vibrations. It follows that we are interested primarily in the transient regime of the seismic waves (where the boundary conditions are practically radiation conditions in the infinite space and the quasi-spherical Earth may be approximated locally by an elastic half-space). The identification of the transient regime in the propagation of the seismic waves is new. We show here (by using energy balance among other arguments) that the intersection of the 'zero-thickness' wavefront of the primary spherical-shell waves with the plane surface of the half-space (or layers interfaces) leads to an interaction which gives rise to additional 'secondary' wave sources, confined (and moving) on the surface. The secondary waves generated by secondary surface sources are estimated here by means of a simplified model. Since the secondary waves are generated by sources moving on the surface, they may be called 'surface seismic radiation'. It is shown that the secondary waves have a time delay with respect to the primary waves and generate a seismic main shock and a long tail, in qualitative agreement with the seismic records. The interaction of the primary waves with the surface has been suggested long ago by Lamb, (**2**), (**18**) and the seismic main shock and long tail have been associated since long with surface phenomena (**19**), (**20**). The estimation done here of the mathematical expression of the main shock is new. The effect of an internal discontinuity in the half-space is also discussed here; it is shown that such a discontinuity reduces appreciably the main shock on the free surface.

The problem dealt with in this article is an old problem, which, in spite of its relevance, received little attention in the recent literature. It consists in realising that the effects of the earthquakes' on the Earth's surface are associated with the transient regime of the seismic waves; this transient regime together with the localised character of the primary waves (which are spherical shells) require a direct-space approach, in contrast with the **k**-space approach, which introduces an artificial delocalised character associated with the Fourier components of the seismic waves (plane waves). In addition, we emphasise that the problem discussed in this article points to another, more complex, mathematical problem, related to inhomogeneous boundary conditions, which may raise appreciable difficulties (and which may be viewed as an open problem).

2. Elementary seismic sources

The seismic load in a point focus consists of opposite forces, usually at (quasi-) equilbrium, so that the total force and angular momentum are vanishing. The load can be accommodated by successive small, (quasi-) static deformations of the Earth's crust and tectonic plates. During the earthquake, the resistance of the rocks in the focus yields, such that we have a localised, active distribution of opposite forces. The seismic tensorial forces (see, for instance, Ref. (**5**), 2nd edn, p. 60, Exercise 3.6) can be derived by estimating the couple produced by a force density $\mathbf{F}(\mathbf{R}, t) = \mathbf{f}(t)w(\mathbf{R})$, where $\mathbf{f}$ is the force and $w(\mathbf{R})$ is a distribution function; a point couple along the i-th direction can be represented as

$$f_i w(x_1 + h_1, x_2 + h_2, x_3 + h_3) - f_i w(x_1, x_2, x_3) \simeq f_i h_j \partial_j w(x_1, x_2, x_3), \tag{2.1}$$

where f_i, $i = 1, 2, 3$, are the components of the force, h_j, $j = 1, 2, 3$, are the components of an infinitesimal displacement $\mathbf{h}$ and x_i are the coordinates of the point with the position vector $\mathbf{R}$. The

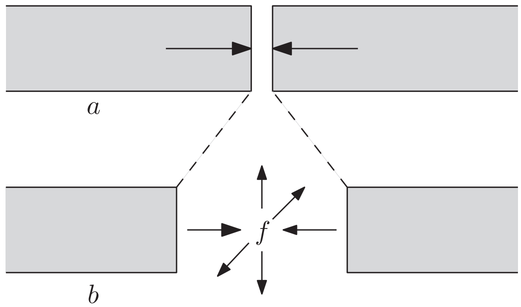

Fig. 1 The load accumulated in two elements of tectonic plates in (quasi-) equilibrium (a) may lead to resistance loss and a localized active focal region f (b)

moments $f_i h_j$ are generalised to a symmetric tensor M_{ij}, which is the seismic moment; in addition, the distribution $w(\mathbf{R})$ is replaced by $\delta(\mathbf{R}-\mathbf{R}_0)$, where δ denotes the Dirac function localised at the point with the position vector $\mathbf{R}_0$. We prefer to use the seismic moment divided by density ρ, $m_{ij}=M_{ij}/\rho$; then, the force distribution per unit mass reads

$$F_i(\mathbf{R},t)=m_{ij}(t)\partial_j\delta(\mathbf{R}-\mathbf{R}_0), \tag{2.2}$$

where $m_{ij}(t)$ has a certain time dependence during the earthquake. Usually, this function is localised over a short, finite duration T, such that we may use the δ-pulse time dependence $m_{ij}(t)=Tm_{ij}\delta(t)$. It is easy to see that the total force and angular momentum associated with the force distribution given by (2.2) are zero (the latter by the symmetry of the tensor m_{ij}). According to our definition, the moment tensor is positive definite for an ‘implosion’, and negative definite for an ‘explosion’ (in general, it is an indefinite tensor). A schematic representation of a tensorial force distribution is shown in Fig. 1. We call the tensorial force distributions given by (2.2) with $m_{ij}(t)=Tm_{ij}\delta(t)$ elementary force distributions; they are produced by elementary seismic sources and generate elementary earthquakes.

Similarly, we can use a model force distribution

$$\mathbf{F}(\mathbf{R},t)=p(t)\frac{\mathbf{R}-\mathbf{R}_0}{|\mathbf{R}-\mathbf{R}_0|}\theta\left(a-|\mathbf{R}-\mathbf{R}_0|\right) \tag{2.3}$$

for an isotropic source localised in a volume with a small radius a, where $p(t)=f(t)/a^3$ is force per unit mass, $f(t)$ is a force (divided by density) and $\theta(x)=1$ for $x>0$, $\theta(x)=0$ for $x<0$ is the step function; for an elementary force distribution $p(t)=Tp\delta(t)$. We show in this article that the waves produced by this isotropic source in the limit $a\to 0$ (for a δ-pulse time dependence) can be obtained from the force given by (2.2) by replacing formally the tensor m_{ij} by $-m\delta_{ij}$, where the scalar seismic moment is of the order $m\simeq fa$.

It is worth attempting an estimation of the order of magnitude of the localisation length l of the focal region. We note that the seismic moment M has the dimension of a mechanical work (energy; we use notation M as a generic notation for the components M_{ij}); it is reasonable to admit that an energy of the order M is spent to destroy the elastic consistency of the material which is ruptured in the focal volume V during the earthquake; this energy density is of the order of the elastic energy density of the material ρc^2, where ρ is the material density and c is a mean value of the velocity of the elastic waves. Therefore, the equality $M/V \simeq \rho c^2$ may hold. For $M = 10^{26} dyn\cdot$ cm (corresponding to an earthquake magnitude $M_w = 7$, from the Gutenberg–Richter definition (**21**)–(**23**) $\lg M = 1.5M_w + 16.1$), $\rho = 5$ g/cm^3 for the average Earth's density and $c = 5$ km/s for a mean value of the velocity of the elastic waves we get a volume $V = 8 \times 10^{13}$cm^3 of the focal region and a localization length $l = V^{1/3} \simeq 400$ m. This spatial uncertainty leads to a time uncertainty of the order $T = l/c = 0.08$ s (for a mean velocity $c = 5$ km/s, though, very likely, the rupture propagation in the focus is slower than the elastic waves). The Dirac delta function used in the representation of the tensorial force may be viewed as being localised over a distance of the order l (volume l^3).

3. Primary waves

The equation of the elastic waves in a homogeneous isotropic body is

$$\ddot{\mathbf{u}} - c_t^2 \Delta \mathbf{u} - (c_l^2 - c_t^2)\mathrm{grad} \cdot \mathrm{div}\mathbf{u} = \mathbf{F}, \tag{3.1}$$

where $\mathbf{u}$ is the displacement vector, $c_{l,t}$ are the wave velocities and $\mathbf{F}$ is the force (per unit mass). (**14**), (**15**) We consider this equation in an isotropic elastic half-space extending in the region $z < 0$ and bounded by the flat surface $z = 0$. The elementary source, which generates the force $\mathbf{F}$, is placed at $\mathbf{R}_0 = (0, 0, z_0)$, $z_0 < 0$; the force is given by (2.2) (with $m_{ij}(t) = Tm_{ij}\delta(t)$), where m_{ij} is the tensor of the seismic moment (divided by density). The coordinates of the position vector $\mathbf{R}$ are denoted by (x_1, x_2, x_3); also, the notation $x = x_1$, $y = x_2$, $z = x_3$ is used. In the transient regime, the waves generated by the elementary force in (3.1) are those propagating in the infinite space. We use the Helmholtz decomposition $\mathbf{F} = \mathrm{grad}\phi + \mathrm{curl}\mathbf{H}$ ($\mathrm{div}\mathbf{H} = 0$), whence

$$\Delta\phi = \mathrm{div}\mathbf{F}, \qquad \Delta\mathbf{H} = -\mathrm{curl}\mathbf{F}. \tag{3.2}$$

similarly, the displacement $\mathbf{u}$ is decomposed as $\mathbf{u} = \mathrm{grad}\Phi + \mathrm{curl}\mathbf{A}$, with the notation $\mathbf{u}^l = \mathrm{grad}\Phi$ and $\mathbf{u}^t = \mathrm{curl}\mathbf{A}$, by using the Helmholtz potentials Φ and $\mathbf{A}$ ($\mathrm{div}\mathbf{A} = 0$); equation (3.1) is transformed into two standard wave equations

$$\ddot{\Phi} - c_l^2 \Delta\Phi = \phi, \quad \ddot{\mathbf{A}} - c_t^2 \Delta\mathbf{A} = \mathbf{H}, \tag{3.3}$$

we can see that the l, t-waves are separated.

From (3.2), and making use of the force distribution given by (2.2), we get immediately

$$\begin{aligned}\phi &= -\frac{1}{4\pi} Tm_{ij}\delta(t) \int d\mathbf{R}_1 \frac{1}{|\mathbf{R} - \mathbf{R}_1|} \partial_i\partial_j\delta(\mathbf{R}_1) \\ &= -\frac{1}{4\pi} Tm_{ij}\delta(t)\partial_i\partial_j \frac{1}{R}\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}H_i &= \frac{1}{4\pi} T\varepsilon_{ijk}m_{kl}\delta(t) \int d\mathbf{R}_1 \frac{1}{|\mathbf{R} - \mathbf{R}_1|} \partial_j\partial_l\delta(\mathbf{R}_1) \\ &= \frac{1}{4\pi} T\varepsilon_{ijk}m_{kl}\delta(t)\partial_j\partial_l \frac{1}{R},\end{aligned} \tag{3.5}$$

where $\mathbf{R}$ stands for $\mathbf{R}-\mathbf{R}_0$ and ε_{ijk} is the totally antisymmetric tensor of rank three. Making use of these sources in (3.3), and using the Kirchhoff retarded solutions, we get the potentials

$$\Phi=-\frac{T}{(4\pi c_l)^2}m_{ij}\partial_i\partial_j\int d\mathbf{R}_1\frac{\delta(t-|\mathbf{R}-\mathbf{R}_1|/c_l)}{|\mathbf{R}-\mathbf{R}_1|}\frac{1}{R_1} \tag{3.6}$$

and

$$A_i=\frac{T}{(4\pi c_t)^2}\varepsilon_{ijk}m_{kl}\partial_j\partial_l\int d\mathbf{R}_1\frac{\delta(t-|\mathbf{R}-\mathbf{R}_1|/c_t)}{|\mathbf{R}-\mathbf{R}_1|}\frac{1}{R_1}. \tag{3.7}$$

We extend the integral

$$I=\int d\mathbf{R}_1\frac{\delta(t-R_1/c)}{R_1\,|\mathbf{R}-\mathbf{R}_1|} \tag{3.8}$$

occurring in the above equations (where c stands for $c_{l,t}$) to the whole space, where it can be effected straightforwardly by using spherical coordinates; we get

$$I=4\pi c\left[\theta\,(ct-R)+\frac{ct}{R}\theta\,(R-ct)\right], \tag{3.9}$$

inserting this result in (3.6) and (3.7) we get the Helmholtz potentials

$$\Phi=-\frac{T}{4\pi c_l}m_{ij}\partial_i\partial_j\left[\theta\,(c_lt-R)+\frac{c_lt}{R}\theta\,(R-c_lt)\right] \tag{3.10}$$

and

$$A_i=\frac{T}{4\pi c_t}\varepsilon_{ijk}m_{kl}\partial_j\partial_l\left[\theta\,(c_tt-R)+\frac{c_tt}{R}\theta\,(R-c_tt)\right]. \tag{3.11}$$

Making use of the notation

$$F_{l,t}=\frac{T}{4\pi c_{l,t}}\left[\theta\left(c_{l,t}t-R\right)+\frac{c_{l,t}t}{R}\theta\left(R-c_{l,t}t\right)\right], \tag{3.12}$$

the potentials can be written as

$$\Phi=-m_{ij}\partial_i\partial_jF_l,\quad A_i=\varepsilon_{ijk}m_{kl}\partial_j\partial_lF_t, \tag{3.13}$$

it follows the displacement

$$\begin{aligned}u_i^l&=\partial_i\Phi=-m_{jk}\partial_i\partial_j\partial_kF_l,\\ u_i^t&=\varepsilon_{ijk}\partial_jA_k=m_{jk}\partial_i\partial_j\partial_kF_t-m_{ij}\partial_j\Delta F_t.\end{aligned} \tag{3.14}$$

We can see that these solutions consist of two parts: spherical waves propagating with velocities $c_{l,t}$, given by δ-functions and derivatives of δ-functions (arising from the derivatives of the θ-functions in

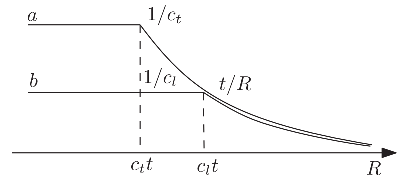

Fig. 2 The functions F_t (*a*) and F_l (*b*) *vs* R

(3.12)), and a quasi-static displacement which includes the functions $\theta(R-c_{l,t}t)$ and extends over the distance $\Delta R=(c_l-c_t)t$. The quasi-static contributions, being proportional to third-order derivatives of t/R, are solutions of the homogeneous wave equation. In the transient regime, the quasi-static contributions should be removed, and we should limit ourselves to the δ-functions and derivatives of δ-functions arising from the derivatives of the θ-functions in (3.12). This is the regularisation (calibration) procedure used for getting the solution. Outside the support of the δ-functions and their derivatives (that is for $R\neq c_{l,t}t$) the displacement is zero. We note also that for $R\neq c_t t$ the function F_t in (3.12) is either $T/4\pi c_t$ or $Tt/4\pi R$; in both cases the term with the laplacian in the second equation (3.14) cancels out, and $\mathbf{u}^t$ acquires the same expression as $-u_i^l$ with c_l replaced by c_t. The functions $F_{l,t}$ are shown schematically in Fig. 2.

The solution is given by the potentials in (3.13), provided we remove the quasi-static displacement; the basic expression $m_{jk}\partial_i\partial_j\partial_k F$ becomes $m_{jk}\partial_i\partial_j\partial_k F$

$$\begin{aligned}
&=\left[\frac{m_{jj}x_i}{2R^3}(1-2ct/R)+\frac{m_{ij}x_j}{R^3}(1-3ct/r)\right]\delta(R-ct)\\
&-\frac{3m_{jk}x_ix_jx_k}{2R^5}(1-4ct/R)]\delta(R-ct)\\
&-\left[\frac{m_{jj}x_i}{2R^2}(1-ct/R)-\frac{m_{jk}x_ix_jx_k}{2R^4}(1-3ct/R)\right]\delta^{'}(R-ct),
\end{aligned} \tag{3.15}$$

where F is a generic notation for $F_{l,t}$ with the pre-factor $1/4\pi c$ omitted; this expression includes only contributions proportional to $\delta(R-ct)$ and $\delta'(R-ct)$; in the pre-factors of the functions δ and δ' we may set $R=ct$. We get $u_i^l=A_i^l+B_i^l$, where

$$A_i^l=\frac{T}{8\pi c_lR^3}\left(m_{jj}x_i+4m_{ij}x_j-\frac{9m_{jk}x_ix_jx_k}{R^2}\right)\delta(R-c_lt) \tag{3.16}$$

and

$$B_i^l=\frac{Tm_{jk}x_ix_jx_k}{4\pi c_lR^4}\delta'(R-c_lt), \tag{3.17}$$

in these expressions c_l and the factor $1/4\pi c_l$ are restored and $x_{i,j,k}$, i, j, $k=1,2,3$, are the coordinates of the position vector $\mathbf{R}=(x_1,x_2,x_3)$. Similarly, we get $u_i^t=-A_i^t-B_i^t+C_i^t$, where

A_i^t, B_i^t are given by A_i^l and, respectively, B_i^l with c_l replaced by c_t and

$$C_i^t = -\frac{Tm_{ij}x_j}{4\pi c_t R^2}\left[\frac{1}{R}\delta(R-c_t t) - \delta'(R-c_t t)\right]. \tag{3.18}$$

For a scalar seismic moment $m_{ij} = m\delta_{ij}$, equation (3.1) with $F_i = Tm\delta(t)\partial_i\delta(\mathbf{R})$ is solved immediately by $\mathbf{u} = \mathrm{grad}\Phi$, where $\ddot{\Phi} - c_l^2\Delta\Phi = Tm\delta(t)\delta(\mathbf{R})$ and $\Phi = (Tm/4\pi c_l)\delta(R - c_l t)/R$; we can check that the regularisation (calibration) procedure used in deriving (3.16), (3.17) and (3.18) leads immediately to this particular solution.

We can see that in the far-field region (wave region) the source generates two (double-shock) spherical-shell waves (derivatives of the δ-function), propagating with velocities $c_{l,t}$, given by

$$u_i^f = \frac{Tm_{ij}x_j}{4\pi c_t R^2}\delta'(R-c_t t) + \frac{Tm_{jk}x_i x_j x_k}{4\pi R^4}\left[\frac{1}{c_l}\delta'(R-c_l t) - \frac{1}{c_t}\delta'(R-c_t t)\right]. \tag{3.19}$$

We note that the mathematical expression of the far-field waves derived here from tensorial forces (3.19) is new in comparison with known, particular cases (**5**), (**6**). We emphasise that these waves are spherical shells, described by zero-support delta-functions; they are localised waves.

The waves propagating with velocity c_l are the primary P waves (compressional waves), while the waves propagating with velocity c_t are the primary S-waves (they include the shear contribution). The second term on the right in (3.19) is longitudinal ($\sim \mathbf{R}$), while the polarisation of the first term depends on the moment tensor. The magnitude of these waves is of the order $u^f \simeq Tm/cRl^2$, where m is a generic notation for the components of the seismic moment (divided by density), c is a mean wave velocity and $l = cT$ is the linear size of the localization of the δ-function (linear size of the earthquake's focus). Making use of a seismic moment $M = 10^{26}$ dyn$\cdot$ cm (earthquake's magnitude 7), density $\rho = 5$ g/cm^3 ($m = M/\rho$), a mean velocity $c = 5$ km/s, $l = 400$ m, for an earthquake's duration $T = 0.08$ s, we get at distance $R = 100$ km a far-field displacement u^f of the order $1m$.

4. Isotropic sources

For an isotropic source of the form $\mathbf{F} = p(t)(\mathbf{R}/R)\theta\,(a - R)$ (2.3) we have $\mathrm{curl}\mathbf{F} = 0$; therefore, $\mathbf{H} = 0$ and $\mathbf{u}_t = 0$. For such a force there exist only l-waves (dilatational waves), given by $\ddot{\mathbf{u}}_l - c_l^2\Delta\mathbf{u}_l = \mathrm{grad}\phi$, where $\Delta\phi = \mathrm{div}\mathbf{F}$; we may take $\mathrm{grad}\phi = \mathbf{F}$, such that we have

$$\mathbf{u}_l = \frac{Tp}{4\pi c^2}\int d\mathbf{R}_1 \frac{\delta\left(t - |\mathbf{R}-\mathbf{R}_1|/c\right)}{|\mathbf{R}-\mathbf{R}_1|}\frac{\mathbf{R}_1}{R_1}\theta(a-R_1) \tag{4.1}$$

for $p(t) = Tp\delta(t)$, where we write c for c_l. It is easy to see that $\mathbf{u}_l = u_l\mathbf{R}/R$, that is an isotropic source generates only longitudinal waves, as expected. From (4.1) we get

$$u_l = \frac{Tp}{2c^2}\int_0^a dR_1 R_1^2 \int_{-1}^1 du \cdot u \frac{\delta\left(t - \sqrt{R^2 + R_1^2 - 2RR_1 u}/c\right)}{\sqrt{R^2 + R_1^2 - 2RR_1 u}}. \tag{4.2}$$

The argument of the δ-function has a zero for

$$-1 \leq u_0 = \frac{R^2 + R_1^2 - c^2t^2}{2RR_1} \leq 1, \tag{4.3}$$

which gives

$$u_l = \frac{Tp}{4cR^2} \int dR_1 (R^2 + {R_1}^2 - c^2t^2). \tag{4.4}$$

The function $u_0(R_1)$ given by (4.3) has a minimum for $R_1 = \sqrt{R^2 - c^2t^2}$ for $R > ct$ and $u_0(R_1) = 1$ for $R_1 = R \mp ct$; for $R < ct$ the function $u_0(R_1)$ has a zero for $R_1 = \sqrt{c^2t^2 - R^2}$ and $u_0(R_1) = \mp 1$ for $R_1 = ct \mp R$; taking into account these conditions, we get

$$u_l = \frac{Tp}{4cR^2} f_a \theta\left(a - |R - ct|\right), \tag{4.5}$$

where

$$f_a = \frac{1}{3}a^3 + (R^2 - c^2t^2)a - (R^2 - c^2t^2)\,|R - ct| - \frac{1}{3}\,|R - ct|^3\,. \tag{4.6}$$

This wave extends inside the region $-a < R - ct < a$ and exhibits a wavefront which moves with velocity c ($R = ct$). The function f_a has two extrema $\mp(a^3/24 - 2cta^2)$ at $R - ct = \mp a/2$. Making use of $p = f/a^3$ (where f is force divided by density), it is easy to see that in the limit $a \to 0$ the displacement u_l given by (4.5) and (4.6) may be represented approximately by

$$u_l \simeq -\frac{Tm}{4\pi cR}\delta'(R - ct), \tag{4.7}$$

where we introduced the scalar seismic moment m (of the order $m \simeq fa$). We can see that the displacement caused by such an isotropic source can be obtained from the far-field displacement generated by a tensorial source (3.19) by replacing formally in the latter the tensor m_{ij} of the seismic moment by an isotropic (scalar) seismic moment m, $m_{ij} \to -m\delta_{ij}$ (this representation amounts to use $-m\mathrm{grad}\delta(\mathbf{R})$ for $p\mathbf{R}\theta(a - R)/R$ in the limit $a \to 0$).

5. Structure factor

It is worth noting that the spherical-shell character of the displacement (involving δ- and δ'-functions) is closely connected to the localisation of the source, that is to the functions $\delta(t)$ and $\delta(\mathbf{R} - \mathbf{R}_0)$ occurring in the mathematical expression of the force. Let us assume that we have a succession of shocks in the source, labelled by i, occurring at times t_i, with duration T_i; then, the displacement given by (3.17) and (3.18) includes summations of the type

$$\sum_i T_i \delta\left(R - c(t - t_i)\right), \qquad \sum_i T_i \delta'\left(R - c(t - t_i)\right), \tag{5.1}$$

where c is a generic notation for the velocities $c_{l,t}$; for a sufficiently dense distribution of such shocks, we may replace the summations over i by integrals:

$$\sum_i T_i \delta\left(R - c(t - t_i)\right) = \frac{1}{\Delta T} \int dt_1 T(t_1) \delta(R - ct + ct_1) = \frac{1}{c \Delta T} T(t - R/c) \tag{5.2}$$

and

$$\sum_i T_i \delta'\left(R - c(t - t_i)\right) = \frac{1}{\Delta T} \int dt_1 T(t_1) \delta'(R - ct + ct_1) = -\frac{1}{c^2 \Delta T} T'(t - R/c), \tag{5.3}$$

where ΔT is the mean separation between the pulses. We can see that the displacement has not a spherical-wave character anymore, but instead it is given now by the functions T(t) and its derivative $T'(t)$ (at a retarded time), which play the role of time signatures of the source. A similar analysis can be done for shocks distributed spatially; we have, for instance

$$u = \sum_{ij} T_i \delta'\left(\left|\mathbf{R} - \mathbf{R}_j\right| - c(t - t_i)\right) g\left(\mathbf{R} - \mathbf{R}_j\right) = -\frac{1}{c^2 \Delta T \Delta v} \int d\mathbf{R}_1 T'(t - \left|\mathbf{R} - \mathbf{R}_1\right| / c) g\left(\mathbf{R} - \mathbf{R}_1\right), \tag{5.4}$$

where $g(\mathbf{R})$ represents the spatial dependence in (3.19) (except the δ'-functions) and Δv is the mean volume associated with individual shocks. The integral in (5.4) reflects the time-space structure of the earthquake's focal region. The factor $1/\Delta v$ can be replaced by a spatial distribution weight $w_s(\mathbf{R}_1)$, a procedure which is also valid for the factor $1/\Delta T$, which may be replaced by a weight function $w_t(t_1)$; a more general situation would imply a weight function $w(t_!, \mathbf{R}_1)$ instead of $T_i/\Delta T \Delta v$, which plays the role of a structure factor for the focal region; then, the displacement can be represented as

$$u = \int d\mathbf{R}_1 dt_1 w(t_!, \mathbf{R}_1) \delta'\left(\left|\mathbf{R} - \mathbf{R}_1\right| - c(t - t_1)\right) g\left(\mathbf{R} - \mathbf{R}_1\right) = -\frac{1}{c^2} \int d\mathbf{R}_1 w'(t - \left|\mathbf{R} - \mathbf{R}_1\right| / c, \mathbf{R}_1) g\left(\mathbf{R} - \mathbf{R}_1\right), \tag{5.5}$$

where the weight function w is localised over the focal region and over the time duration of the earthquake; such weight functions can be derived, in principle, from recorded seismograms, as an imprint of the structure of the focal region, by de-convoluting equations of the type given by (5.5). The occurence of shocks in succession is reflected in the irregular oscillations exhibited usually by the weight function (and by the displacement, velocity and acceleration recorded in seismograms). In general, the source of an earthquake may be viewed as a spatial-temporal succession of elementary events (i, j) of the form $\sim \delta(t - t_i)\delta(\mathbf{R} - \mathbf{R}_j)$, localised in the focal region. This succession of elementary ('primitive') earthquakes contribute to the oscillations which are a prominent feature in all seismic records (**2**), (**4**).

6. Energy balance

Multiplying the waves equation (3.1) by $\dot{\mathbf{u}}$ we get the energy conservation law

$$\frac{\partial \mathcal{E}}{\partial t} = -\mathrm{div}\mathbf{S} + \mathcal{W}, \tag{6.1}$$

where

$$\mathcal{E} = \frac{1}{2}\dot{u}_i^2 + \frac{1}{2}c_t^2(\partial_j u_i)^2 + \frac{1}{2}(c_l^2 - c_t^2)(\partial_i u_i)^2 \tag{6.2}$$

is the energy density (per unit mass),

$$S_i = -c_t^2(\dot{u}_j \partial_i u_j) - (c_l^2 - c_t^2)(\dot{u}_i \partial_j u_j) \tag{6.3}$$

are the components of the energy flux density (per unit mass) and $\mathcal{W} = \dot{u}_i F_i$ is the density of the mechanical work done by the external force $\mathbf{F}$ per unit time (and unit mass). It is easy to see that for a force localised for a short time T in the focal point the mechanical work $\mathcal{W}$ is non-vanishing only at this point and for the short time T, while for a spherical-shell wave the continuity equation $\partial\mathcal{E}/\partial t + \mathrm{div}\mathbf{S} = 0$ is satisfied identically at any point outside the focus, the energy density $\mathcal{E}$ and the energy flux density $\mathbf{S}$ being zero outside the support of the wave. The mechanical work done by the external force for a short period of time in the focus is transferred to the wave energy, which is carried through the space by the propagating wave without loss.

For an order of magnitude estimation we may use $F \simeq M/\rho l^4$ for the force given by (2.2) and $u \simeq M/\rho c^2 Rl$ for a spherical wave of the form $u = (MT/\rho cR)\delta'(R - ct)$, with $l = cT$. The density of the mechanical work per unit time is $\mathcal{W} \simeq M^2/\rho^2 cl^7$ and the total mechanical work is $W \simeq M^2/\rho c^2 l^3$. The energy density is $\mathcal{E} \simeq M^2/\rho^2 c^2 R^2 l^4$ and the total energy is $E_0 = M^2/\rho c^2 l^3 = W$ (similarly, the energy flux density is $S \simeq M^2/\rho^2 cR^2 l^4$, and $\mathrm{div}\mathbf{S}$ can be represented as $M^2/\rho^2 cR^2 l^5$; we can check the continuity equation $\partial\mathcal{E}/\partial t + \mathrm{div}\mathbf{S} = 0$). It is worth noting that the energy $E_0 = W$ transferred to the waves is smaller than the energy M released in the focal region by the factor $W/M = M/\rho c^2 l^3 = u_0/l$, where $u_0 = M/\rho c^2 l^2$ is the displacement in the focal region (at distance $R = l$). Making use of $M = 10^{26}$ dyn · cm, $\rho = 5\,\mathrm{g/cm}^3$, $c = 5$ km/s and $l = 1$ km, we get a focal displacement of the order $u_0 \simeq 80$ m (for $l = 400$ m we get $u_0 = l$).

The wavefront of the spherical-shell waves given by (3.19) intersects the surface $x_3 = z = 0$ along a circular line defined by $\overline{\mathbf{R}} = (x_1, x_2, -z_0)$, $\overline{R} = \left(r^2 + z_0^2\right)^{1/2}$, where $r = \left(x_1^2 + x_2^2\right)^{1/2}$ is the distance from the origin (placed on the surface, the epicentre) to the intersection points (we recall that $\mathbf{R}$ and $\overline{\mathbf{R}}$ are in fact $\mathbf{R} - \mathbf{R}_0$ and $\overline{\mathbf{R}} - \mathbf{R}_0$). The radius $\overline{R}$ moves with velocity c, $\overline{R} = ct$, $t > \mid z_0 \mid /c$, and the in-plane radius r moves according to the law $r = \sqrt{\overline{R}^2 - z_0^2} = \sqrt{c^2t^2 - z_0^2}$, where c stands for the velocities $c_{l,t}$; its velocity $v = dr/dt = c^2 t/r$ is infinite for $r = 0$ ($\overline{R} = ct = \mid z_0 \mid$) and tends to c for large distances.

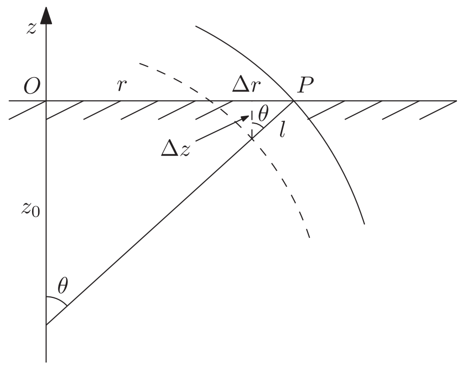

Fig. 3 Spherical-shell wave intersecting the surface $z = 0$ at P

The finite duration T of the source makes the δ'-functions in equation (3.19) to be viewed as functions with a finite spread $l = \Delta R = cT \ll R$; consequently, the intersection line of the waves with the surface has a finite spread Δr, which can be calculated from

$$\overline{R}^2 = r^2 + z_0^2, \quad (\overline{R} + l)^2 = (r + \Delta r)^2 + z_0^2, \tag{6.4}$$

hence,

$$\Delta r \simeq \frac{2\overline{R}l}{r + \sqrt{r^2 + 2\overline{R}l}}. \tag{6.5}$$

we can see that for $r \to 0$ the width $\Delta r \simeq \sqrt{2 \mid z_0 \mid l}$ of the seismic spot on the surface is much larger than the width of the spot for large distances $\Delta r \simeq l$ $(2 \mid z_0 \mid \gg l)$. For values of r not too close to the epicentre we may use the approximation $\Delta r \simeq \overline{R}l/r$. A spherical wave intersecting the surface $z = 0$ is shown in Fig. 3.

As long as the spherical wave is fully included in the half-space its total energy E_0 is given by the energy density $\mathcal{E}$ integrated over the spherical shell of radius R and thickness l. If the wave intersects the surface of the half-space, its energy E is given by the energy density integrated over the spherical sector which subtends the solid angle $2\pi(1 + \cos\theta)$, where $\cos\theta = \mid z_0 \mid /\overline{R}$ (see Fig. 3). It follows $E = \frac{1}{2}E_0\,(1+ \mid z_0 \mid /ct)$ for $ct > \mid z_0 \mid$. We can see that the energy of the wave decreases by the amount $E_s = \frac{1}{2}E_0\,(1- \mid z_0 \mid /ct)$, $ct > \mid z_0 \mid$. This amount of energy is transferred to the surface, which generates secondary waves (according to Huygens principle).

7. Interaction with the surface

In the seismic spot with the width Δr generated on the surface by the far-field primary waves given by (3.19) we may expect a reaction of the (free) surface, such as to compensate the force exerted by the incoming spherical waves. This localised reaction force generates secondary waves, distinct from the incoming, primary spherical waves. The secondary waves can be viewed as waves scattered off the surface, from the small region of contact of the surface seismic spot (practically, a circular line).

If the reaction force is strictly limited to the zero-thickness surface (as, for instance, a surface force), it would not give rise to waves, since its source has a zero integration measure. We assume that this reaction appears in a surface layer with thickness Δz ($\Delta z \ll \mid z_0 \mid$) and with a surface extension $2\pi r \Delta r$, where it is produced by volume forces. The thickness Δz of the superficial layer activated by the incoming primary wave may depend on $\overline{R}$ (and r), as the surface spread Δr does (7.2); for instance, from Fig. 3 we have $\Delta z = l \mid z_0 \mid /\overline{R}$. Since for an intermediate, limited region of the variable r (and $\overline{R}$) (that is, for a region not very close to the epicentre and not extending to infinity), the dependence on r of the product $\Delta r \Delta z$ is weak, we may neglect this dependence in what follows.

The volume elastic force per unit mass is given by $\partial_j \sigma_{ij}/\rho$, where $\sigma_{ij} = \rho \left[2c_t^2 u_{ij} + (c_l^2 - 2c_t^2) u_{kk}\delta_{ij}\right]$ is the stress tensor and u_{ij} is the strain tensor. The reaction force which compensates this elastic force is

$$f_i = -\partial_j \sigma_{ij}/\rho = -\partial_j \left[2c_t^2 u_{ij} + (c_l^2 - 2c_t^2) u_{kk}\delta_{ij}\right]. \tag{7.1}$$

We calculate the strain tensor from the displacement given by (3.19) and use it in (7.1); to compute the secondary waves we use the decomposition in Helmholtz potentials. We denote by $\mathbf{u}_s$ the displacement vector in the secondary waves, and introduce the Helmholtz potentials ψ and $\mathbf{B}$ ($\mathrm{div}\mathbf{B} = 0$) by $\mathbf{u}_s = \mathrm{grad}\psi + \mathrm{curl}\mathbf{B}$; then, we decompose the force $\mathbf{f}$ according to $\mathbf{f} = \mathrm{grad}\chi + \mathrm{curl}\mathbf{h}$ ($\mathrm{div}\mathbf{h} = 0$), where $\Delta\chi = \mathrm{div}\mathbf{f}$ and $\Delta\mathbf{h} = -\mathrm{curl}\mathbf{f}$; by the equation of the elastic waves, the Helmholtz potentials satisfy the wave equations (3.3); by straightforward calculations we get $\chi = -c_l^2 u_{ii}$ and $\mathbf{h} = c_t^2 \mathrm{curl}\mathbf{u}$, where $\mathbf{u}$ is $\mathbf{u}^f$ given by (3.19):

$$\chi = -\frac{c_l T m_{jk} x_j x_k}{4\pi R^3}\delta''(R - c_l t), \quad h_i = \varepsilon_{ijk}\frac{c_t T m_{kl} x_j x_l}{4\pi R^3}\delta''(R - c_t t), \tag{7.2}$$

we can see that the potentials χ and $\mathbf{h}$ 'move' with velocities c_l and, respectively, c_t (v_l and, respectively, v_t in the plane $z = 0$).

We can calculate the displacement in the secondary waves $\mathbf{u}_s = \mathrm{grad}\psi + \mathrm{curl}\mathbf{B}$, by solving the wave (3.3)

$$\ddot{\psi} - c_l^2 \Delta\psi = \chi, \quad \ddot{\mathbf{B}} - c_t^2 \Delta\mathbf{B} = \mathbf{h}, \tag{7.3}$$

with $\chi = -c_l^2 u_{ii}$ and $\mathbf{h} = c_t^2 \mathrm{curl}\mathbf{u}$ restricted to the superficial layer of thickness Δz and surface spread $2\pi r \Delta r$. Apart from appreciable technical complications, this procedure brings many superfluous features which obscure the relevant physical picture. This is why we prefer to use a simplified model which makes use of potentials of the form

$$\chi = \chi_0(r)\delta(z)\delta(r - v_l t), \quad \mathbf{h} = \mathbf{h}_0(r)\delta(z)\delta(r - v_t t) \tag{7.4}$$

($div\mathbf{h}_0 = 0$); equation (7.4) describe wave sources, distributed uniformly along circular lines on the surface, propagating on the surface with constant velocities $v_{l,t}$ and limited to a superficial layer with zero thickness and a circular line with zero width; their magnitudes $\chi_0(r)$ and $\mathbf{h}_0(r)$ have an approximate $1/\overline{R}$-dependence, which has a slow variation for $r \leq \mid z_0 \mid$ (and r not very close to the epicentre); for this range of the variable r we may consider χ_0 and $\mathbf{h}_0$ as being constant. The velocities $v_{l,t}$ in correspond to the velocities $v_{l,t} = dr/dt = c_{l,t}^2 t/r$ calculated above, which are greater than $c_{l,t}$, depend on r and tends to $c_{l,t}$ for large values of the distance r. We make a further simplification

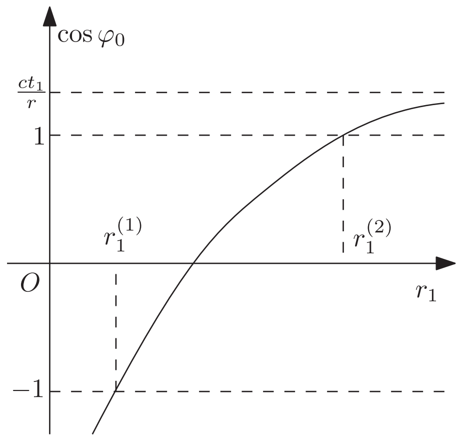

Fig. 4 The function $\cos\varphi_0$ *vs* r' for $C > 0$ (8.10)

and consider them as constant velocities slightly greater than $c_{l,t}$ (over an intermediate, limited range of variation of r). Also, in the subsequent calculations we consider the origin of the time at $r = 0$ (the epicentre) for each primary wave and the associated secondary source. The simplified model of secondary sources introduced here retains the main features of the exact problem, incorporated in the surface localisation and propagation of the sources with velocities $v_{l,t}$ greater than wave velocities $c_{l,t}$; on the other hand, by using this model we lose the anisotropy induced by the tensor of the seismic moment and specific details regarding the dependence on the distance. Since the secondary seismic sources are moving sources on the surface we may call the secondary waves produced by these sources 'surface seismic radiation'.

8. Secondary waves

Making use of the potentials given by (7.4), the solutions of (7.3) can be represented as

$$\psi = \frac{1}{4\pi c_l^2} \int dt_1 \int d\mathbf{R}_1 \frac{\chi_0(r_1)\delta(z_1)\delta(r_1 - v_l t_1)}{|\mathbf{R} - \mathbf{R}_1|} \delta\left(t - t_1 - |\mathbf{R} - \mathbf{R}_1|/c_l\right) \tag{8.1}$$

and a similar equation for $\mathbf{B}$. First, we focus on the potential ψ, which can be written as

$$\psi = \frac{1}{4\pi v c^2} \int d\mathbf{r}_1 \frac{\chi_0(r_1)\delta\left[t - r_1/v - \left(r^2 + r_1^2 - 2rr_1\cos\varphi + z^2\right)^{1/2}/c\right]}{\left(r^2 + r_1^2 - 2rr_1\cos\varphi + z^2\right)^{1/2}}, \tag{8.2}$$

where φ is the angle between the vectors $\mathbf{r}$ and $\mathbf{r}_1$ and we use c and v for c_l and, respectively, v_l, for the sake of simplicity. To calculate the integral with respect to the angle φ in (8.2) we introduce the function

$$F(\cos\varphi) = t - r_1/v - \left(r^2 + r_1^2 - 2rr_1\cos\varphi + z^2\right)^{1/2}/c \tag{8.3}$$

and look for its zeroes, $F_0 = F(\cos\varphi_0) = 0$ ($r_1 < vt$); we note that, if there exists one root of this equation, there exists another one at least, in view of the symmetry $\cos\varphi = \cos(2\pi - \varphi)$. Then, we expand the function F in a Taylor series in the vicinity of its zero, according to

$$F = F_0 + (\cos\varphi - \cos\varphi_0)\,F' + ... = (\cos\varphi - \cos\varphi_0)\,F' + ..., \tag{8.4}$$

where F' is the derivative of the function F with respect to $\cos\varphi$ for $\cos\varphi = \cos\varphi_0$. It is easy to see that the integral reduces to

$$\psi = \frac{1}{2\pi cvr}\int_0^\infty dr_1 \frac{\chi_0(r_1)}{\sin\varphi_0}, \tag{8.5}$$

where φ_0 is the root of the equation $F(\cos\varphi_0) = 0$, lying between 0 and π.

The root $\cos\varphi_0$ is given by

$$F(\cos\varphi_0) = t - r_1/v - \left(r^2 + r_1^2 - 2rr_1\cos\varphi_0 + z^2\right)^{1/2}/c = 0, \tag{8.6}$$

or

$$\left(1 - c^2/v^2\right)r_1^2 - 2\left(r\cos\varphi_0 - c^2t/v\right)r_1 - \left(c^2t^2 - r^2 - z^2\right) = 0 \tag{8.7}$$

for $r_1 < vt$. The important feature brought by the diference between the two velocities c and v can be accounted for conveniently by assuming that the two velocities are close to one another; we set $v = c(1+\varepsilon)$, $0 < \varepsilon \ll 1$ (as for sufficiently large distances). In this circumstance we may neglect the quadratic term $\sim r_1^2$ in (8.7) and replace t by the 'advanced' time $\tau = t(1-\varepsilon)$ (*i.e.*, $\tau_{l,t} = t(1-\varepsilon_{l,t})$); we get

$$\cos\varphi_0 \simeq \frac{2c\tau r_1 - C}{2rr_1}, \quad C = c^2\tau^2 - r^2 - z^2 \tag{8.8}$$

for $r_1 < vt = c\tau(1+2\varepsilon)$. It is easy to see that this equation has no solution for $C < 0$ (because of the condition $r_1 < vt$); for $C > 0$ ($c^2\tau^2 - r^2 - z^2 > 0$) it has two solutions

$$r_1^{(1)} = \frac{C}{2(c\tau + r)}, \quad r_1^{(2)} = \frac{C}{2(c\tau - r)} \tag{8.9}$$

corresponding to $\cos\varphi_0 = -1$ ($\varphi_0 = \pi$) and, respectively, $\cos\varphi_0 = 1$ ($\varphi_0 = 0$) (Fig. 4). For $z = 0$ the two roots $r_1^{(1,2)}$ reduce to $r_1^{(1,2)} = (c\tau \mp r)/2$; we can see that the sources of the secondary waves which arrive at r lie inside an anullus with radii $r_1^{(1,2)}$ and a constant width r, which expands on the surface with velocity $c/2$, after a time interval $\tau = r/c$. In the integral given by (8.7) we pass from the variable r_1 to the variable φ_0; for a limited range of integration r (from $r_1^{(1)}$ to $r_1^{(2)}$), we may take χ_0 out of the integral sign; we get

$$\psi \simeq \frac{C\chi_0}{4\pi c^2}\int_0^\pi d\varphi_0 \frac{1}{(r\cos\varphi_0 - c\tau)^2}, \tag{8.10}$$

or

$$\psi \simeq \frac{C\chi_0}{4\pi c^2 r^2}\frac{\partial}{\partial x}\int_0^{\pi/2} d\varphi_0 \left(\frac{1}{\cos\varphi_0 - x} - \frac{1}{\cos\varphi_0 + x}\right), \quad x = c\tau/r > 1. \tag{8.11}$$

The integrals in (8.11) can be effected immediately; we get the potential

$$\psi \simeq \frac{\chi_0}{4c_l^2}\frac{(c_l^2\tau_l^2 - r^2 - z^2)c_l\tau_l}{\left(c_l^2\tau_l^2 - r^2\right)^{3/2}}, \tag{8.12}$$

where the velocity c_l is restored. Similarly, we get the vector potential

$$\mathbf{B} \simeq \frac{\mathbf{h}_0}{4c_t^2}\frac{(c_t^2\tau_t^2 - r^2 - z^2)c_t\tau_t}{\left(c_t^2\tau_t^2 - r^2\right)^{3/2}}; \tag{8.13}$$

these equations are valid for $C_{l,t} = c_{l,t}^2\tau_{l,t}^2 - r^2 - z^2 > 0$.

We can see that the wavefronts $r^2 + z^2 = c_{l,t}^2\tau_{l,t}^2$ defines two spherical perturbations which move with velocities $c_{l,t}$. The singular behaviour of these waves for $z = 0$ resembles the algebraic singularity of the waves in two dimensions produced by localised sources (**18**), (**24**). The discontinuities exhibited by these functions are present irrespective of the particular dependence on r of the source potentials, as long as these potentials remain localised; they are related to a constant, finite velocity of propagation of the waves.

Making use of $\mathbf{u}_s = \mathrm{grad}\psi + \mathrm{curl}\mathbf{B}$ we can compute the displacement vector $\mathbf{u}_s$ in the secondary waves. We are interested mainly in the waves propagating on the surface ($z = 0$). First, we note that the displacement is singular at $c_{l,t}\tau_{l,t} = r$; this indicates the existence of two main shocks, occcurring after the arrival of the primary waves. Indeed, the primary waves arrive at the observation point $\mathbf{r}$ at the time $t_p = r/v_{l,t} = (r/c_{l,t})(1 - \varepsilon_{l,t})$, while the main shocks occur at $t_m = \tau_{l,t}/(1 - \varepsilon_{l,t}) \simeq (r/c_{l,t})(1 + \varepsilon_{l,t})$; we can see that there exists a time delay $\Delta t \simeq t_m - t_p \simeq 2(r/c_{l,t})\varepsilon_{l,t}$ between the primary waves and the wavefronts of the secondary waves (the main shocks). The sharp singularity in (8.12) and (8.13) is related to our using constant velocities $v_{l,t}$; actually, an uncertainty of the form $\Delta v \simeq c\varepsilon$ exists in these velocities, which entails an uncertainty $\tau\varepsilon$ in the time τ, such that the smallest value of the denominator in (8.12) and (8.13) is of the order $c^2\tau^2\varepsilon$. In the vicinity of the two main shocks, the leading contributions to the components of the surface displacement ($z = 0$, in polar cylindrical coordinates) are given by

$$u_{sr} \simeq \frac{\chi_0\tau_l}{4c_l}\cdot\frac{r}{\left(c_l^2\tau_l^2 - r^2\right)^{3/2}}, \quad u_{s\varphi} \simeq -\frac{h_{0z}\tau_t}{4c_t}\cdot\frac{r}{\left(c_t^2\tau_t^2 - r^2\right)^{3/2}} \tag{8.14}$$

and

$$u_{sz} \simeq \frac{h_{0\varphi}\tau_t}{4c_t}\cdot\frac{c_t^2\tau_t^2}{r\left(c_t^2\tau_t^2 - r^2\right)^{3/2}}, \tag{8.15}$$

we can see that there exists a horizontal component of the displacement perpendicular to the propagation direction ($u_{s\varphi}$) and both the r-component and the φ, z-components, which make right

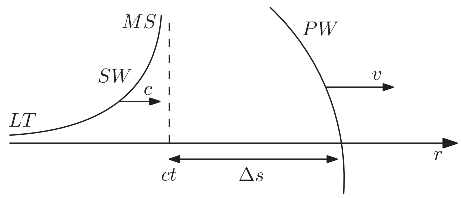

Fig. 5 Primary wave (PW), moving with velocity v on the Earth's surface, secondary wave (SW), moving with velocity $c < v$, the main shock (MS) and the long tail (LT); the separation between the two wavefronts is $\Delta s = 2(v - c)t$ and the time delay is $\Delta t = (2r/c)(v/c - 1)$, where r is the distance on the surface from the epicentre

angles with the propagation direction, are of the same order of magnitude (**4**). For long times ($c_{l,t}\tau_{l,t} \gg r$) the displacement (from (8.14) and (8.15)) goes like

$$u_{sr} \simeq \frac{\chi_0 r}{4c_l^4 \tau_l^2}, \quad u_{s\varphi} \simeq -\frac{h_{0z} r}{4c_t^4 \tau_t^2}, \quad u_{sz} \simeq \frac{h_{0\varphi}}{4c_t^2 r} \tag{8.16}$$

which show that the displacement exhibits a long tail, especially the z-component; it subsides as a consequence of the time-dependence induced in the potential $\mathbf{h}_0$ by the integration variable r_1, a circumstance which is neglected in the calculations presented here. The main shock and its long tail obtained here, in qualitative agreement with the recorded seismograms, are new. Primary and secondary waves, the main shock and the long tail are shown in Fig. 5.

9. Internal discontinuity

Let us consider a homogeneous isotropic elastic half-space extending in the region $-\infty < z < z_1$ with a superposed homogeneous isotropic elastic layer extending from $z = z_1$ to $z = 0$, in welded contact with the half-space at the plane surface $z = z_1$; we assume $z_1 < 0$. The elastic properties of the half-space and the layer are distinct. This model can serve as a representation of an internal discontinuity in the elastic properties of the half-space investigated above. An elementary seismic source as given by (2.2) is located at depth z_0, either above ($z_0 > z_1$) or beneath the discontinuity ($z_0 < z_1$). In the subsequent calculations we assume $z_0 < z_1$. We denote the half-space by 1 and the superposed layer by 2. A primary spherical wave generated by the elementary z_0-source arrives at the z_1-interface, along a circular line of contact, where it generates secondary waves; the secondary waves propagate both in the half space 1 and in the layer 2, where they arrive at the surface $z = 0$; we estimate here these secondary waves generated by the z_1-interface.

By analogy with (8.2) we assume that the primary waves in the half-space 1 generate on the interface $z = z_1$ the force Helmholtz potentials

$$\chi = \chi_0(r)\delta(z - z_1)\delta(r - v_{l1}t), \quad \mathbf{h} = \mathbf{h}_0(r)\delta(z - z_1)\delta(r - v_{t1}t), \tag{9.1}$$

the velocities $v_{l,t1}$ are considered constant and the r-dependence in $\chi_0(r)$, $\mathbf{h}_0(r)$ is weak for a finite, intermediate range of distances r. It is easy to see, by analogy with the calculations described in the previous section, that the Helmholtz potentials ψ and $\mathbf{B}$ of the secondary displacement are given by

$$\psi \simeq \frac{\chi_0}{4c_{l2}^2} \frac{\left[c_{l2}^2\tau_l^2 - r^2 - (z - z_1)^2\right] c_{l2}\tau_l}{\left(c_{l2}^2\tau_l^2 - r^2\right)^{3/2}}. \tag{9.2}$$

Similarly, we get from the wave equation the vector potential

$$\mathbf{B} \simeq \frac{\mathbf{h}_0}{4c_{t2}^2} \frac{\left[c_{t2}^2\tau_t^2 - r^2 - (z - z_1)^2\right] c_{t2}\tau_t}{\left(c_{t2}^2\tau_t^2 - r^2\right)^{3/2}}, \tag{9.3}$$

these potentials differ from the potentials given above by (8.13) and (8.14) by the presence of $z_1 \neq 0$. The origin of the time is considered here the moment when the primary wave touches the z_1-interface. The above formulae are valid for $C_{l,t} = c_{l,t2}^2\tau_{l,t}^2 - r^2 - (z - z_1)^2 > 0$; for $C_{l,t} < 0$ the potentials are equal to zero.

We are interested mainly in the surface $z = 0$. The presence of $z_1 \neq 0$ in (9.2) and (9.3) gives rise to a qualitatively different behaviour of the secondary waves generated by the discontinuity. The difference arises fom the condition $C_{l,t} = c_{l,t2}^2\tau_{l,t}^2 - r^2 - z_1^2 > 0$, which prevents the singularity at $c_{l,t2}^2\tau_{l,t}^2 - r^2 = 0$ to be reached; consequently, the secondary waves in the presence of the discontinuity do not exhibit the singular main shock on the surface $z = 0$; the main shock is reduced appreciably in this case.

We note also the advanced time $\tau_{l,t} = t_{l,t}(1 - \varepsilon_{l,t})$ in the above formulae (where $\tau_{l,t}$ is measured from the moment the primary waves touches the z_1-interface). For $z_0 < z_1$ the primary waves do not arrive on the surface and the $\mathbf{u}_{s2}$-waves generated by the interface are the only secondary waves which arrive on the surface $z = 0$. For $z_1 < z_0 < 0$, primary waves arrive on the surface $z = 0$, (delayed) secondary waves are generated on the surface $z = 0$ and, afterwards, much reduced secondary waves generated by the interface arrive on the surface $z = 0$.

We emphasise also that the results given above are valid for small values of $\varepsilon_{l,t}$, that is for the elastic properties of the layer 2 differing slightly from the elastic properties of the half-space 1. In addition, the secondary waves $\mathbf{u}_{s2}$ generate in their turn additional waves an the surface $z = 0$, which, however, are too small to present any further interest here (they may be called ‘tertiary’ waves).

10. Concluding remarks

Tensorial point forces governed by the tensor of the seismic moment are derived here and used for a homogeneous isotropic elastic half-space; such forces are placed at an inner point in the half-space. Endowed with a δ-like time dependence (temporal pulses), where δ is the Dirac delta function, they are termed here elementary seismic forces; they are generated by elementary seismic sources and produce elementary earthquakes. A weighted superposition of such forces (sources) can be performed by using a structure factor introduced here for the earthquakes’s focal region. All these concepts introduced here, and their mathamatical expressions, are new. The (double shock) P and S spherical-shell seismic waves generated by such forces are derived here by solving the equation of the elastic motion in the direct space; unphysical quasi-static contributions are removed by a regularisation (calibration) procedure. These waves are called here primary waves. They are associated with the feeble tremor exhibited usually by the seismic records (**4**), (**13**). Also, the primary waves produced by an isotropic source are derived here by solving the equation of the elastic motion; it is shown that these waves correspond to an isotropic (scalar) seismic tensor, as expected (such waves may correspond to the seismic waves produced by an explosion). The mathematical expression derived here for the primary waves produced by elementary tensorial point forces is new; it differs from known, particular cases.

It is shown, mainly by using energy-balance arguments, that the primary waves interact with the surface of the half-space and transfer part of their energy to the surface; consequently, additional, secondary wave sources occur on the surface, which generate secondary waves. Since the secondary sources move on the surface, the secondary waves they generate may be called 'surface seismic radiation'. A similar suggestion was implied long ago by Lamb (**2**), (**18**). The secondary wave sources are localised on the surface along circular lines. It is worth noting that the secondary sources move on the surface with velocities greater than the elastic waves velocities. A simplified model is put forward here for secondary waves sources, which allows the estimation of the secondary waves. The model assumes a uniform distribution of sources along circular lines, moving with constant velocities greater than the velocities of the elastic waves; it does not account for the anisotropy of the sources, and gives only a qualitative dependence of the waves on the distance. The secondary waves generated by the surface sources are estimated within this model, with emphasis on the secondary waves propagating on the surface. It is shown that these secondary waves are responsible for the seismic main shock and the long tail exhibited usually by earthquakes in the seismic records. These two latter items have indeed been associated long ago to waves generated and propagating on the surface (**19**), (**20**). The secondary waves generated by an internal discontinuity of the half-space are also estimated; it is shown that they produce a much reduced main shock. The precise formulation of the concept of secondary waves, generated by sources moving on the surface, the mathematical expression of the secondary waves obtained here in a simplified model and the expression of the main shock and its long tail are new.

Finally, a special situation deserves attention. If the source of the primary waves is located on the surface, the primary waves it generates are those given above for $z_0 = 0$. The interaction of these primary waves with the surface is null, since the thickness $\Delta z = l \mid z_0 \mid /\overline{R}$ of the intersecting layer is zero for $z_0 = 0$ (Fig. 3). The support of the interaction force with the surface reduces to zero and, consequently, the secondary waves are absent.

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