Letter

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Scattering of Non-Relativistic Charged Particles by Electromagnetic Radiation

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Abstract: The cross-section is computed for non-relativistic charged particles (like electrons and ions) scattered by electromagnetic radiation confined to a finite region (like the focal region of optical laser beams). The cross-section exhibits maxima at scattering angles given by the energy and momentum conservation in multi-photon absorption or emission processes. For convenience, a potential scattering is included and a comparison is made with the well-known Kroll-Watson scattering formula. The scattering process addressed in this paper is distinct from the process dealt with in previous studies, where the scattering is immersed in the radiation field.

Keywords: Electromagnetic Radiation; Non-Relativistic Charged Particles; Scattering.

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The potential scattering of charged particles in the presence of electromagnetic radiation has been enjoying much interest [1–10]. In the context of laser development, previous studies revealed the potential scattering assisted by multiple-photon exchange, as shown by the well-known Kroll-Watson cross-section [1, 2]. In the formulation of this problem, the scattering is immersed in the radiation field, that is, the asymptotic incoming and outgoing particle states and the scattering potential are included in the region containing the radiation. The duration of the laser pulse is much longer than the scattering time. The starting point of these approaches is the standard non-relativistic Hamiltonian where the particle momentum $\mathbf{p}=m\mathbf{v}+e\mathbf{A}/c$ includes the electromagnetic contribution $e\mathbf{A}/c$ beside the purely mechanical contribution $m\mathbf{v}$ (the notations are the usual ones, i.e. m and e denote the particle mass and charge, respectively, $\mathbf{v}$ is the particle velocity, $\mathbf{A}$ is the vector potential of the radiation field and c denotes the speed of light in a vacuum). The Kroll-Watson cross-section corresponds to radiation-assisted potential scattering, that is, it shows how the potential cross-section is modified by the presence of the radiation; it becomes zero when the potential is removed.

With the advent of high-intensity lasers and strongly focused laser beams [11–18], it appears there is the possibility of scattering charged particles by the radiation field confined to the focal region of the beam (the radiation is vanishing smoothly ouside the focal region). In this case, the asymptotic scattering states are radiation free; they are the eigenstates of the quantum-mechanical momentum corresponding to the purely mechanical momentum, without including the electromagnetic contribution. This is the scattering problem addressed in this paper. It resembles, to some extent, the electron diffraction from standing light waves, where the scattering proceeds by spontaneous emission of Compton photons (the Kapitsa-Dirac effect) [19]. We envisage charged particles like electrons or ions, with non-relativistic energies, scattered off electromagnetic radiation confined to the focal region (in a vacuum) of an optical laser beam. Usually, the order of magnitude of the dimension of the focal region is a few tens of radiation wavelengths, and the radiation field has a reasonably high intensity, such that the non-relativistic character of the particle motion is preserved. For convenience, we include in the focal region a static potential. We assume first a laser pulse much longer than the radiation period.

We consider a non-relativistic particle with mass m and charge e scattered by an electromagnetic radiation field with the vector potential $\mathbf{A}=\mathbf{A}_0\cos(\omega t-\mathbf{kr})$ (linear polarisation), where $\mathbf{A}_0$ is the amplitude, and ω and $\mathbf{k}$ are the radiation frequency and wavevector, respectively; t and $\mathbf{r}$ denote the time and position, respectively. Since the phase velocity of the non-relativistic charge is much smaller than the speed of light c in a vacuum ($\omega=ck$), we may neglect the spatial phase $\mathbf{kr}$ in comparison with the temporal phase ωt; consequently, the vector potential may be approximated by $\mathbf{A}\simeq\mathbf{A}_0\cos\omega t$. The Hamiltonian of the particle in the radiation field becomes the well-known dipole Hamiltonian [20–25]

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$$H=\frac{1}{2m}p^2-e\mathbf{rE}(t)+V(\mathbf{r}), \quad (1)$$

where $\mathbf{E}(t)=\mathbf{E}_0\sin\omega t$, $\mathbf{E}_0=\omega\mathbf{A}_0/c$, is the electric field, and $V(\mathbf{r})$ is a potential which does not depend on time. The non-relativistic character of the motion is preserved in the radiation field provided $eA_0/mc^2\ll 1$. Making use of the notations $H_0=p^2/2m$ and $U=-e\mathbf{rE}$, the wavefunction ψ_i of the incident particle satisfies the Schrodinger equation

$$i\hbar\frac{\partial\psi_i}{\partial t}=(H_0+U+V)\psi_i \quad (2)$$

with the initial condition (incoming state)

$$\psi_i^0=\frac{1}{\sqrt{v}}e^{-\frac{i}{\hbar}E_it+\frac{i}{\hbar}\mathbf{p}_i\mathbf{r}}, \quad (3)$$

where $\mathbf{p}_i$ is the initial momentum, E_i is the initial energy and v denotes the volume ($\hbar$ is the Planck's constant); the solution of (2) is given by

$$\psi_i=\psi_i^0-\frac{i}{\hbar}e^{-\frac{i}{\hbar}H_0t}\int_{-\infty}^{t}\mathrm{d}t'e^{\frac{i}{\hbar}H_0t'}(U+V)\psi_i. \quad (4)$$

The wavefunction of the final scattering state (outgoing state) is

$$\psi_f=\frac{1}{\sqrt{v}}e^{-\frac{i}{\hbar}E_ft+\frac{i}{\hbar}\mathbf{p}_f\mathbf{r}}, \quad (5)$$

where $\mathbf{p}_f$ is the final momentum, and E_f is the final energy. The transition amplitude (S-matrix) reads

$$a_{fi}=-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t(\psi_f,\,(U+V)\psi_i). \quad (6)$$

In (2), we successively insert the Goeppert-Mayer transform [26]

$$\psi_i=e^{iS_1}\phi_i,\; S_1=-\frac{1}{\hbar}\int_{-\infty}^{t}\mathrm{d}t'U(t')=-\frac{e}{\hbar c}\mathbf{rA},$$
$$i\hbar\frac{\partial\phi_i}{\partial t}=\left[\frac{1}{2m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2+V\right]\phi_i \quad (7)$$

and the Kramers-Henneberger transform [27–30]

$$\phi_i=e^{iS_2}\chi_i,\; S_2=\frac{e}{\hbar mc}\int_{-\infty}^{t}\mathrm{d}t'\mathbf{pA}-\frac{e^2}{2\hbar mc^2}\int_{-\infty}^{t}\mathrm{d}t'A^2$$
$$=\frac{e}{\hbar mc\omega}\mathbf{pA}_0\sin\omega t-\frac{e^2}{8\hbar mc^2\omega}A_0^2(\sin 2\omega t+2\omega t),$$
$$i\hbar\frac{\partial\chi_i}{\partial t}=(H_0+\tilde{V})\chi_i,\; \tilde{V}(\mathbf{r})=V(\mathbf{r}-e\mathbf{E}/m\omega^2). \quad (8)$$

We recognise in (7) the standard non-relativistic Hamiltonian of the particle in the radiation field. With regards to the potential V, we limit ourselves to the Born approximation; consequenty, the wavefunction χ_i can be written as

$$\chi_i=\psi_i^0-\frac{i}{\hbar}e^{-\frac{i}{\hbar}H_0t}\int_{-\infty}^{t}\mathrm{d}t'e^{\frac{i}{\hbar}H_0t'}\tilde{V}\psi_i^0 \quad (9)$$

and the transition amplitude becomes

$$a_{fi}=-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t(\psi_f,\,(U+V)e^{iS_1}e^{iS_2}\psi_i^0)$$
$$+\left(-\frac{i}{\hbar}\right)^2\int_{-\infty}^{+\infty}\mathrm{d}t\left(\psi_f,\,Ue^{iS_1}e^{iS_2}e^{-\frac{i}{\hbar}H_0t}\int_{-\infty}^{t}\mathrm{d}t'e^{\frac{i}{\hbar}H_0t'}\tilde{V}\psi_i^0\right). \quad (10)$$

The advantage of the canonical transformations carried out above consists of the possibility of applying the Born approximation. The U term in (10) corresponds to the scattering by the radiation field; we can see in (10) that the scattering by the potential V is dressed with the radiation (terms $\sim Ve^{iS_1}e^{iS_2}$, $\sim\tilde{V}$). In addition, there appears an interference term, which includes the product $U\tilde{V}$.

We may make certain simplifications in (10). It is easy to see that the phase S_1 (7) is of the order (r/λ) $(eA_0/\hbar\omega)\gg eA_0/\hbar\omega$, where λ is the radiation wavelength, while the phase S_2 (8) is of the order $(p/mc,\,eA_0/mc^2)$ $(eA_0/\hbar\omega)=eA_0/\hbar\omega$; consequently, we may neglect the phase S_2 in (10). In addition, the position given by the argument of $\tilde{V}$ (8) is of the order $r-\lambda_c(eA_0/\hbar\omega)$, where $\lambda_c=\hbar/mc$ is the Compton wavelength of the particle; it takes the potential $\tilde{V}$ far away from its short range in very short times, especially for (reasonably) high-intensity radiation. Similarly, the Coulomb potential is rapidly reduced to an appreciable extent by the radiation, such that we may neglect the potential $\tilde{V}$ in (10).

Making use of these simplifications, the transition amplitude given by (10) becomes

$$a_{fi}\simeq-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t(\psi_f,\,Ue^{iS_1}\psi_i^0)-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t(\psi_f,\,Ve^{iS_1}\psi_i^0)$$
$$=\int_{-\infty}^{+\infty}\mathrm{d}t\left(\psi_f,\,\left(\frac{\partial}{\partial t}e^{iS_1}\right)\psi_i^0\right)-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t(\psi_f,\,Ve^{iS_1}\psi_i^0), \quad (11)$$

or

$$a_{fi}=\frac{1}{v}\int_{-\infty}^{+\infty}\mathrm{d}t\mathrm{d}\mathbf{r}e^{\frac{i}{\hbar}(E_f-E_i)t}\frac{\partial}{\partial t}e^{-\frac{ie}{\hbar\omega}\mathbf{rE}_0\cos\omega t}e^{\frac{i}{\hbar}\mathbf{pr}}$$
$$-\frac{i}{\hbar v}\int_{-\infty}^{+\infty}\mathrm{d}t\mathrm{d}\mathbf{r}e^{\frac{i}{\hbar}(E_f-E_i)t}Ve^{-\frac{ie}{\hbar\omega}\mathbf{rE}_0\cos\omega t}e^{\frac{i}{\hbar}\mathbf{pr}}, \quad (12)$$

where $\mathbf{p}=\mathbf{p}_i-\mathbf{p}_f$ is the momentum transfer. In (12), we use the decomposition

$$e^{-\frac{ie}{\hbar\omega}\mathbf{rE}_0\cos\omega t}=\sum_{n=-\infty}^{+\infty}(-i)^n J_n(e\mathbf{rE}_0/\hbar\omega)e^{-in\omega t}, \tag{13}$$

where J_n is the Bessel function (n being any integer) and get

$$a_{fi}=-\frac{2\pi i}{v}\sum_{n=-\infty}^{+\infty}(-i)^n\delta(E_f-E_i-n\hbar\omega)\int d\mathbf{r}[n\hbar\omega+V(\mathbf{r})]$$
$$J_n(e\mathbf{rE}_0/\hbar\omega)e^{\frac{i}{\hbar}\mathbf{pr}}. \tag{14}$$

We can see the occurrence of multiple-photon scattering processes with energy conservation $E_f=E_i+n\hbar\omega$. The transition probability per unit time is given by

$$w_{fi}=\frac{2\pi}{\hbar v^2}\sum_{n=-\infty}^{+\infty}\delta(E_f-E_i-n\hbar\omega)$$
$$\left|\int d\mathbf{r}[n\hbar\omega+V(\mathbf{r})]J_n(e\mathbf{rE}_0/\hbar\omega)e^{\frac{i}{\hbar}\mathbf{pr}}\right|^2. \tag{15}$$

We multiply w_{fi} by the density of the final states $vp_f^2dp_f d\Omega/(2\pi\hbar)^3$, where $d\Omega$ is the element of the solid angle, divide by the current density v_i/v, where v_i is the initial velocity, and integrate over the final momentum p_f in order to get the differential cross-section for the n-process

$$d\sigma_n=\frac{p_{fn}}{p_i}\left|\frac{m}{2\pi\hbar^2}\int d\mathbf{r}[n\hbar\omega+V(\mathbf{r})]J_n(e\mathbf{rE}_0/\hbar\omega)e^{\frac{i}{\hbar}\mathbf{pr}}\right|^2 d\Omega. \tag{16}$$

The momentum p_{fn} is given by the energy conservation $p_{fn}^2/2m=p_i^2/2m+n\hbar\omega$.

We can see in (16) the cross-section of the scattering by the radiation and the cross-section of the scattering from the potential V. In addition, there are mixed radiation-potential scattering terms; for $n=0$, we get from (16) the elastic Born scattering in the field of the potential V affected by the radiation, due to the presence of the function J_0. The argument of the Bessel function in (16) varies rapidly over the integration domain; therefore, we may use the asymptotic expression for the Bessel function. In doing so, we can easily see that the $\mathbf{r}$ integration in the contribution of the radiation scattering is non-vanishing for $\mathbf{p}\simeq e\mathbf{E}_0/\omega$, that is, the momentum is trasferred along the direction of the electric field, as expected; this is the momentum conservation. Making use of this relation and the energy conservation, we get $p_{fn}\simeq p_i(1+n\hbar\omega/2E_i)$ (for $n\hbar\omega\ll E_i$) and

$$-1<\cos\theta_n=\frac{1-e\mathbf{p}_i\mathbf{E}_0/p_i^2\omega}{1+n\hbar\omega/2E_i}<1, \tag{17}$$

where θ_n is the scattering angle. For $n>0$ (photon absorption), the scattering angle θ_n increases, while for $n<0$ (photon emission), the scattering angle decreases, with respect to the elastic scattering angle θ_0. Equation (17) indicates that there exists a limitation which can be written as $-n\hbar\omega/2E_i<(eA_0\cos\alpha/2E_i)(v/c)<2+n\hbar\omega/2E_i$, where α is the angle the initial momentum makes with the electric field; for very low incident energies, the scattering occurs only for angles close to the right angle made by the incident momentum with the electric field. Finally, we note that the order of magnitude of the cross-section of the scattering due to the radiation is $\simeq d^2[n\hbar\omega/(\hbar^2/md^2)]^2(\hbar c/eA_0 d)$, where $(\hbar c/eA_0 d)\ll 1$ and d is the dimension of the region where the radiation is confined to. We can see from (16) that the cross-section due to the radiation may acquire large values as a consequence of the large dimension of the region containing the radiation; the cross-section increases with increasing n, that is, for large scattering angles, where, however, the scattering maxima coalesce.

As regards the modification brought about by the presence of the radiation in the scattering produced by the potential $V(\mathbf{r})$, there are two differences in the present case, where the radiation is limited to a finite region, in comparison with the case where the scattering is immersed in radiation. If, in the latter case, we neglect the interaction U, leave aside the corresponding transformation given by U_1 and start directly with the standard non-relativistic Hamiltonian given by (7), then it is easy to see that we get from (11) and (16) the Kroll-Watson formula $d\sigma_n=(p_{fn}/p_i)J_n^2(e\mathbf{pE}_0/m\hbar\omega^2)d\sigma_B$, where $d\sigma_B$ is the elastic Born cross-section [1, 2, 31]. This formula shows that the Born cross-section is modulated by the function J_n^2, whose argument includes the energy conservation with the exchange of n photons. In our case, the first difference consists of the presence of the interference term $\sim n\hbar\omega V(\mathbf{r})J_n(e\mathbf{rE}_0/\hbar\omega)$ in (16), arising from both potentials U and V, which is absent in the Kroll-Watson formula. The second difference is related to the Bessel function $J_n(e\mathbf{rE}_0/\hbar\omega)$, which is included in the integral in (16) and depends on the range of the interaction $V(\mathbf{r})$ through its argument $e\mathbf{rE}_0/\hbar\omega$, while in the Kroll-Watson formula, the Bessel function depends on $e\mathbf{pE}_0/m\hbar\omega^2$ and is placed outside the integral. These differences are important, especially for the Coulomb potentials. In addition, while in the Kroll-Watson formula the momentum transfer $\mathbf{p}$ is given by the potential $V(\mathbf{r})$, in our case it is given both by the potential $V(\mathbf{r})$ and the radiation potential $U(\mathbf{r})$ through the function $J_n(e\mathbf{rE}_0/\hbar\omega)$.

The above calculations are done for a sufficiently long laser pulse. In practice, the pulse has a finite duration τ and a repetition time Δt. In these conditions, the expansion given by (13) remains valid, but the function $\delta(\Delta E)$ in the scattering amplitude a_{fi} (14), where $\Delta E = E_f - E_i - n\hbar\omega$, is replaced by the function $\zeta(\Delta E) = e^{\frac{i}{\hbar}\Delta E \cdot t_i} \sin\alpha\Delta E / \pi\Delta E$, where $\alpha = \tau/2\hbar$ and t_i denotes the time moment of the pulse (the pulse lasts from $t_i - \tau/2$ to $t_i + \tau/2$). For a large α, the function $\zeta(\Delta E)$ has a maximum for $\Delta E = 0$ and extends approximately over a bandwidth $\delta E \simeq \pi/\alpha = 2\pi\hbar/\tau$. Since usually τ is much longer than the radiation period $T = 2\pi/\omega$, the energy separation δE is much smaller than the radiation quanta of the energy $\hbar\omega$. It follows that the function $\zeta(\Delta E)$ for different n values can be viewed as being well separated. In these conditions, the cross-section $d\sigma_n$ (16) preserves its form, except that it is multiplied by the reduction factor $\tau/(\tau + \Delta t)$.

In conclusion, it is shown in this paper that non-relativistic charged particles may suffer scattering as a result of their interaction with the electromagnetic radiation in the focal region of laser beams. The cross-section of this scattering process is computed in this paper for a single-mode radiation with linear polarisation. As expected, the cross-section exhibits maxima at certain scattering (diffraction) angles θ_n, as given by (17), determined by the energy and the momentum conservation in multiple-photon exchange processes. The calculations can be easily extended to any polarisation; for realistic laser beams or for multi-mode radiation, we should take into account the particular beam shape [32] and the amplitude and frequency fluctuations [33]. For convenience, we included also the scattering from a potential placed in the radiation field in the Born approximation. The cross-section of the potential scattering is modified by the presence of the radiation, because the scattering states are dressed by radiation. The cross-section is reduced for high-intensity radiation (preserving the non-relativistic character of the particle motion), and, similarly, the multi-photon scattering is diminished for the high energy of the particle, when the process reduces to elastic scattering in the forward direction. In contrast with previous studies, where the scattering process is immersed in the radiation field, in the scattering process addressed in this paper the radiation field is confined to a finite region. The modifications brought by the radiation in this circumstance to the potential scattering are different from the well-known Kroll-Watson formula.

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