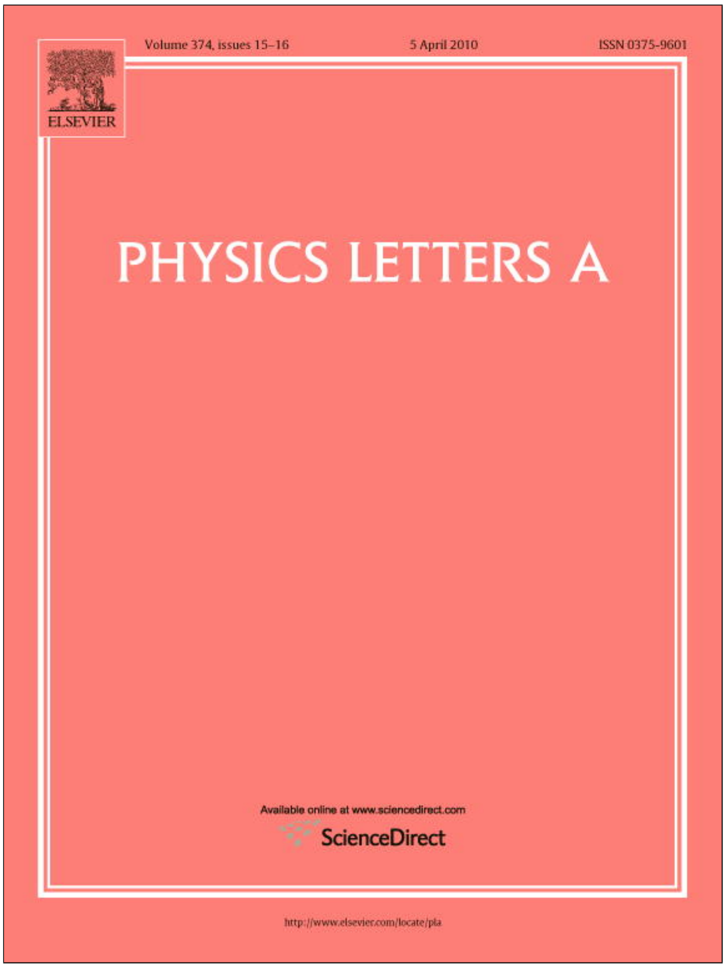
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Elastic waves in a semi-infinite body

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ABSTRACT

A new method is introduced for studying the propagation of elastic waves in isotropic bodies, based on the Kirchhoff potentials borrowed from electromagnetism. By means of this method we identify and characterize the elastic waves generated in a semi-infinite (half-space) body by the action of an external force localized on, or beneath, the body surface. The method implies coupled integral equations for the wave amplitudes, which we solve for both cases mentioned above. For a force localized on the body surface we identify two transverse waves, corresponding to the two polarizations (normal and parallel to the propagation plane). The longitudinal waves appear as eigenmodes. The waves produced by a force localized beneath the surface are stationary waves along the normal to the surface. We compute the surface displacement in both cases and the force exerted on the surface by a force localized beneath. All these quantities exhibit a characteristic decrease with the distance on the body surface and an oscillatory behaviour. We discuss briefly some possibilities of extending the present method to include the effect of the inhomogeneities on the waves propagation.

1. Introduction

As it is well known, the propagation of elastic waves in bodies with special, restricted geometries has ever enjoyed a great deal of interest [1–12]. The problem was originally introduced by Rayleigh [13] and Lamb [14], and underwent various developments during the time. It exhibits a certain complexity related to difficulties arising mainly from the lack of an adequate treatment of the boundary conditions. These difficulties are increased for the problem of determining the waves produced in such elastic bodies by external forces, either localized on the body surface, or within the bulk, or extended over certain spatial volumes. Even more interesting, and more difficult, is the problem of treating the effect of the inhomogeneities, either localized or extended, on the wave propagation in finite elastic bodies. Apart from their practical importance in engineering, such problems are of great relevance for the effect of the seismic waves on the Earth's surface [15–20].

The propagation of elastic waves in isotropic solids is governed by the well-known Navier–Stokes equation [21]. In the absence of external forces, solving this (homogeneous) equation amounts to an eigenmodes problem. For bodies with restricted, special geometries the eigenmodes equation is supplemented with adequate boundary conditions, which, basically, means the continuity of the elastic force (given by the stress tensor) at boundaries. For a semi-infinite body (half-space), for instance, with a free surface the component of the elastic force normal to the surface must vanish. In addition, boundary conditions are imposed at infinity, leading, for instance, to propagating or surface (Rayleigh) waves. Similar boundary conditions are employed, for instance, for a layer superposed over an elastic half-space, leading to Love's waves [22]. This way, we get the eigenmodes (and the corresponding eigenfrequencies), which we may call "free waves", in the absence of an external force. Localized external forces are treated also by considering sui generis "boundary conditions" at the location of these forces, in order to account for the discontinuities implied by such forces. This method, employed extensively by Lamb [14], is traditionally in use today. We can say that the particular solution obtained for the Navier–Stokes equation in this way give the "forced waves", *i.e.* the waves generated by the external force. The general solution, consisting of a superposition of "free" and "forced waves", should then obey the boundary conditions, related to the particular geometry of the body. We can see that such an approach is pretty complicate, and its application to problems of practical interest is limited. Various approximate methods, both analytical and numerical, have been developed for such problems [23]. The degree of mathematical complexity increases significantly for distributed external forces, or for the presence of inhomogeneities and defects. Consequently, it is not a surprise that little progress has been recorded in studying these matters from the classical works of Rayleigh [13], Lamb [14] or Love [22].

We present here a new method for studying the wave propagation in isotropic elastic bodies with a finite (or partially finite) structure, based on the Kirchhoff potentials of the wave equation with sources, borrowed from electromagnetism. The novelty of our method consists in viewing the compression term in the Navier–Stokes equation as a source term. We apply this method to determine the elastic waves produced in a semi-infinite (half-

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space) body by external forces localized either on the body surface or beneath it. For the force localized on the surface we determine two transverse waves propagating in the body, and a longitudinal one which appears as an eigenmode. For the force localized beneath the body surface the elastic waves are stationary waves along the direction perpendicular to the body surface. We compute the surface displacement in both cases as well as the force exerted on the surface by a force localized beneath the surface. All these quantities exhibit a characteristic decrease and an oscillatory behaviour along the in-plane distance on the body surface. In both cases, the present method leads to coupled integral equations for the wave amplitudes, which we solve. By means of the method presented here we generalize one of Lamb's problem (force localized on the surface) [14] and get new results for a point-like force localized under the surface. The generalization consists in treating a general distribution of forces with a general orientation acting on the surface. To the author knowledge, for a force localized beneath the surface the results presented here are new. Finally, we give a brief discussion of how the present method can be extended to include the effect of the inhomogeneities on the wave propagation in elastic bodies with finite geometries.

The elastic waves in isotropic bodies are governed by the equation of motion (Navier–Stokes equation) [21]

$$\rho\ddot{\mathbf{u}} = \mu\Delta\mathbf{u} + (\lambda+\mu)\, grad\cdot div\,\mathbf{u} + \rho\mathbf{f}, \tag{1}$$

where ρ is the density, $\mathbf{u}$ is the displacement field, μ and λ are the Lame coefficients and $\mathbf{f}$ is an external force per unit mass. By a Fourier transform of the form

$$\mathbf{u}(\mathbf{R},t) = \sum_{\mathbf{K}}\int d\omega\,\mathbf{u}(\mathbf{K},\omega)e^{i\mathbf{KR}-i\omega t} \tag{2}$$

and a similar one for the force $\mathbf{f}$, Eq. (1) becomes

$$\left(-\rho\omega^2+\mu K^2\right)\mathbf{u} = -(\lambda+\mu)(\mathbf{Ku})\mathbf{K} + \rho\mathbf{f}, \tag{3}$$

where we dropped out the arguments $\mathbf{K}$, ω for simplicity. In Eqs. (2) and (3), as well as in all subsequent cases, the juxtaposition of two bold-faced vectors (like $\mathbf{KR}$, $\mathbf{Ku}$, etc.) means the scalar product. Eq. (3) can easily be solved. Its solutions are given by

$$\mathbf{u} = -\frac{(v_l^2-v_t^2)(\mathbf{Kf})}{(\omega^2-v_t^2K^2)(\omega^2-v_l^2K^2)}\mathbf{K} - \frac{\mathbf{f}}{\omega^2-v_t^2K^2}, \tag{4}$$

where

$$v_t = \sqrt{\frac{\mu}{\rho}}, \qquad v_l = \sqrt{\frac{\lambda+2\mu}{\rho}} \tag{5}$$

are the velocities of the transverse and, respectively, longitudinal waves. We can see from Eq. (4) that for a longitudinal force $\mathbf{f} = f\mathbf{K}/K$ the displacement field is longitudinal and has the eigenfrequencies $\omega = v_l K$, while for a transverse force, $\mathbf{Kf} = 0$, the field is transverse and has the eigenfrequencies $\omega = v_t K$. As it is well known, the Lame coefficients can be expressed by the Young modulus E and the Poisson ratio σ,

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \qquad \mu = \frac{E}{2(1+\sigma)}, \tag{6}$$

and, for reasons of stability, $E > 0$ and $-1 < \sigma < 1/2$ (actually, for usual bodies, $0 < \sigma < 1/2$). In particular, the ratio

$$q = \frac{v_l^2}{v_t^2} - 1 = \frac{\lambda}{\mu} + 1 = \frac{1}{1-2\sigma} \tag{7}$$

satisfies the inequality $q > 1/3$ (actually $q > 1$) [21]. In general, the solution of the homogeneous equation (1) ("free waves") must be added to the particular solution given by Eq. (4) ("forced waves").

Making use of these notations we write Eq. (1) as

$$\frac{1}{v_t^2}\ddot{\mathbf{u}} - \Delta\mathbf{u} = q\cdot grad\cdot div\,\mathbf{u} + \frac{\mathbf{f}}{v_t^2}, \tag{8}$$

where we can recognize the wave equation with sources $q\cdot grad\cdot div\,\mathbf{u}$ and $\mathbf{f}/v_t^2$. As it is well known, its solution is given by the retarded (Kirchhhoff) potential

$$\mathbf{u}(\mathbf{R},t) = \frac{q}{4\pi}\int d\mathbf{R}'\frac{grad\cdot div\,\mathbf{u}(\mathbf{R}',t-|\mathbf{R}-\mathbf{R}'|/v_t)}{|\mathbf{R}-\mathbf{R}'|} + \frac{1}{4\pi v_t^2}\int d\mathbf{R}'\frac{\mathbf{f}(\mathbf{R}',t-|\mathbf{R}-\mathbf{R}'|/v_t)}{|\mathbf{R}-\mathbf{R}'|}. \tag{9}$$

Indeed, making use of the Fourier transform given by Eq. (2) and using also the well-known integral

$$\int d\mathbf{R}\frac{1}{R}e^{i\mathbf{KR}+i\omega R/v_t} = -\frac{4\pi v_t^2}{\omega^2-v_t^2K^2} \tag{10}$$

we get easily the solution given by Eqs. (3) and (4). We apply here this method of Kirchhoff potential, inspired from the theory of electromagnetism [24], to the elastic waves generated in a semi-infinite body by forces localized either on the body surface or beneath it.

2. Force localized on the surface

We consider a semi-infinite isotropic elastic body extending over the region $z > 0$ and assume a localized force

$$\mathbf{f}(\mathbf{R},t) = a\sum_{\mathbf{k}}\int d\omega\,\mathbf{f}(\mathbf{k},\omega)e^{i\mathbf{kr}-i\omega t}\delta(z) \tag{11}$$

acting on the body plane surface $z = 0$, where a is a characteristic length, $\mathbf{R} = (\mathbf{r}, z)$ and $\mathbf{k}$ is the in-plane wavevector. This is a generalization of one of Lamb's problems [14]. The generalization consists in assuming a general distribution of the force acting on the surface and a general orientation of the force. The length a, much smaller than the relevant distances and wavelengths, is introduced on the one hand for reasons of dimensionality and, on the other, in order to have a representation for the thickness of the surface. Eq. (11) is the standard form for a Fourier transform of a function (force) of position $\mathbf{R} = (\mathbf{r}, z)$ and time t with respect to the in-plane position $\mathbf{r}$ and time t. The dependence of the z-coordinate is maintained in the form of a Dirac delta function $\delta(z)$ (an "impulse" force). In this respect the "function" defined by Eq. (11) is in fact a distribution. It is "local", in the sense that along the z-coordinate the force is localized over a small distance a, where it has the value $1/a$, and is zero outside that region.

We represent the displacement field as

$$\mathbf{u} = (\mathbf{v}, u_3)\theta(z), \tag{12}$$

where $\mathbf{v}$ is the in-plane component (parallel to the surface), u_3 is the transverse component (perpendicular to the surface) and $\theta(z) = 0$ for $z < 0$, $\theta(z) = 1$ for $z > 0$ is the step function. The divergence occurring in Eq. (9) can then be written as

$$div\,\mathbf{u} = \left(div\,\mathbf{v} + \frac{\partial u_3}{\partial z}\right)\theta(z) + u_3(0)\delta(z), \tag{13}$$

where we can see the occurrence of specific surface contributions associated with $u_3(0) = u_3(z=0)$. We use a Fourier transform of the form

$$\mathbf{v}(\mathbf{r},z;t) = \sum_{\mathbf{k}}\int d\omega\,\mathbf{v}(\mathbf{k},\omega;z)e^{i\mathbf{kr}-i\omega t}, \tag{14}$$

and a similar one for $u_3(\mathbf{r}, z; t)$. Usually, we leave aside the arguments $\mathbf{k}$, ω, while preserving explicitly the z-dependence of the functions $\mathbf{v}(\mathbf{k}, \omega; z)$ and $u_3(\mathbf{k}, \omega; z)$. We compute $grad \cdot div\, \mathbf{u}$ according to Eqs. (13) and (14) and introduce it, together with the force given by Eq. (11), in Eq. (9). The intervening integrals reduce to the known integral [25]

$$\int_{|z|}^{\infty} dx\, J_0\left(k\sqrt{x^2 - z^2}\right) e^{i\omega x/v_t} = \frac{i}{\kappa} e^{i\kappa|z|}, \tag{15}$$

where

$$\kappa = \sqrt{\frac{\omega^2}{v_t^2} - k^2}. \tag{16}$$

In addition, we introduce the convenient notations $v_1 = \mathbf{v}\mathbf{k}/k$, $v_2 = \mathbf{v}\mathbf{k}_\perp/k$ and similar ones for $f_{1,2}$, where $\mathbf{k}_\perp$ is a vector perpendicular to $\mathbf{k}$, $\mathbf{k}\mathbf{k}_\perp = 0$, and of the same magnitude k. Then, Eq. (9) reduces to

$$v_2 = \frac{iaf_2}{2v_t^2\kappa} e^{i\kappa z} \tag{17}$$

and to a set of two coupled integral equations which read

$$v_1 = -\frac{iqk^2}{2\kappa}\int_0 dz'\, v_1(z') e^{i\kappa|z-z'|} - \frac{qk}{2\kappa}\frac{\partial}{\partial z}\int_0 dz'\, u_3(z') e^{i\kappa|z-z'|} + \frac{iaf_1}{2v_t^2\kappa} e^{i\kappa z} \tag{18}$$

and

$$u_3 = -\frac{qk}{2\kappa}\frac{\partial}{\partial z}\int_0 dz'\, v_1(z') e^{i\kappa|z-z'|} + \frac{iq}{2\kappa}\frac{\partial^2}{\partial z^2}\int_0 dz'\, u_3(z') e^{i\kappa|z-z'|} + \frac{iaf_3}{2v_t^2\kappa} e^{i\kappa z}. \tag{19}$$

In deriving these equations it is worth noting the non-invertibility of the derivatives and the integrals, according to the identity

$$\frac{\partial}{\partial z}\int_0 dz'\, f(z')\frac{\partial}{\partial z'} e^{i\kappa|z-z'|} = \kappa^2 \int_0 dz'\, f(z') e^{i\kappa|z-z'|} - 2i\kappa f(z) \tag{20}$$

for any function $f(z)$, $z > 0$; it is due to the discontinuity in the derivative of the function $e^{i\kappa|z-z'|}$ for $z = z'$. From Eqs. (18) and (19) we get easily

$$u_3 = -\frac{i}{k}\frac{\partial v_1}{\partial z} - \frac{ia(\kappa f_1 - kf_3)}{2v_t^2\kappa k} e^{i\kappa z}. \tag{21}$$

Eq. (17) gives the transverse wave v_2 (for κ real) propagating with the velocity v_t, according to Eq. (16). Its polarization is normal to the plane of propagation (the plane determined by the vectors $\mathbf{k}$ and κ). This wave is usually known as the s-wave in the theory of electromagnetism (from the German word “senkrecht” which means “perpendicular”). From Eq. (16) we can see that the s-wave becomes singular for $\kappa = 0$, *i.e.* for in-plane propagation, as expected for waves generated by such localized forces.

We pass now to the system of coupled equations (18) and (19), and the relationship given by Eq. (21). We introduce u_3 from Eq. (21) into Eq. (18) and get

$$(1+q)v_1 = -\frac{iq\omega^2}{2v_t^2\kappa}\int_0 dz'\, v_1(z') e^{i\kappa|z-z'|} + \frac{iaq}{4v_t^2\kappa^2}(\kappa f_1 - kf_3)\frac{\partial}{\partial z}\int_0 dz'\, e^{i\kappa z'} e^{i\kappa|z-z'|} + \frac{1}{2}\left[\frac{iaf_1}{v_t^2\kappa} + qv_1(0)\right] e^{i\kappa z}. \tag{22}$$

This equation can easily be solved by taking the second derivative with respect to z and using the non-invertibility equation (20). We get

$$\frac{\partial^2 v_1}{\partial z^2} + \kappa'^2 v_1 = -\frac{iaq}{2v_t^2(1+q)}(\kappa f_1 - kf_3) e^{i\kappa z}, \tag{23}$$

where

$$\kappa' = \sqrt{\frac{\omega^2}{v_l^2} - k^2}. \tag{24}$$

For a longitudinal force $\kappa f_1 - kf_3 = 0$ we obtain from Eq. (23) longitudinal waves propagating with wavevector κ' (for κ' real) and with the velocity v_l. For a general force, Eq. (23) has the particular solution

$$v_1 = \frac{ia}{2\omega^2}(\kappa f_1 - kf_3) e^{i\kappa z} \tag{25}$$

and

$$u_3 = -\frac{iak}{2\omega^2\kappa}(\kappa f_1 - kf_3) e^{i\kappa z}. \tag{26}$$

We can see that v_1 and u_3 given above correspond to a transverse wave, $kv_1 + \kappa u_3 = 0$, whose polarization lies in the plane of propagation. This is called the p-wave, where p stands for “parallel”. We can see also that v_1 is a continuous function, while u_3 may exhibit the same singularity as v_2 does for $\kappa = 0$.

Since we are mainly interested in the effects produced by a localized force acting on the surface we restrict ourselves to the particular solution to the displacement as given here by Eqs. (17), (25) and (26) (“forced waves”). For a point-like force acting on the surface for instance this particular solution gives the force exerted on the surface, which is compensated by the force brought about by the solution of the homogeneous equation (“free waves”), such as to get a vanishing total force, in accordance with the “free surface” condition, as incorporated in Eq. (1).

3. Surface displacement

The displacement of the surface $z = 0$ can be computed by using the inverse Fourier transforms of $v_{1,2}(\mathbf{K})$ and $u_3(\mathbf{K})$ given by Eqs. (17), (25) and (26), where $\mathbf{K} = (\mathbf{k}, \kappa)$. As usually, we leave aside for the moment the argument ω in these expressions. It is worth noting that $\kappa = \sqrt{\omega^2/v_t^2 - k^2}$ is not an independent variable. First, we consider a δ-type force localized on the surface, $\mathbf{f}(\mathbf{R}) = ab^2\mathbf{f}\delta(\mathbf{r})\delta(z)$, where $\mathbf{g}$ is a constant vector and b is a characteristic localization length on the surface. This is one of Lamb's problems [14], with the difference that the force has here a general orientation. The Fourier components $\mathbf{f}(\mathbf{K}) = ab^2\mathbf{f}$ of this force do not depend on $\mathbf{K}$ (but they may have an ω-dependence). We choose an in-plane reference frame with one axis oriented along the in-plane radius $\mathbf{r}$ (radial axis r) and another perpendicular to the former (tangential axis t). We denote by α the angle between the force vector $\mathbf{f}$ and radius $\mathbf{r}$. Then, the force vector can be written as $\mathbf{f} = (f\cos\alpha, f\sin\alpha, f_v)$, where f denotes the in-plane (horizontal) force and f_v denotes the vertical force. Similarly, we denote by φ the angle between the in-plane wavevector $\mathbf{k}$ and radius $\mathbf{r}$, such that $\mathbf{k} = k(\cos\varphi, \sin\varphi)$ and $\mathbf{k}_\perp = k(-\sin\varphi, \cos\varphi)$. Then, the force projections $f_{1,2,3}$ entering Eqs. (17), (25) and (26) can be written as

$$f_1 = b^2 f\cos(\alpha-\varphi), \qquad f_2 = b^2 f\sin(\alpha-\varphi),$$
$$f_3 = b^2 f_v. \tag{27}$$

It is worth noting that on changing $\mathbf{k} \to -\mathbf{k}$, *i.e.* $\varphi \to \pi+\varphi$, the quantities $f_{1,2}$ change sign, as they should do; similarly, f_3, being the projection of the force along the wavevector component κ, must change sign under the reversal of the direction of this component, $\kappa \to -\kappa$. Making use of the above notations and of $\mathbf{v}(\mathbf{K}) = v_1\mathbf{k}/k + v_2\mathbf{k}_\perp/k$ we can obtain immediately the radial and tangential components of the displacement, $v_r(\mathbf{K})$ and $v_t(\mathbf{K})$, respectively. However, it is worth noting that for a real displacement the Fourier transforms must satisfy the symmetry relationship $\mathbf{v}^*(-\mathbf{K}) = \mathbf{v}(\mathbf{K})$, and, similarly, $\mathbf{u}_3^*(-\mathbf{K}) = \mathbf{u}_3(\mathbf{K})$. The displacement, as function of position and time, should be a real-valued function. If we look at the Fourier transform given by Eq. (14), and solutions given by Eqs. (17), (25) and (26), and take therein the complex conjugation, we can see that we get a real-valued function only by reversing the sign of the wavevector K and of frequency ω. This leads to the symmetry property discussed here. Taking into account the change of sign of the force components $f_{1,2,3}$ under this operation, we can see that the quantities κ and k in Eqs. (17), (25) and (26) must bear the factor $sgn(\pi-\varphi)$. We can write down the Fourier components of the displacement as

$$\begin{aligned}
v_r(\mathbf{k}) &= \left[\frac{iab^2}{2\omega^2}\kappa f\cos(\alpha-\varphi)\cos\varphi \right.\\
&\quad \left. - \frac{iab^2 f}{2v_t^2\kappa}\sin(\alpha-\varphi)\sin\varphi\right] sgn(\pi-\varphi)\\
&\quad - \frac{iab^2}{2\omega^2} k f_v\cos\varphi,\\
v_t(\mathbf{k}) &= \left[\frac{iab^2}{2\omega^2}\kappa f\cos(\alpha-\varphi)\sin\varphi \right.\\
&\quad \left. + \frac{iab^2 f}{2v_t^2\kappa}\sin(\alpha-\varphi)\cos\varphi\right] sgn(\pi-\varphi)\\
&\quad - \frac{iab^2}{2\omega^2} k f_v\sin\varphi,\\
u_3(\mathbf{k}) &= -\frac{iab^2}{2\omega^2} k f\cos(\alpha-\varphi) + \frac{iab^2k^2}{2\omega^2\kappa} f_v\, sgn(\pi-\varphi)
\end{aligned} \tag{28}$$

(for $z = 0$). Now we can take the inverse Fourier transforms of these quantities. It is easy to see that the integrals over angle φ which contain factors $\sin^2\varphi$ and $\cos^2\varphi$ are vanishing. For the radial component we are left with

$$\begin{aligned}
v_r(\mathbf{r}) &= -\frac{iab^2 f}{2(2\pi)^2\omega^2}\sin\alpha \int_0^{\omega/v_t} dk\frac{k^3}{\kappa}\int_0^{2\pi} d\varphi\, sgn(\pi-\varphi)\\
&\quad \times \sin\varphi\cos\varphi e^{ikr\cos\varphi}\\
&\quad - \frac{iab^2 f_v}{2(2\pi)^2\omega^2}\int_0^{\omega/v_t} dk\, k^2 \int_0^{2\pi} d\varphi \cos\varphi e^{ikr\cos\varphi}.
\end{aligned} \tag{29}$$

The integrals in Eq. (29) can be performed straightforwardly, by making use of the properties of the Bessel functions [25,26]. We get

$$v_r(\mathbf{r}) = -\frac{ab^2}{4\pi v_t^2 r}(f\sin\alpha + f_v)\left[J_0\left(\frac{\omega r}{v_t}\right) - \frac{2v_t}{\omega r}J_1\left(\frac{\omega r}{v_t}\right)\right], \tag{30}$$

where $J_{0,1}$ are Bessel functions of the first kind and zeroth and, respectively, first order. In the limit $\omega r/v_t \gg 1$ we get

$$v_r(\mathbf{r}) \sim_{\omega r/v_t \gg 1} -\frac{ab^2}{\omega^{1/2}}(f\sin\alpha + f_v)\frac{1}{(2\pi v_t r)^{3/2}}\cos\left(\frac{\omega r}{v_t} - \frac{\pi}{4}\right). \tag{31}$$

We can see that the radial component of the surface displacement attains its maximum value along a direction perpendicular to the direction of the force ($\alpha = \pi/2$), as expected for a transverse wave generated by such a localized force. It has a characteristic oscillatory behaviour with the in-plane distance and goes like $r^{-3/2}$ for long distances. The temporal Fourier transform of the spectrum given by Eq. (31) for f and f_v independent of ω (related to Fresnel integrals) exhibits a characteristic oscillatory wave front of the form $\sim (r - v_t t)^{-1/2}$, as expected. Such qualitative characteristics of the solution to this problem are similar with those indicated long time ago by Lamb [14] (see also Ref. [12]).

Similar calculations can be done for the tangential component $v_t(\mathbf{r})$ and the vertical component $u_3(\mathbf{r})$. The result for $v_t(\mathbf{r})$ can be obtained from Eqs. (30) and (31) by putting formally $f_v = 0$ and replacing $\sin\alpha$ by $\cos\alpha$. The vertical component can be obtained from Eqs. (30) and (31) by replacing $\sin\alpha$ by 1 and putting $f_v = 0$.

Next, we consider an in-plane localized pressure $pb^2\delta(\mathbf{r})$. The Fourier components of the force are given by $f_1 = (-ib^2p/\rho)k$, $f_2 = f_3 = 0$ and the Fourier components of the displacement are

$$\begin{aligned}
v_1(\mathbf{k}) &= \frac{ab^2p}{2\rho\omega^2}\kappa k(\cos\varphi, \sin\varphi)\, sgn(\pi-\varphi),\\
u_3(\mathbf{k}) &= -\frac{ab^2p}{2\rho\omega^2}k^2.
\end{aligned} \tag{32}$$

The inverse Fourier transforms of these displacements give $v_r(\mathbf{r}) = 0$ and

$$\begin{aligned}
v_t(r) &= \frac{ab^2p\omega}{16\pi\rho v_t^3 r}\left[J_1\left(\frac{\omega r}{v_t}\right) + J_3\left(\frac{\omega r}{v_t}\right)\right],\\
u_3(r) &= -\frac{ab^2p\omega}{4\pi\rho v_t^3 r}\left[J_1\left(\frac{\omega r}{v_t}\right) - \frac{2v_t}{\omega r}J_2\left(\frac{\omega r}{v_t}\right)\right].
\end{aligned} \tag{33}$$

The leading term ($\sim r^{-3/2}$) in v_t is vanishing in the limit $\omega r/v_t \gg 1$, while u_3 behaves like

$$u_3(r) \sim_{\omega r/v_t \gg 1} -\frac{ab^2p}{\rho v_t}\frac{\omega^{1/2}}{(2\pi v_r r)^{3/2}}\cos\left(\frac{\omega r}{v_t} - \frac{3\pi}{4}\right). \tag{34}$$

The vertical component of the surface displacement has a wave front of the form $\sim (r - v_t t)^{-3/2}$.

Additional surface displacements (including the longitudinal one) occur from the contribution of the "free waves". They do not change the asymptotic r-dependence, but introduce an additional directional character.

4. Force localized beneath the surface

We consider a force

$$\mathbf{f}(\mathbf{R}, t) = a^3\mathbf{f}(t)\delta(\mathbf{R} - \mathbf{R}_0) \tag{35}$$

localized at depth d beneath the plane surface $z = 0$ of a semi-infinite elastic body extending to the region $z < 0$, such as $\mathbf{R}_0 = (0, 0, -d)$. The characteristic length a is much smaller than the relevant distances. The propagating spherical waves produced by this point-like force in an infinite body are well known [17]. We derive here the waves produced by such a source in a semi-infinite body. We use again a displacement field

$$\mathbf{u} = (\mathbf{v}, u_3)\theta(-z), \tag{36}$$

Fourier transforms of the form given by Eq. (14) and the reference frame defined by the in-plane vectors $\mathbf{k}$, $\mathbf{k}_\perp$ such that $v_1 = \mathbf{vk}/k$, $v_2 = \mathbf{vk}_\perp/k$, and similarly for $\mathbf{f}$. The force term in Eq. (9), which we denote by $\mathbf{F}$, can easily be evaluated. Its Fourier transform is given by

$$\mathbf{F} = -\frac{a^3\mathbf{f}}{2v_t^2\kappa}\sin\kappa|z+d|, \tag{37}$$

where $\kappa^2 = \omega^2/v_t^2 - k^2 > 0$ (we limit ourselves to the propagating waves). From Eq. (9) we get straightforwardly

$$v_2 = F_2 = -\frac{a^3 f_2}{2v_t^2\kappa}\sin\kappa|z+d| \tag{38}$$

and the set of coupled integral equations

$$\begin{aligned}
v_1 &= -\frac{iqk^2}{2\kappa}\int^0 dz'\, v_1(z')e^{i\kappa|z-z'|} \\
&\quad - \frac{qk}{2\kappa}\frac{\partial}{\partial z}\int^0 dz'\, u_3(z')e^{i\kappa|z-z'|} + F_1, \\
u_3 &= -\frac{qk}{2\kappa}\frac{\partial}{\partial z}\int^0 dz'\, v_1(z')e^{i\kappa|z-z'|} \\
&\quad + \frac{iq}{2\kappa}\frac{\partial^2}{\partial z^2}\int^0 dz'\, u_3(z')e^{i\kappa|z-z'|} + F_3.
\end{aligned} \tag{39}$$

These equations imply the relationship

$$u_3 = -\frac{i}{k}\frac{\partial v_1}{\partial z} - \frac{i}{k}\frac{\partial F_1}{\partial z} - F_3. \tag{40}$$

Introducing u_3 from this equation into the first equation (39) and performing the integrations by parts, we get a single integral equation

$$\begin{aligned}
(1+q)v_1 &= -\frac{iq\omega^2}{2v_t^2\kappa}\int^0 dz'\, v_1(z')e^{i\kappa|z-z'|} + \frac{q}{2}v_1(0)e^{-i\kappa z} \\
&\quad + (1-q)F_1 - \frac{iq\kappa}{2}\int^0 dz'\, F_1(z')e^{i\kappa|z-z'|} \\
&\quad + \frac{q}{2}F_1(0)e^{-i\kappa z} + \frac{qk}{2\kappa}\frac{\partial}{\partial z}\int^0 dz'\, F_3(z')e^{i\kappa|z-z'|}.
\end{aligned} \tag{41}$$

Taking the second derivative with respect to z in this equation we find

$$\frac{\partial^2 v_1}{\partial z^2} + \kappa'^2 v_1 = \frac{q}{1+q}\left(\kappa^2 F_1 + ik\frac{\partial F_3}{\partial z}\right), \tag{42}$$

where $\kappa'^2 = \omega^2/v_l^2 - k^2$. Now, it is easy to get the solution for v_1. It is given by

$$v_1 = \frac{a^3}{2\omega^2}\left[\kappa f_1 \sin\kappa|z+d| + ikf_3\, sgn(z+d)\cos\kappa(z+d)\right] \tag{43}$$

and, by Eqs. (37) and (40),

$$u_3 = \frac{a^3 k}{2\omega^2\kappa}\left[kf_3 \sin\kappa|z+d| + i\kappa f_1\, sgn(z+d)\cos\kappa(z+d)\right]. \tag{44}$$

We can see that all these solutions $v_{1,2}$, u_3 are stationary waves along the direction perpendicular to the surface, as generated by the stationary oscillating force given by Eq. (37). In addition, they are continuous functions for $\kappa \to 0$, though v_2 and u_3 may increase indefinitely, $v_2(\kappa \to 0), u_3(\kappa \to 0) \sim |z+d|$; this increase indicates the transition to the damped regime. It is also worth noting the discontinuity at $z = -d$.

5. Surface displacement caused by a force localized beneath the surface

We take the inverse in-plane spatial Fourier transforms of Eqs. (38), (43) and (44) for $z = 0$, using the same frame oriented along the radial, tangential and vertical directions. In this reference frame the force is given by $(f\cos\alpha, f\sin\alpha, f_3)$ and the in-plane wavevector is $k(\cos\varphi, \sin\varphi)$. The in-plane displacement in this reference frame is obtained by $\mathbf{v} = v_1\mathbf{k}/k + v_2\mathbf{k}_\perp/k$, where $\mathbf{k}_\perp = k(-\sin\varphi, \cos\varphi)$. The integrals with respect to angle φ in the Fourier transforms imply the Bessel functions $J_{0,1}$. The surface displacement can be written as

$$\begin{aligned}
v_r(\mathbf{r}) &= \frac{a^3 f}{4\pi\omega^2}\left(I_1 - \frac{1}{r}I_2\right)\cos\alpha - \frac{a^3 f}{4\pi v_t^2 r}I_3\cos\alpha - \frac{a^3 f_3}{4\pi\omega^2}I_4, \\
v_t(\mathbf{r}) &= \frac{a^3 f}{4\pi\omega^2 r}I_2\sin\alpha - \frac{a^3 f}{4\pi v_t^2}\left(I_5 - \frac{1}{r}I_3\right)\sin\alpha, \\
u_3(\mathbf{r}) &= \frac{a^3 f_3}{4\pi\omega^2}I_6 - \frac{a^3 f}{4\pi\omega^2}I_4\cos\alpha,
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
I_1 &= \int_0^{\omega/v_t} dk\,\kappa k\sin\kappa d\cdot J_0(kr), & I_2 &= \int_0^{\omega/v_t} dk\,\kappa\sin\kappa d\cdot J_1(kr), \\
I_3 &= \int_0^{\omega/v_t} dk\frac{1}{\kappa}\sin\kappa d\cdot J_1(kr), & I_4 &= \int_0^{\omega/v_t} dk\,k^2\cos\kappa d\cdot J_1(kr), \\
I_5 &= \int_0^{\omega/v_t} dk\frac{k}{\kappa}\sin\kappa d\cdot J_0(kr), & I_6 &= \int_0^{\omega/v_t} dk\frac{k^3}{\kappa}\sin\kappa d\cdot J_0(kr).
\end{aligned} \tag{46}$$

We estimate these integrals in the fast oscillating limit $\omega r/v_t$, $\omega d/v_t \gg 1$. In this case, the main contribution comes from $k \sim 0$ and extends over a range $\Delta k \sim 1/r$ for $r \gg d$ or $\Delta k \sim 1/d$ for $d \gg r$. The leading contributions for $r \gg d$ are given by

$$\begin{aligned}
v_r(\mathbf{r}) &\sim \frac{a^3 f}{\omega v_t r^2}\cos\alpha, \qquad v_t(\mathbf{r}) \sim \frac{a^3 f}{\omega v_t r^2}\sin\alpha, \\
u_3(\mathbf{r}) &\sim \frac{a^3 f}{\omega^2 r^3}\cos\alpha,
\end{aligned} \tag{47}$$

where oscillating factors of the form $\sin\omega d/v_t, \cos\omega d/v_t$ are left aside. We can see the directional character of the surface displacement (through angle α) and the vertical component (u_3) which is much smaller (by a factor $\omega r/v_t$) than the horizontal components. It is also worth noting that the leading contribution to the vertical displacement is caused by the in-plane force f, and, in general, the vertical component of the force brings a smaller contribution.

Let us assume now a force derived from a localized pressure p. The force components are then given by $f_1 = ipk/\rho$, $f_2 = 0$ and $f_3 = (-ip\kappa/\rho)e^{i\kappa d}$. In computing the Fourier transforms of the surface displacement we must take care now of the symmetry relations $\mathbf{v}^*(-\mathbf{K}) = \mathbf{v}(\mathbf{K})$ and $\mathbf{u}_3^*(-\mathbf{K}) = \mathbf{u}_3(\mathbf{K})$. We get

$$v_r(r) = \frac{a^3 p}{4\pi\omega^2\rho}\int_0^{\omega/v_t} dk\,\kappa k^2\sin\kappa d(1-\cos\kappa d)J_1(kr),$$

$$v_t(r) = \frac{a^3 p}{2\pi^2\omega^2\rho r}\int_0^{\omega/v_t} dk\,\kappa k\cos^2\kappa d\sin(kr),$$

$$u_3(r) = \frac{a^3 p}{4\pi\omega^2\rho}\int_0^{\omega/v_t} dk\,k^3\left(\sin^2\kappa d + \cos\kappa d\right)J_0(kr). \tag{48}$$

In the same limit $\omega d/v_t$, $\omega r/v_t$, $r/d \gg 1$ the leading contributions to the above displacements are given by

$$v_r(r), v_t(r) \sim \frac{a^3 p}{\omega v_t\rho r^3}, \qquad u_3(r) \sim \frac{a^3 p}{\omega^2\rho r^4}. \tag{49}$$

We can see that the displacements produced by pressure fall off faster with distance than the corresponding displacements caused by a force (Eq. (47)).

6. Force exerted on the surface

We are interested now in the force exerted on the surface $z = 0$ by the elastic waves produced beneath the surface. As it is well known, the force exerted by a displacement field $\mathbf{u}$ per unit area of a surface with unit normal $\mathbf{n}$ is given (in our notations) by $\rho f_i^s = \sigma_{ik}n_k$, where $\sigma_{ik} = \lambda u_{ll}\delta_{ik} + 2\mu u_{ik}$ is the stress tensor and $u_{ik} = (1/2)(\partial u_i/\partial x_k + \partial u_k/\partial x_i)$ is the strain tensor. Using the reference frame defined by $\mathbf{k}$, $\mathbf{k}_\perp$ and $\boldsymbol{\kappa}$ we get

$$f_1^s(\mathbf{k},\omega) = \frac{a^3 v_t^2\kappa}{\omega^2}[\kappa f_1\cos\kappa d - ikf_3\sin\kappa d],$$
$$f_2^s(\mathbf{k},\omega) = -a^3 f_2\cos\kappa d,$$
$$f_3^s(\mathbf{k},\omega) = \frac{a^3 v_t^2 k}{\omega^2}[kf_3\cos\kappa d - i\kappa f_1\sin\kappa d]. \tag{50}$$

We note that the dilatation vanishes, $v_{11} + v_{22} + u_{33} = 0$ (this property holds also for the force localized on the body surface).

We compute the inverse Fourier transforms of these forces with respect to the wavevector $\mathbf{k}$ according to the procedure described above for the surface displacements. The asymptotic expressions ($\omega d/v_t, \omega r/v_t, r/d \gg 1$) are given by

$$f_r^s(\mathbf{r}) \sim \frac{a^3 f}{r^2}\cos\alpha, \qquad f_t^s(\mathbf{r}) \sim \frac{a^3 f}{r^2}\sin\alpha,$$
$$f_3^s(\mathbf{r}) \sim \frac{a^3 f v_t}{\omega r^3}\cos\alpha; \tag{51}$$

they are similar with the surface displacements given by Eq. (47), except for an additional factor ω.

Similarly, we can compute the force exerted on the surface by elastic waves produced by a point-like force localized on the surface. The results are similar with the corresponding displacements. For instance, the asymptotic expression for the radial component of such a force is given by

$$f_r^2(\mathbf{r}) \sim_{\omega r/v_t \gg 1} -\frac{ab^2}{4\pi r}(\omega/v_t r)^{1/2} f\cos\alpha\cos\left(\frac{\omega r}{v_t} - \frac{3\pi}{4}\right), \tag{52}$$

which is similar with Eq. (31). In the same manner, we can compute the force exerted on the surface by a localized pressure.

7. Conclusions

In conclusion, we may say that we have introduced here a new method of studying the propagation of the elastic waves in isotropic bodies, based on the Kirchhoff potentials for wave equation with sources, borrowed from the theory of electromagnetism. The novelty consists in viewing the compression term in the Navier–Stokes equation as a source term for the wave equation. The method implies coupled integral equations for the waves amplitudes, which we have solved. Making use of this method we have determined the waves produced in an elastic semi-infinite body by an external force localized either on the body surface or beneath the surface at some distance d. In the latter case the waves are stationary along the direction perpendicular to the body surface. We have also computed the surface displacements produced by these forces as well as the force exerted on the surface as caused by a force localized beneath. We have estimated these quantities in the fast oscillating regime ($\omega d/v_t, \omega r/v_t \gg 1$, where ω denotes the frequency and v_t is the velocity of the transverse waves) and for in-plane distances r much longer than the depth d. These quantities exhibit a characteristic decrease along the in-plane distance on the body surface and a characteristic oscillatory behaviour. We have limited ourselves to particular solutions produced by localized force-sources, as we were mainly interested in the effects produced by such forces in a semi-infinite elastic body. By making use of this method we have generalized one of Lamb's problem (force localized on the surface of the body) and obtained new results for a point-like force localized beneath the body surface. Various other results can be obtained by means of this method, for various other geometries and force distributions.

The present approach can be extended to determine the waves propagating in elastic bodies with special, finite geometry, either as eigenmodes or caused by some external forces (both localized or extended). More interesting, we can extend the present approach to include the effect of various inhomogeneities placed in elastic bodies, as caused by local variations in the body density or elastic constants.

This latter point deserves a brief comment here. Indeed, suppose for instance that a small irregularity $\delta\rho$ occurs in the density ρ in Eq. (1). The corresponding term $\delta\rho\ddot{u}$ can be transferred into the *rhs* of Eq. (8) and can be treated as a "wave source". It will bring an additional contribution to the "potential" given by Eq. (9), which allows one to compute the changes brought by this inhomogeneity both in the eigenmodes and the elastic response of the body. Obviously, a similar treatment can be applied to inhomogeneities occurring in the elastic coefficients λ and μ, both on the body surface or in the bulk. Some results in this direction will be reported in a forthcoming publication.

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