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NOTE

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On unphysical terms in the elastic Hertz potentials

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Abstract Unphysical terms in the elastic Hertz potentials are identified, and a regularization procedure is devised for removing them. The solutions of the equation of elastic motion are given for tensorial forces (seismic moment forces) and vectorial forces (Stokes problem) concentrated in both space and time.

Although the investigation of the propagation of elastic waves in various complex media and under various circumstances enjoys a continuous interest [1–10], the interesting method of the elastic Hertz potentials receives comparatively less attention. A particular problem arising in using these potentials is presented in this Note.

We consider the equation for the elastic motion [11]

$$\ddot{\mathbf{u}} - c_t^2 \Delta \mathbf{u} - (c_l^2 - c_t^2)\mathrm{grad} \cdot \mathrm{div}\mathbf{u} = \mathbf{F}, \tag{1}$$

where $\mathbf{u}$ is the displacement, $c_{l,t}$ are the velocities of the (longitudinal and transverse) elastic waves and $\mathbf{F}$ is the force (per unit mass); we consider a force concentrated (localized) in both space and time, given by [12]

$$F_i = m_{ij} T \delta(t) \partial_j \delta(\mathbf{R}), \tag{2}$$

where T is the short duration of the time impulse $\delta(t)$ and m_{ij} is the tensor of the seismic moment. We follow the standard procedure for introducing the Hertz potentials for the solution of this equation [13]. To this end, we introduce the notation $f_i = -(1/4\pi) m_{ij} T \partial_j$ and write the equation as

$$\ddot{\mathbf{u}} - c_t^2 \Delta \mathbf{u} - (c_l^2 - c_t^2)\mathrm{grad} \cdot \mathrm{div}\mathbf{u} = -4\pi \delta(t) \mathbf{f} \delta(\mathbf{R}); \tag{3}$$

then, we write $\delta(\mathbf{R}) = -(1/4\pi)\Delta \frac{1}{R}$; the equation becomes

$$\ddot{\mathbf{u}} - c_t^2 \Delta \mathbf{u} - (c_l^2 - c_t^2)\mathrm{grad} \cdot \mathrm{div}\mathbf{u} = \delta(t) \Delta \left(\mathbf{f} \frac{1}{R}\right); \tag{4}$$

further on, we use $\Delta\left(\mathbf{f}\frac{1}{R}\right) = \mathrm{grad} \cdot \mathrm{div}\left(\mathbf{f}\frac{1}{R}\right) - \mathrm{curl} \cdot \mathrm{curl}\left(\mathbf{f}\frac{1}{R}\right)$ and get

$$\ddot{\mathbf{u}} - \mathrm{grad} \cdot \mathrm{div}\left[c_l^2 \mathbf{u} + \delta(t) \mathbf{f} \frac{1}{R}\right] + \mathrm{curl} \cdot \mathrm{curl}\left[c_t^2 \mathbf{u} + \delta(t) \mathbf{f} \frac{1}{R}\right] = 0; \tag{5}$$

we can see that $\mathbf{u}$ can be written as

$$\mathbf{u} = \mathrm{grad} \cdot \mathrm{div}\mathbf{B} + \mathrm{curl} \cdot \mathrm{curl}\mathbf{C} = -\Delta \mathbf{C} + \mathrm{grad} \cdot \mathrm{div}(\mathbf{B} + \mathbf{C}), \tag{6}$$

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where

$$\ddot{\mathbf{B}} - c_l^2 \Delta \mathbf{B} = \delta(t)\mathbf{f}\frac{1}{R}, \quad \ddot{\mathbf{C}} - c_t^2 \Delta \mathbf{C} = -\delta(t)\mathbf{f}\frac{1}{R}; \tag{7}$$

the vectors **B** and **C** are known in Electromagnetism as the Hertz vectors (potentials) [14–16].

We are led to study the potential equation

$$\ddot{\Psi} - c^2 \Delta \Psi = \delta(t)\frac{1}{R}, \tag{8}$$

where the vectors **B**, **C** are given by $\mathbf{f}\Psi$; c is a generic notation for $c_{l,t}$. For the tensorial force given in Eq. (2), we get the potentials **B** and **C** by applying the operator $f_i = -(1/4\pi)m_{ij}T\partial_j$ to the scalar potential Ψ; from Eq. (6) the displacement is given by

$$u_i = -\frac{T}{4\pi}m_{ij}\partial_j \Delta \Psi_t - \frac{T}{4\pi}m_{jk}\partial_i\partial_j\partial_k(\Psi_l - \Psi_t), \tag{9}$$

where $\Psi_{l,t}$ correspond to $c_{l,t}$ in Eq. (8). For a vectorial force $\mathbf{F} = \mathbf{f}T\delta(t)\delta(\mathbf{R})$ (Stokes problem [17]), we get the potentials **B** and **C** by applying the constant vector $-(1/4\pi)T\mathbf{f}$ to Ψ; from Eq. (6) the displacement is given by

$$\mathbf{u} = -\frac{T}{4\pi}\mathbf{f}\Delta\Psi_t - \frac{T}{4\pi}\mathrm{grad}(\mathbf{f}\,\mathrm{grad})(\Psi_l - \Psi_t). \tag{10}$$

The solution of Eq. (8) is of the form $\Psi = \chi(R, t)/R$, where

$$\ddot{\chi} - c^2\chi'' = \delta(t); \tag{11}$$

we get

$$\Psi = \frac{t\theta(t)}{R} + \theta(ct - R)\frac{\chi(R - ct)}{R}, \tag{12}$$

where χ is an arbitrary function and the factor $\theta(ct - R)$ is introduced to satisfy the natural boundary condition $\Psi = 0$ for $t < 0$ (causality condition). The function χ is determined by imposing the boundary condition for $R \to 0$. It is natural to assume that the time dependence of Ψ is much slower than its spatial dependence, such that $\Psi \to 0$ for $R \to 0$ and

$$\frac{t}{R} + \frac{\chi(-ct)}{R} = 0 \tag{13}$$

from Eq. (12). We can see that this condition amounts to assuming that the focal perturbation occurs with a slower velocity than the wave velocity c (it follows that $l < cT$, where l is the localization length of the focus, i.e., the localization length of the function $\delta(\mathbf{R})$). It follows from Eq. (13) that $\chi(x) = x/c$ as the leading term; we get the solution

$$\begin{aligned}\Psi &= \frac{t}{R} + \frac{1}{c}\left(1 - \frac{ct}{R}\right)\theta(ct - R) + \mathrm{const.}\\ &= \frac{1}{c}\left[\theta(ct - R) + \frac{ct}{R}\theta(R - ct)\right] + \mathrm{const.},\end{aligned} \tag{14}$$

where const. $= -1/c$; up to this constant, this is precisely the (retarded) Kirchhoff solution

$$\Psi = \frac{1}{4\pi c^2}\int d\mathbf{R}'\frac{\delta(t - \left|\mathbf{R} - \mathbf{R}'\right|/c)}{\left|\mathbf{R} - \mathbf{R}'\right| R'} = \frac{1}{4\pi c^2}\int d\mathbf{r}\frac{\delta(t - r/c)}{r\left|\mathbf{R} - \mathbf{r}\right|}. \tag{15}$$

Indeed, the integral in Eq. (15) gives the θ-functions in Eq. (14).

The potential Ψ given by Eq. (14) satisfies the homogeneous equation $\ddot{\Psi} - c^2\Delta\Psi = 0$, except for $ct = R$, where the solution is not determined (the functions θ are not determined for $R = ct$). It follows that we should disregard contributions to Ψ for $R \neq ct$ and determine the solution in the vicinity of $R = ct$ by other means; this is a regularization (calibration) procedure. It is known that the potentials have not a direct physical relevance; in particular, for $m_{ij} = m\delta_{ij}$ the solution of Eq. (1) is $\mathbf{u} = \mathrm{grad}\Phi$, where

$$\Phi = \frac{Tm}{4\pi c_l^2}\int d\mathbf{R}'\frac{\delta(t - \left|\mathbf{R} - \mathbf{R}'\right|/c_l)}{\left|\mathbf{R} - \mathbf{R}'\right|}\delta(\mathbf{R}') = \frac{Tm}{4\pi c_l}\frac{\delta(R - c_l t)}{R}, \tag{16}$$

while Eqs. (9) and (14) give $\mathbf{u} = -\frac{Tm}{4\pi}\mathrm{grad}\Delta\Psi_l$, i.e., $\Phi = -\frac{Tm}{4\pi}\Delta\Psi_l$,

$$\Phi = -\frac{Tm}{4\pi}\Delta\Psi_l = \frac{Tm}{2\pi c_l}\frac{\delta(R-c_l t)}{R}; \tag{17}$$

we can see that these two expressions given by Eqs. (16) and (17) differ by a factor 1/2.

Since second-order derivatives are relevant (Eqs. (9), (10)), we apply the calibration procedure to expressions like

$$\begin{aligned} c\partial_i\partial_j\Psi = &-\frac{\delta_{ij}}{R}(1-ct/R)\delta + \frac{x_i x_j}{R^3}(1-3ct/R)\delta \\ &-\frac{x_i x_j}{R^2}(1-ct/R)\delta' - \frac{ct\delta_{ij}}{R^3}\theta + \frac{3ct x_i x_j}{R^5}\theta \end{aligned} \tag{18}$$

(where we omit the argument $R-ct$); hence,

$$c\Delta\Psi = -\frac{2}{R}\delta - (1-ct/R)\delta'. \tag{19}$$

If we apply the Laplacian to Eq. (8), we get

$$\frac{\partial^2}{\partial t^2}\Delta\Psi - c^2\Delta(\Delta\boldsymbol{\Psi}) = -4\pi\delta(t)\delta(\mathbf{R}), \tag{20}$$

whence, with the Kirchhoff solution (Green function),

$$\Delta\Psi = -\frac{\delta(R-ct)}{cR}. \tag{21}$$

Comparing Eq. (19) against (21), we can see that the δ'-contributions should be neglected in Eq. (18), as well as the θ-functions, and a factor 1/2 should be inserted; we get the calibrated expression

$$c\partial_i\partial_j\Psi = -\frac{\delta_{ij}}{2R}(1-ct/R)\delta + \frac{x_i x_j}{2R^3}(1-3ct/R)\delta; \tag{22}$$

by construction, it is unique. Making use of Eqs. (9) and (22), we get immediately the displacement $\mathbf{u} = \mathbf{u}^{nf} + \mathbf{u}^{ff}$, where the near-field displacement is

$$\begin{aligned} u_i^{nf} = &-\frac{Tm_{ij}x_j}{4\pi c_t R^3}\delta(R-c_t t) \\ &+\frac{T}{8\pi R^3}\left(m_{jj}x_i + 4m_{ij}x_j - \frac{9m_{jk}x_i x_j x_k}{R^2}\right)\left[\frac{1}{c_l}\delta(R-c_l t) - \frac{1}{c_t}\delta(R-c_t t)\right] \end{aligned} \tag{23}$$

and the far-field displacement is

$$u_i^{ff} = \frac{Tm_{ij}x_j}{4\pi c_t R^2}\delta'(R-c_t t) + \frac{Tm_{jk}x_i x_j x_k}{4\pi R^4}\left[\frac{1}{c_l}\delta'(R-c_l t) - \frac{1}{c_t}\delta'(R-c_t t)\right]. \tag{24}$$

These are spherical-shell waves. Similarly, using Eqs. (10) and (22) we get the solution

$$\mathbf{u} = \frac{T\mathbf{f}}{4\pi c_t R}\delta(R-c_t t) + \frac{T\mathbf{R}(\mathbf{R}\mathbf{f})}{4\pi R^3}\left[\frac{1}{c_l}\delta(R-c_l t) - \frac{1}{c_t}\delta(R-c_t t)\right] \tag{25}$$

for the Stokes problem.

In conclusion, we may say that spurious (unphysical) contributions may appear in the Hertz potentials of the equation of the elastic waves with forces concentrated in both time and space; these terms have the appearance of static or quasi-static contributions which are solutions of the homogeneous equation (θ-functions). We cannot simply dismiss them, because their discontinuous nature (for $R = c_{l,t}t$) contributes to the genuine solution, which is singular. Consequently, a regularization (calibration) procedure is needed, which should recover the known solution for isotropic force sources. This regularization procedure is worked out in this paper; by construction, it is unique.

The investigation reported in this Note was motivated by solving the equation of the elastic waves with point tensorial forces concentrated in time (seismic moment force), and the related Stokes problem, by direct integration of the equations of the elastic Hertz potentials. It should be emphasized that the tensorial force is a generalization of the Stokes problem. Both problems are relevant for the seismic waves. Although unphysical terms are known to occur in potentials, the occurrence of spurious terms rather came as a surprise. Using the regularization (calibration) procedure, the solutions are derived here for both the more general point tensorial forces and the vectorial forces (Stokes problem) concentrated in time. The identification of the unphysical terms, the regularization procedure for removing them and the solutions for the concentrated forces as given in this Note may be viewed as elements of novelty.

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