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On the vertex function method in the Tomonaga–Luttinger model

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Abstract. The Ward identities relating the three-point vertex functions to the single-particle Green functions as well as the generalised Ward identities between the four- and three-point vertex functions are derived analytically for the Tomonaga–Luttinger model by using explicitly the boson algebra of the particle–hole operators.

1. Introduction

It was recognised long ago that the weakly interacting fermions in one dimension can be described in terms of the particle–hole excitations which are present in the neighbourhood of the two Fermi points. Tomonaga (1950) showed that the operators corresponding to these pair excitations satisfy boson-like commutation relations. The free-fermion energy levels near the Fermi points may approximately be taken as linear in the wavevector and this linearised kinetic Hamiltonian was shown to be equivalent to a bilinear form in the aforementioned boson operators. As the two-particle interaction has the same structure it follows that the whole Hamiltonian of the system is bilinear in boson operators and its diagonalisation is a straightforward matter. A field-theoretical version of this model was given by Luttinger (1963) who introduced two types of fermions with linear energy levels and extended the allowable fermion states in the momentum space to $-\infty$ for the first type of fermions and to $+\infty$ for the second one. It was shown (Mattis and Lieb 1965) that, due to this infinite filling of the Fermi sea, the Fourier components of the fermion-density operators satisfy boson-like commutation relations. This is why the Tomonaga–Luttinger model was originally formulated in terms of these boson operators. The boson algebra was fully exploited when Luther and Peschel (1974) and Mattis (1974) introduced a boson representation for the fermion fields in one dimension. Remarkable progress was achieved in studying the Tomonaga–Luttinger model by means of the bosonisation technique (see, for example, Sólyom 1979, Haldane 1979, 1981).

An alternative way for treating the Tomonaga–Luttinger model was given by Dzyaloshinsky and Larkin (1973). Their method is based upon the diagrammatic analysis of the relevant quantities appearing in the theoretical-perturbation approach, in particular the three-point vertex function. The two main features of the Tomonaga–Luttinger model, namely (i) the conservation of the particle number for the fermions of each type and (ii) the linear energy levels, were used by these authors for deriving diagrammatically the Ward identities which relate the three-point vertex functions to the single-particle Green functions. These Ward identities were employed to get the single-particle Green functions and the effective polarisations of the system. The method given by Dzyaloshinsky and Larkin (1973) was extended by Sólyom (1978) who derived diagrammatically a set of generalised Ward identities which relate the four-point to the three-point vertex functions. The method may also be extended to the higher-order vertex functions. The generalised Ward identities are very useful for calculating the response of the system to an external field. Obviously the diagrammatic method does not make use explicitly of the boson-like commutation relations of the particle–hole pair operators. It seems, at first sight, that there are two unrelated ways of treating the Tomonaga–Luttinger model: bosonisation technique and diagrammatic analysis. However the attempts of formalising the diagrammatic method (Fogedby 1976, Bohr 1981) revealed, as might be expected, a close relationship between the diagrammatic analysis and the associated boson fields. Though interesting enough, this rather abstruse relationship has not been fully investigated so far.

The aim of the present paper is to derive analytically the Ward identities for the Tomonaga–Luttinger model by making use explicitly of the properties of the boson operators. A basic set of equations of motion is obtained for the density correlation functions. These equations incorporate the essential features of the model and they are of great use in studying the equations of motion of more complicated three- and four-point vertex functions. The Ward identities between the three-point vertex functions and the single-particle Green functions are obtained as well as the generalised Ward identities relating the four- to three-point vertex functions. The equation-of-motion method employed here was suggested by Everts and Schulz (1974) who used it to investigate the spectral properties of the single-particle Green functions. The same method was used (Apostol 1980, Apostol and Bârsan 1981) for deriving the Ward identities between the three-point vertex functions and the single-particle Green functions in an approximate way that does not emphasise the connection with the boson features of the problem.

2. Polarisations and Ward identities

The Tomonaga–Luttinger model is described by the Hamiltonian $H = H_0 + H_1$,

$$H_0 = \sum_{s,p>0} p a^+_{1ps} a_{1ps} + \sum_{s,p\leqslant 0} p(a^+_{1ps}a_{1ps} - 1) - \sum_{s,p<0} p a^+_{2ps}a_{2ps} - \sum_{s,p\geqslant 0} p(a^+_{2ps}a_{2ps} - 1),$$

$$\begin{aligned} H_1 = g_{4\parallel} \sum_{s,k>0} &[\rho^+_{1s}(-k)\rho_{1s}(-k) + \rho^+_{2s}(k)\rho_{2s}(k)] \\ &+ g_{4\perp} \sum_{s,k>0} [\rho^+_{1s}(-k)\rho_{1-s}(-k) + \rho^+_{2s}(k)\rho_{2-s}(k)] \\ &+ g_{2\parallel} \sum_{s,k>0} [\rho_{1s}(-k)\rho_{2s}(k) + \rho^+_{1s}(-k)\rho^+_{2s}(k)] \\ &+ g_{2\perp} \sum_{s,k>0} [\rho_{1s}(-k)\rho_{2-s}(k) + \rho^+_{1s}(-k)\rho^+_{2-s}(k)], \end{aligned} \tag{1}$$

where $a_{jps}(a^+_{jps})$ are the destruction (creation) operators of the one-fermion states labelled by $j = 1, 2$ (type of fermions), wavevector p and the spin index $s = \pm 1$; the operators

$$\rho_{js}(-k) = \rho^+_{js}(k) = \sum_p a^+_{jps}a_{jp+ks} \qquad (k \neq 0)$$

are the Fourier components of the particle-density operators. The Fermi sea of this system is filled with $j = 1$ particles from $p = -\infty$ to $p = +k_F$ and with $j = 2$ particles from $p = -k_F$ to $p = +\infty$, k_F being the Fermi momentum. Due to this infinite filling of the Fermi sea, infinite constants have to be subtracted from the kinetic Hamiltonian H_0 in order to get finite values of the kinetic energy.

The Tomonaga model (Tomonaga 1950) assumes that the interparticle force is long-ranged (though not necessarily very weak) and causes momentum transfers much smaller than k_F. The fermions are forbidden to be scattered across the Fermi sea, from states with wavevectors $p \sim +k_F$ (or $p \sim -k_F$) to state with $p \sim k_F$ (or, respectively, $p \sim +k_F$). Consequently, the fermion states are practically separated into two species, those with wavevectors p near $+k_F$ (which correspond to $j = 1$ in the Tomonaga–Luttinger model) and those with p near $-k_F$ ($j = 2$). As the low excited states of the system can be built up by superposing particle–hole pairs in the neighbourhood of the $\pm k_F$ points a bandwidth cut-off k_0 is introduced in the model, much smaller than k_F, which restricts the single-particle states participating in the dynamics of the system within the range $2k_0$ around $\pm k_F$, $\pm k_F - k_0 < p < \pm k_F + k_0$. Under these assumptions Tomonaga (1950) showed that the Fourier components $\rho_{js}(-k)$ of the particle-density operators satisfy the boson-like commutation relations

$$[\rho_{js}(\mp k), \rho^+_{j's'}(\mp k')] = (2\pi)^{-1} k \delta_{jj'} \delta_{ss'} \delta_{kk'}, \tag{2}$$

where the upper (lower) sign corresponds to $j = 1(2)$. The system is confined to a box of length equal to unit so that $p, k = 2\pi n$ with n an integer. Tomonaga's original derivation does not include the spin but the extension is straightforward. The proof of (2) is based upon the evaluation of the matrix elements of the commutator between any two allowable states of the model. The boson-like commutation relations (2) represent a good approximation for the interacting fermion systems whose Fermi sea is not too strongly distorted by interaction (see, for example, Apostol 1981, Apostol *et al* 1981) and, in fact, it is equivalent to the random-phase approximation. Similar boson-like commutation relations are the starting point of the Sawada model Hamiltonian used in treating the three-dimensional electron gas (Sawada 1957, Brout and Carruthers 1963).

The Tomonaga–Luttinger model as defined above (equation (1)) consists of two types of fermions with energy levels $\pm p$ so that the Fermi sea is infinitely filled with particles and the system has no well-defined ground state. Therefore some operators might yield infinite values when acting upon the states of the system. Indeed, the direct evaluation of the commutator (2) leads, for example, to

$$[\rho_{1s}(-k), \rho^+_{1s}(-k')] = \sum_p a^+_{1ps} a_{1p+k-k's} - \sum_p a^+_{1p+k's} a_{1p+ks}$$

and one may be tempted to replace $p + k'$ in the second sum by p, whereby the result would be zero. However, this procedure is not allowed since the sums diverge, when acting upon the states of the system, and the result might be wrong. In order to deal consistently with these objects we have to perform the normal-ordering of the operators, as, practically, was done in the kinetic Hamiltonian H_0 (equation (1)). In fact this point was noticed by Mattis and Lieb (1965) who have shown that the solution given by Luttinger (1963) to this model is incorrect as he failed to work properly with these commutators. One of the ways of dealing with this problem was pointed out by Mattis and Lieb (1965). It consists of introducing a particle–hole representation, as is usual in the field-theoretical literature, by defining two new types of fermion operators

$$b_{ps}=\begin{cases}a_{1ps} & p\geqslant 0\\ a_{2ps} & p<0\end{cases}\qquad c_{ps}=\begin{cases}a^{+}_{2ps} & p\geqslant 0\\ a^{+}_{1ps} & p<0.\end{cases}$$

Again the original formulation does not include the spin label but the extension is immediate. Doing so, the infinitely filled Fermi sea of the system becomes equivalent to the ground state of the c operators which is completely empty and to the ground state of the b operators which is filled from $-k_{\mathrm{F}}$ to $+k_{\mathrm{F}}$. In the new representation the operators $\rho_{js}(\mp k)$ take the form $(k>0)$

$$\rho_{1s}(-k)=\sum_{p<-k}c_{ps}c^{+}_{p+ks}+\sum_{-k\leqslant p<0}c_{ps}b_{p+ks}+\sum_{p\geqslant 0}b^{+}_{ps}b_{p+ks}$$

$$\rho_{2s}(k)=\sum_{p<-k}b^{+}_{p+ks}b_{ps}+\sum_{-k\leqslant p<0}c_{p+ks}b_{ps}+\sum_{p\geqslant 0}c_{p+ks}c^{+}_{ps}.$$

These expressions can easily be normal-ordered and, doing so, it is a simple matter to get, for example,

$$[\rho_{1s}(-k),\rho^{+}_{1s}(-k)]=\sum_{p<-k}c^{+}_{p+ks}c_{p+ks}-\sum_{p<-k}c^{+}_{ps}c_{ps}+\sum_{-k\leqslant p<0}c_{ps}c^{+}_{ps}$$
$$-\sum_{-k\leqslant p<0}b^{+}_{p+ks}b_{p+ks}+\sum_{p\geqslant 0}b^{+}_{ps}b_{ps}-\sum_{p\geqslant 0}b^{+}_{p+ks}b_{p+ks}.$$

The sums written above have finite values on the states of the system so that we may safely shift the summation labels to obtain

$$[\rho_{1s}(-k),\rho^{+}_{1s}(-k)]=\sum_{-k\leqslant p<0}1=(2\pi)^{-1}k.$$

One can derive similarly all the remaining commutation relations (2). The boson-like commutation relations are the starting point of the bosonisation technique (Luther and Peschel 1974, Mattis 1974, Haldane 1979, 1981).

Let us define the density response functions (polarisations)

$$\Pi_{jj'\parallel}(k,t)=-\mathrm{i}\langle 0|T\rho_{js}(-k,t)\rho^{+}_{j's}(-k,0)|0\rangle,$$
$$\Pi_{jj'\perp}(k,t)=-\mathrm{i}\langle 0|T\rho_{j-s}(-k,t)\rho^{+}_{j's}(-k,0)|0\rangle, \qquad (3)$$

where $k>0$ and the Heisenberg picture is used. Owing to the obvious invariance of the Hamiltonian (1) to the reversal of spin the functions $\Pi_{jj'\parallel,\perp}(k,t)$ do not depend on the spin index s. The diagrams corresponding to these response functions are made of a succession of bubbles (see figure 1); a closer examination of these diagrams leads to the Dyson equations

$$\Pi_{jj'\parallel}(\boldsymbol{k})=\Pi^{*}_{j}(\boldsymbol{k})[\delta_{jj'}+v_{jj'\parallel}(\boldsymbol{k})\Pi^{*}_{j'}(\boldsymbol{k})],$$
$$\Pi_{jj'\perp}(\boldsymbol{k})=\Pi^{*}_{j}(\boldsymbol{k})v_{jj'\perp}(\boldsymbol{k})\Pi^{*}_{j'}(\boldsymbol{k}), \qquad (4)$$

where $\Pi^{*}_{j}(\boldsymbol{k})$ are the irreducible polarisations, $v_{jj'\parallel,\perp}(\boldsymbol{k})$ are the effective interactions and $\boldsymbol{k}$ stands for the pair (k,ω). The effective interactions appearing in (4) may also be analysed in terms of the polarisations; we get, for example, from the Dyson equations of the effective interactions (figure 1(c))

$$v_{jj\parallel}(\boldsymbol{k})=g_{4\parallel}+g_{4\parallel}\Pi^{*}_{j}(\boldsymbol{k})v_{jj\parallel}(\boldsymbol{k})+g_{4\perp}\Pi^{*}_{j}(\boldsymbol{k})v_{jj\perp}(\boldsymbol{k})+g_{2\parallel}\Pi^{*}_{\bar{j}}(\boldsymbol{k})v_{\bar{j}j\parallel}(\boldsymbol{k})+g_{2\perp}\Pi^{*}_{\bar{j}}(\boldsymbol{k})v_{\bar{j}j\perp}(\boldsymbol{k})$$

and multiplying by $\Pi_j^*(k)$ and making use of (4) (see also figure 1(*b*)) we obtain

$$v_{jj\|}(k)\Pi_j^*(k) = g_{4\|}\Pi_{jj\|}(k) + g_{4\perp}\Pi_{jj\perp}(k) + g_{2\|}\Pi_{\bar{j}j\|}(k) + g_{2\perp}\Pi_{\bar{j}j\perp}(k), \tag{5}$$

where $\bar{j} = 1(2)$ for $j = 2(1)$. Introducing (5) into (4) we get at once

$$\Pi_j^*(k) = \Pi_{jj\|}(k)[1 + g_{4\|}\Pi_{jj\|}(k) + g_{4\perp}\Pi_{jj\perp}(k) + g_{2\|}\Pi_{\bar{j}j\|}(k) + g_{2\perp}\Pi_{\bar{j}j\perp}(k)]^{-1}, \tag{6}$$

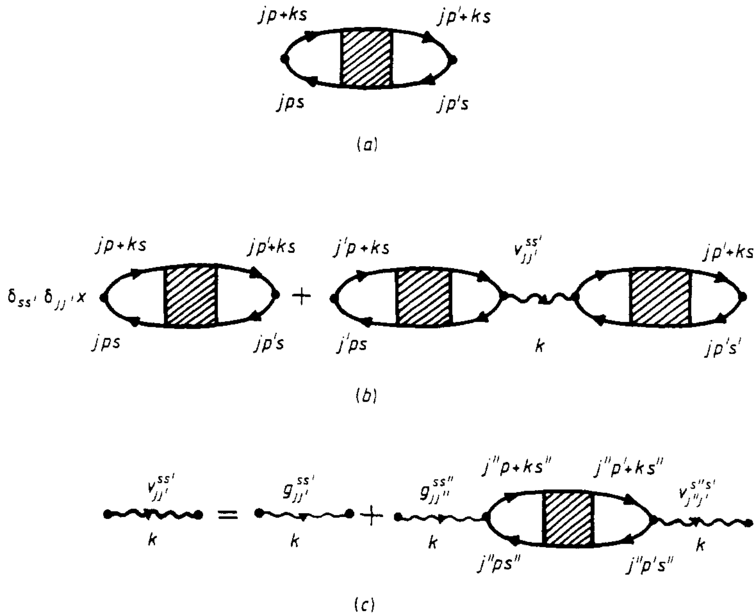


Figure 1. (*a*) Typical diagram of the irreducible polarisation. Full lines represent the particle propagator (dressed) and the hatched region includes all the insertions that preserve the irreducible character: the diagram cannot be split into two distinct parts by cutting a single interaction line. Summation over internal variables (p, p', etc) is assumed.

(*b*) Dyson equation for the (total) polarisations $\Pi_{jj'}^{ss'}(k) = \Pi_{jj'\|}(k)\delta_{ss'} + \Pi_{jj'\perp}(k)\delta_{s-s'}$. The wavy line represents the effective interaction $v_{jj'}^{ss'}(k) = v_{jj'\|}(k)\delta_{ss'} + v_{jj'\perp}(k)\delta_{s-s'}$.

(*c*) Dyson equation for the effective interaction $v_{jj'}^{ss'}(k)$. The light wavy line represents the bare interaction $g_{jj'}^{ss'} = g_{4\|}\delta_{jj'}\delta_{ss'} + g_{4\perp}\delta_{jj'}\delta_{s-s'} + g_{2\|}\delta_{\bar{j}j'}\delta_{ss'} + g_{2\perp}\delta_{\bar{j}j'}\delta_{s-s'}$ (where $\bar{j} = 1(2)$ for $j = 2(1)$). Summation over $j''s''$ is implied.

so that we can obtain the irreducible polarisations if we know the density response functions.

The density response functions can be obtained as follows. Taking the time derivative of $\Pi_{11\|}(k, t)$ and using the equal-time commutators and the equation of motion for $\rho_{1s}(-k, t)$ we get

$$\mathrm{i}(\partial/\partial t)\Pi_{11\|}(k, t) = (k/2\pi)\delta(t) + k(1 + g'_{4\|})\Pi_{11\|}(k, t) + kg'_{4\perp}\Pi_{11\perp}(k, t) + kg'_{2\|}\Pi_{21\|}(k, t) + kg'_{2\perp}\Pi_{21\perp}(k, t), \tag{7}$$

where the prime on the coupling constants means the factor $(2\pi)^{-1}$. On the other hand

$$\mathrm{i}\frac{\partial}{\partial t}\Pi_{11\|}(k, t) = (2\pi)^{-1}\int \mathrm{d}\omega\, \omega\Pi_{11\|}(k)\, \mathrm{e}^{-\mathrm{i}\omega t}, \tag{8}$$

so that comparing (7) with (8) we get

$$[(\omega/k) - (1 + g'_{4\|})]\Pi_{11\|}(\boldsymbol{k}) - g'_{4\perp}\Pi_{11\perp}(\boldsymbol{k}) - g'_{2\|}\Pi_{21\|}(\boldsymbol{k}) - g'_{2\perp}\Pi_{21\perp}(\boldsymbol{k}) = (2\pi)^{-1}. \tag{9a}$$

Similarly we obtain three more equations

$$-g'_{4\perp}\Pi_{11\|}(\boldsymbol{k}) + [(\omega/k) - (1 + g'_{4\|})]\Pi_{11\perp}(\boldsymbol{k}) - g'_{2\perp}\Pi_{21\|}(\boldsymbol{k}) - g'_{2\|}\Pi_{21\perp}(\boldsymbol{k}) = 0,$$

$$g'_{2\|}\Pi_{11\|}(\boldsymbol{k}) + g'_{2\perp}\Pi_{11\perp}(\boldsymbol{k}) + [(\omega/k) + (1 + g'_{4\|})]\Pi_{21\|}(\boldsymbol{k}) + g'_{4\perp}\Pi_{21\perp}(\boldsymbol{k}) = 0,$$

$$g'_{2\perp}\Pi_{11\|}(\boldsymbol{k}) + g'_{2\|}\Pi_{11\perp}(\boldsymbol{k}) + g'_{4\perp}\Pi_{21\|}(\boldsymbol{k}) + [(\omega/k) + (1 + g'_{4\|})]\Pi_{21\perp}(\boldsymbol{k}) = 0. \tag{9b}$$

The remaining polarisations $\Pi_{22\|,\perp}(\boldsymbol{k})$, $\Pi_{12\|,\perp}(\boldsymbol{k})$ obey that system of equations which is obtained from (9*a*, *b*) by changing k into $-k$. Introducing the quantities

$$\Pi^{\rho,\sigma}_{jj'}(\boldsymbol{k}) = \Pi_{jj'\|}(\boldsymbol{k}) \pm \Pi_{jj'\perp}(\boldsymbol{k}) \qquad u^0_{\rho,\sigma} = 1 + g'_{4\|} \pm g'_{4\perp} \qquad \beta_{\rho,\sigma} = g'_{2\|} \pm g'_{2\perp} \tag{10}$$

where the upper (lower) sign correspond to $\rho(\sigma)$, the system (9*a*, *b*) reads

$$(\omega/k - u^0_{\rho,\sigma})\Pi^{\rho,\sigma}_{11}(\boldsymbol{k}) - \beta_{\rho,\sigma}\Pi^{\rho,\sigma}_{21}(\boldsymbol{k}) = (2\pi)^{-1},$$

$$\beta_{\rho,\sigma}\Pi^{\rho,\sigma}_{11}(\boldsymbol{k}) + (\omega/k + u^0_{\rho,\sigma})\Pi^{\rho,\sigma}_{21}(\boldsymbol{k}) = 0 \tag{11}$$

whose solution is

$$\Pi^{\rho,\sigma}_{11}(k,\omega) = \Pi^{\rho,\sigma}_{22}(-k,\omega) = (2\pi)^{-1}k(\omega + u^0_{\rho,\sigma}k)(\omega^2 - u^2_{\rho,\sigma}k^2)^{-1},$$

$$\Pi^{\rho,\sigma}_{12}(k,\omega) = \Pi^{\rho,\sigma}_{21}(k,\omega) = -(2\pi)^{-1}k^2\beta_{\rho,\sigma}(\omega^2 - u^2_{\rho,\sigma}k^2)^{-1}, \tag{12}$$

where $u^2_{\rho,\sigma} = u^{02}_{\rho,\sigma} - \beta^2_{\rho,\sigma}$. One can see that the charge-density degrees of freedom are decoupled from the spin-density ones and the equations (12) are the well-known polarisations of the Tomonaga–Luttinger model (see, for example, Sólyom 1979). The system of equations (9*a*, *b*) is a direct consequence of the continuity equations of the $\rho_{js}(\mp k)$ operators. Using the results (12) we get from (6) the irreducible polarisations

$$\Pi^*_1(k,\omega) = \Pi^*_2(-k,\omega) = (2\pi)^{-1}k(\omega - k)^{-1} \tag{13}$$

which will be of great use further below.

We pass now to the Ward identities which relate the three-point vertex functions to the single-particle Green functions. In order to get them we define the three-point vertex functions

$$T_{jj'\|}(p,k;t_3 - t_2, t_1 - t_2) = \langle 0|T\rho_{js}(-k_1,t_1)a^+_{j'p+ks}(t_2)a_{j'ps}(t_3)|0\rangle$$

$$T_{jj'\perp}(p,k;t_3 - t_2, t_1 - t_2) = \langle 0|T\rho_{j-s}(-k,t_1)a^+_{j'p+ks}(t_2)a_{j'ps}(t_3)|0\rangle \tag{14}$$

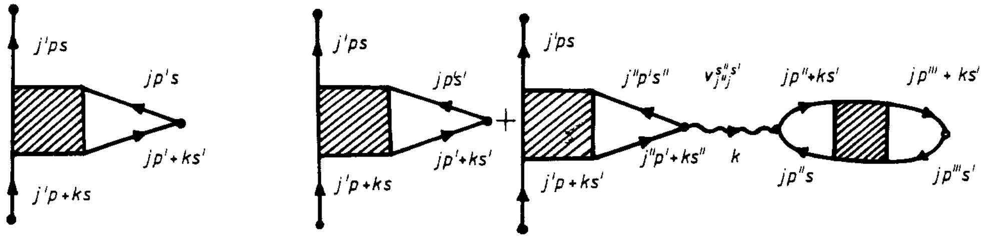
whose Fourier transforms are given by

$$T_{jj'\|,\perp}(p,k;t_3 - t_2, t_1 - t_2) = (2\pi)^{-2}\int \mathrm{d}\varepsilon\,\mathrm{d}\omega\, T_{jj'\|,\perp}(\boldsymbol{p},\boldsymbol{k}) \times \exp[-\mathrm{i}\varepsilon(t_3 - t_2)]\exp[-\mathrm{i}\omega(t_1 - t_2)]$$

$\boldsymbol{p}$ and $\boldsymbol{k}$ denoting the pairs (p, ε) and (k, ω), respectively. The diagrammatic representation of these vertex functions is very simple (see figure 2): their diagrams contain a particle line starting at t_2 with momentum $\boldsymbol{p} + \boldsymbol{k}$ and ending at t_3 with $\boldsymbol{p}$; an interaction line carrying momentum $\boldsymbol{k}$ is connected with this diagram at t_1. An examination of these three-legged diagrams reveals the existence of a subset of reducible diagrams, that is diagrams which can be split into two distinct parts by cutting a single interaction line.

Denoting the contribution of the irreducible diagrams by $T^{\text{ir}}_{jj'\|,\perp}(p,k)$ we can write down a set of Dyson equations

$$T_{jj'\|}(p,k) = T^{\text{ir}}_{jj'\|}(p,k) + \sum_{j''} T^{\text{ir}}_{j''j'\|}(p,k)v_{j''j\|}(k)\Pi^*_j(k)$$
$$+ \sum_{j''} T^{\text{ir}}_{j''j'\perp}(p,k)v_{j''j\perp}(k)\Pi^*_j(k),$$
$$T_{jj'\perp}(p,k) = T^{\text{ir}}_{jj'\perp}(p,k) + \sum_{j''} T^{\text{ir}}_{j''j'\|}(p,k)v_{j''j\perp}(k)\Pi^*_j(k)$$
$$+ \sum_{j''} T^{\text{ir}}_{j''j'\perp}(p,k)v_{j''j\|}(k)\Pi^*_j(k).$$



(a) (b)

Figure 2. (*a*) Diagrammatic representation of the irreducible three-point vertex function $T^{\text{ir}ss'}_{jj'}(p,k) = T^{\text{ir}}_{jj'\|}(p,k)\delta_{ss'} + T^{\text{ir}}_{jj'\perp}(p,k)\delta_{s-s'}$.

(*b*) Dyson equation for the (total) three-point vertex function $T^{ss'}_{jj'}(p,k)$. Summation over $j''s''$ is implied.

The first of these equations can be written as

$$T_{jj'\|}(p,k) = T^{\text{ir}}_{jj'\|}(p,k)[1 + v_{jj\|}(k)\Pi^*_j(k)] + T^{\text{ir}}_{jj'\perp}(p,k)v_{jj\perp}(k)\Pi^*_j(k)$$
$$+ T^{\text{ir}}_{\bar{j}j'\|}(p,k)v_{\bar{j}j\|}(k)\Pi^*_j(k) + T^{\text{ir}}_{\bar{j}j'\perp}(p,k)v_{\bar{j}j\perp}(k)\Pi^*_j(k),$$

or, by making use of (4),

$$T_{jj'\|}(p,k) = (\Pi^*_j(k))^{-1}[\Pi_{jj\|}(k)T^{\text{ir}}_{jj'\|}(p,k) + \Pi_{jj\perp}(k)T^{\text{ir}}_{jj'\perp}(p,k)]$$
$$+ (\Pi^*(k))^{-1}[\Pi_{\bar{j}j\|}(k)T^{\text{ir}}_{\bar{j}j'\|}(p,k) + \Pi_{\bar{j}j\perp}(k)T^{\text{ir}}_{\bar{j}j'\perp}(p,k)] \qquad (15a)$$

where $\bar{j} = 1(2)$ for $j = 2(1)$. Similarly the second Dyson equation for $T_{jj'\perp}(p,k)$ becomes

$$T_{jj'\perp}(p,k) = (\Pi^*_j(k))^{-1}[\Pi_{jj\|}(k)T^{\text{ir}}_{jj'\perp}(p,k) + \Pi_{jj\perp}(k)T^{\text{ir}}_{jj'\|}(p,k)]$$
$$+ (\Pi^*_j(k))^{-1}[\Pi_{\bar{j}j\|}(k)T^{\text{ir}}_{\bar{j}j'\perp}(p,k) + \Pi_{\bar{j}j\perp}(k)T^{\text{ir}}_{\bar{j}j'\|}(p,k). \qquad (15b)$$

Taking the derivative of $T_{11\|}(p,k;t_3-t_2,t_1-t_2)$ given by (14) with respect to the first time argument t_1, Fourier transforming and using the equal-time commutators we get

$$[(\omega/k) - (1+g'_{4\|})]T_{11\|}(p,k) - g'_{4\perp}(p,k)\,T_{11\perp}(p,k) - g'_{2\|}T_{21\|}(p,k) - g'_{2\perp}T_{21\perp}(p,k)$$
$$= k^{-1}[G_1(p) - G_1(p+k)], \qquad (16)$$

whose LHS has much the same structure as that of equation (9*a*). If we introduce here the Dyson equations (15*a*, *b*) and make use of the system (9*a*, *b*) satisfied by the polarisations we obtain simply

$$T^{\mathrm{ir}}_{11\|}(\boldsymbol{p},\boldsymbol{k})(\Pi_1^*(\boldsymbol{k}))^{-1} = 2\pi k^{-1}[G_1(\boldsymbol{p}) - G_1(\boldsymbol{p}+\boldsymbol{k})]. \tag{17}$$

By means of a similar calculation we have for $j = 2$

$$T^{\mathrm{ir}}_{22\|}(\boldsymbol{p},\boldsymbol{k})(\Pi_2^*)^{-1}(\boldsymbol{k}) = -2\pi k^{-1}[G_2(\boldsymbol{p}) - G_2(\boldsymbol{p}+\boldsymbol{k})],$$

whence, making use of the irreducible polarisations (13) and removing the external legs by $T^{\mathrm{ir}}_{jj\|}(\boldsymbol{p},\boldsymbol{k}) = G_j(\boldsymbol{p})G_j(\boldsymbol{p}+\boldsymbol{k})\Gamma_j(\boldsymbol{p},\boldsymbol{k})$, we obtain the Ward identities

$$\Gamma_j(\boldsymbol{p},\boldsymbol{k}) = [G_j^{-1}(\boldsymbol{p}+\boldsymbol{k}) - G_j^{-1}(\boldsymbol{p})]/(\omega \mp k), \tag{18}$$

the upper (lower) sign corresponding to $j = 1(2)$. The remaining irreducible parts of the three-point vertex functions, $T^{\mathrm{ir}}_{jj\perp}(\boldsymbol{p},\boldsymbol{k})$, $T^{\mathrm{ir}}_{jj'\|,\perp}(\boldsymbol{p},\boldsymbol{k})$, $j \neq j'$, are all equal to zero, so that the Dyson equations (15a, b) have a much simpler form. The vanishing of these irreducible contributions is but another form of a statement previously made by Dzyaloshinsky and Larkin (1973) according to which the diagrams containing loops with more than two fermion lines do not contribute to the single-particle Green functions. It is noteworthy here that (18) is satisfied by the lowest-order contributions of $T_{jj\|}(\boldsymbol{p},\boldsymbol{k})$ calculated from the definition (14). This agreement is based upon the identity

$$G_j^0(\boldsymbol{p}) - G_j^0(\boldsymbol{p}+\boldsymbol{k}) = (\omega \mp k)G_j^0(\boldsymbol{p})G_j^0(\boldsymbol{p}+\boldsymbol{k}), \tag{19}$$

where $G_j^0(\boldsymbol{p}) = [\varepsilon \mp p \pm \mathrm{i}0^+ \operatorname{sgn}(p)]^{-1}$ is the free Green function of the j-fermions.

3. Generalised Ward identities

The response of the Tomonaga–Luttinger model to various external fields may be invesigated by means of more complicated three-point vertex functions containing operators which couple to these external fields. Such operators correspond to the spin flip

$$\sigma_{js}(k) = \sum_p a^+_{jp-s}a_{jp+ks} \tag{20}$$

changing of the type of particles

$$\tau_{js}(k) = \sum_p a^+_{jps}a_{\bar{j}p+ks} \tag{21}$$

and to the combination

$$\pi_{js}(k) = \sum_p a^+_{jp-s}a_{\bar{j}p+ks} \tag{22}$$

which changes both the spin and the type of particles ($\bar{j} = 1(2)$ for $j = 2(1)$). The corresponding three-point vertex functions are

$$\begin{aligned} T_j^\sigma(p,k;t_3-t_2,t_1-t_2) &= \langle 0|T\sigma_{js}(k,t_1)a^+_{jp+ks}(t_2)a_{jp-s}(t_3)|0\rangle \\ T_j^\tau(p,k;t_3-t_2,t_1-t_2) &= \langle 0|T\tau_{js}(k,t_1)a^+_{\bar{j}p+ks}(t_2)a_{jps}(t_3)|0\rangle \\ T_j^\pi(p,k;t_3-t_2,t_1-t_2) &= \langle 0|T\pi_{js}(k,t_1)a^+_{\bar{j}p+ks}(t_2)a_{jp-s}(t_3)|0\rangle. \end{aligned} \tag{23}$$

Their diagrammatic structure is similar to that of the functions $T_{jj'\|,\perp}$ (see figure 2) except for the absence of the reducible contributions. Unfortunately these vertex functions cannot be related to the single-particle Green functions through simple Ward identities. However Sólyom (1978) showed that there exist generalised Ward identities connecting the vertex functions (23) to four-point vertex functions. Let us introduce the four-point vertex functions

$$L_{jj'\parallel}^{\sigma}(p,k,q;t_4-t_3,t_2-t_3,t_1-t_4)$$
$$= -\mathrm{i}\langle 0|\,T\rho_{js}(-q,t_1)\sigma_{j's}(k,t_2)a_{j'p+ks}^{+}(t_3)a_{j'p-q-s}(t_4)\,|0\rangle$$
$$L_{jj'\perp}^{\sigma}(p,k,q;t_4-t_3,t_2-t_3,t_1-t_4)$$
$$= -\mathrm{i}\langle 0|\,T\rho_{j-s}(-q,t_1)\sigma_{j's}(k,t_2)a_{j'p+ks}^{+}(t_3)a_{j'p-q-s}(t_4)\,|0\rangle \tag{24a}$$
$$L_{jj'\parallel}^{\tau}(p,k,q;t_4-t_3,t_2-t_3,t_1-t_4)$$
$$= -\mathrm{i}\langle 0|\,T\rho_{js}(-q,t_1)\tau_{j's}(k,t_2)a_{\bar{j}'p+ks}^{+}(t_3)a_{j'p-qs}(t_4)\,|0\rangle$$
$$L_{jj'\perp}^{\tau}(p,k,q;t_4-t_3,t_2-t_3,t_1-t_4)$$
$$= -\mathrm{i}\langle 0|\,T\rho_{j-s}(-q,t_1)\tau_{j's}(k,t_2)a_{\bar{j}'p+ks}^{+}(t_3)a_{j'p-qs}(t_4)\,|0\rangle \tag{24b}$$
$$L_{jj'\parallel}^{\pi}(p,k,q;t_4-t_3,t_2-t_3,t_1-t_4)$$
$$= -\mathrm{i}\langle 0|\,T\rho_{js}(-q,t_1)\pi_{j's}(k,t_2)a_{\bar{j}'p+ks}^{+}(t_3)a_{j'p-q-s}(t_4)\,|0\rangle$$
$$L_{jj'\perp}^{\pi}(p,k,q;t_4-t_3,t_2-t_3,t_1-t_4)$$
$$= -\mathrm{i}\langle 0|\,T\rho_{j-s}(-q,t_1)\pi_{j's}(k,t_2)a_{\bar{j}'p+ks}^{+}(t_3)a_{j'p-q-s}(t_4)\,|0\rangle \tag{24c}$$

whose diagrammatic representation (see figure 3) leads to the following Dyson

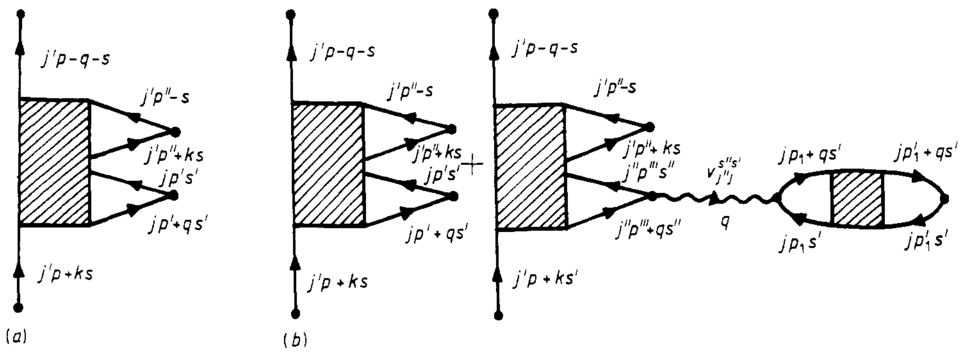


Figure 3. (*a*) Diagrammatic representation of the irreducible four-point vertex function $L_{jj'}^{\sigma\mathrm{ir}ss'}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}) = L_{jj'\parallel}^{\sigma\mathrm{ir}}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})\delta_{ss'} + L_{jj'\perp}^{\sigma\mathrm{ir}}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})\delta_{s-s'}$.
(*b*) Dyson equation for $L_{jj'}^{\sigma ss'}$. Similar diagrammatic representations are valid also for the functions $L_{jj'}^{\tau,\pi ss'}$

equations:

$$L_{jj'\parallel}^{\mu}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}) = (\Pi_j^*(\boldsymbol{q}))^{-1}[\Pi_{jj\parallel}(\boldsymbol{q})L_{jj'\parallel}^{\mu\mathrm{ir}}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}) + \Pi_{jj\perp}(\boldsymbol{q})L_{jj'\perp}^{\mu\mathrm{ir}}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})]$$
$$+ (\Pi_{\bar{j}}^*(\boldsymbol{q}))^{-1}[\Pi_{j\bar{j}\parallel}(\boldsymbol{q})L_{\bar{j}j'\parallel}^{\mu\mathrm{ir}}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}) + \Pi_{j\bar{j}\perp}(\boldsymbol{q})L_{\bar{j}j'\perp}^{\mu\mathrm{ir}}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})] \tag{25}$$
$$L_{jj'\perp}^{\mu}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}) = (\Pi_j^*(\boldsymbol{q}))^{-1}[\Pi_{jj\parallel}(\boldsymbol{q})L_{jj'\perp}^{\mu\mathrm{ir}}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}) + \Pi_{jj\perp}(\boldsymbol{q})L_{jj'\parallel}^{\mu\mathrm{ir}}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})]$$
$$+ (\Pi_{\bar{j}}^*(\boldsymbol{q}))^{-1}[\Pi_{j\bar{j}\parallel}(\boldsymbol{q})L_{\bar{j}j'\perp}^{\mu\mathrm{ir}}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}) + \Pi_{j\bar{j}\perp}(\boldsymbol{q})L_{\bar{j}j'\parallel}^{\mu\mathrm{ir}}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})],$$

where $\mu = \sigma, \tau$ or π, $\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q}$ stand for the pairs (p, ε_1), (k, ω) and (q, ε), respectively and $L^{\mu\mathrm{ir}}_{jj'\|,\perp}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q})$ denote the contribution of the irreducible diagrams. The Dyson equations (25) have the same structure as those corresponding to the three-point vertex functions (see equations (15*a*, *b*)). The Fourier transforms of the four-point vertex functions are defined by

$$L^{\mu}_{jj'\|,\perp}(p, k, q; t_y - t_3, t_2 - t_3, t_1 - t_4) = (2\pi)^{-3} \int \mathrm{d}\varepsilon_1\, \mathrm{d}\omega\, \mathrm{d}\varepsilon\, L^{\mu}_{jj'\|,\perp}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q})$$
$$\times \exp[-\mathrm{i}\varepsilon_1(t_4 - t_3)] \exp[-\mathrm{i}\omega(t_2 - t_3)] \exp[-\mathrm{i}\varepsilon(t_1 - t_4)]. \qquad (26)$$

The generalised Ward identities can be derived by means of a technique entirely analogous with that used in the preceding section. Let us take the derivative of $L^{\sigma}_{11\|}(p, k, q; t_4 - t_3, t_2 - t_3, t_1 - t_4)$ with respect to the first time argument; using the equation of motion for $\rho_{1s}(-q, t_1)$ and the equal-time commutators we get

$$\mathrm{i}\frac{\partial}{\partial t_1} L^{\sigma}_{11\|}(p, k, q; t_y - t_3, t_2 - t_3, t_1 - t_4) = \delta(t_1 - t_3)\, T^{\sigma}_{1}(p - q, k; t_4 - t_1, t_2 - t_1)$$
$$- \delta(t_1 - t_2)\, T^{\sigma}_{1}(p - q, k + q; t_4 - t_3, t_1 - t_3)$$
$$+ q(1 + g'_{4\|}) L^{\sigma}_{11\|}(p, k, q; t_4 - t_3, t_2 - t_3, t_1 - t_4)$$
$$+ q g'_{4\perp} L^{\sigma}_{11\perp}(p, k, q; t_4 - t_3, t_2 - t_3, t_1 - t_4)$$
$$+ q g'_{2\|} L^{\sigma}_{21\|}(p, k, q; t_4 - t_3, t_2 - t_3, t_1 - t_4)$$
$$+ q g'_{2\perp} L^{\sigma}_{21\perp}(p, k, q; t_4 - t_3, t_2 - t_3, t_1 - t_4)$$

and its Fourier transform

$$[\varepsilon/q) - (1 + g'_{4\|})]\, L^{\sigma}_{11\|}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q}) - g'_{4\perp} L^{\sigma}_{11\perp}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q}) - g'_{2\|} L^{\sigma}_{21\|}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q}) - g'_{2\perp} L^{\sigma}_{21\perp}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q})$$
$$= q^{-1}[T^{\sigma}_{1}(\boldsymbol{p} - \boldsymbol{q}, \boldsymbol{k}) - T^{\sigma}_{1}(\boldsymbol{p} - \boldsymbol{q}, \boldsymbol{k} + \boldsymbol{q})], \qquad (27)$$

which has the same structure as (16). Using the Dyson equation (25) and the basic system of equations (9*a*, *b*) we obtain from (27)

$$L^{\sigma\mathrm{ir}}_{11\|}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q})\, (\Pi^{*}_{1})^{-1}(\boldsymbol{q}) = 2\pi q^{-1}[T^{\sigma}_{1}(\boldsymbol{p} - \boldsymbol{q}, \boldsymbol{k}) - T^{\sigma}_{1}(\boldsymbol{p} - \boldsymbol{q}, \boldsymbol{k} + \boldsymbol{q})],$$

that is

$$L^{\sigma\mathrm{ir}}_{11\|}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q}) = [T^{\sigma}_{1}(\boldsymbol{p} - \boldsymbol{q}, \boldsymbol{k}) - T^{\sigma}_{1}(\boldsymbol{p} - \boldsymbol{q}, \boldsymbol{k} + \boldsymbol{q})]/(\varepsilon - q), \qquad (28)$$

which is the generalised Ward identity. Similarly we get other three identities, so that we may write

$$L^{\sigma\mathrm{ir}}_{jj\|}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q}) = [T^{\sigma}_{j}(\boldsymbol{p} - \boldsymbol{q}, \boldsymbol{k}) - T^{\sigma}_{j}(\boldsymbol{p} - \boldsymbol{q}, \boldsymbol{k} + \boldsymbol{q})/(\varepsilon \mp q),$$
$$L^{\sigma\mathrm{ir}}_{jj\perp}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q}) = [T^{\sigma}_{j}(\boldsymbol{p} - \boldsymbol{q}, \boldsymbol{k} + \boldsymbol{q}) - T^{\sigma}_{j}(\boldsymbol{p}, \boldsymbol{k})/(\varepsilon \mp q), \qquad (29a)$$

the upper (lower) corresponding to $j = 1(2)$; the remaining irreducible vertex functions are equal to zero, $L^{\sigma\mathrm{ir}}_{jj'\|,\perp}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q}) = 0$. By using a similar technique it can be shown that the remaining vertex functions satisfy the following generalised Ward identities:

$$L^{\tau\mathrm{ir}}_{jj\|}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q}) = [T^{\tau}_{j}(\boldsymbol{p} - \boldsymbol{q}, \boldsymbol{k}) - T^{\tau}_{j}(\boldsymbol{p} - \boldsymbol{q}, \boldsymbol{k} + \boldsymbol{q})]/(\varepsilon \mp q),$$
$$L^{\pi\mathrm{ir}}_{jj\|}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q}) = [T^{\tau}_{j}(\boldsymbol{p} - \boldsymbol{q}, \boldsymbol{k} + \boldsymbol{q}) - T^{\tau}_{j}(\boldsymbol{p}, \boldsymbol{k})]/(\varepsilon \mp q) \qquad (29b)$$
$$L^{\pi\mathrm{ir}}_{jj'\perp}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q}) = 0$$

and

$$
\begin{aligned}
L^{\bar{\pi}\pi r}_{jj\parallel}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}) &= [T^{\bar{\pi}}_{j}(\boldsymbol{p}-\boldsymbol{q},\boldsymbol{k}) - T^{\bar{\pi}}_{j}(\boldsymbol{p}-\boldsymbol{q},\boldsymbol{k}+\boldsymbol{q})]/(\varepsilon \mp q),\\
L^{\bar{\pi}\pi r}_{jj\perp}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}) &= [T^{\bar{\pi}}_{j}(\boldsymbol{p}-\boldsymbol{q},\boldsymbol{k}+\boldsymbol{q}) - T^{\bar{\pi}}_{j}(\boldsymbol{p},\boldsymbol{k})]/(\varepsilon \mp q),\\
L^{\bar{\pi}\bar{\pi} r}_{jj\parallel}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}) &= L^{\bar{\pi}\bar{\pi} r}_{jj\perp}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}) = 0.
\end{aligned} \tag{29c}
$$

We should remark that $L^{\mu\nu r}_{jj'\parallel,\perp}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})$ include the two external legs of single-particle Green functions. The generalised Ward identities (29*b*, *c*) have been derived diagrammatically by Sólyom (1978). The method given here for obtaining these identities allows us to derive Ward identities between even more complicated vertex functions of higher orders.

4. Conclusion

The Ward identities relating the three-point vertex functions to the single-particle Green functions as well as the generalised Ward identities between the four- and three-point vertex functions have been derived analytically by means of the equation of motion method for the Tomonaga–Luttinger model. These Ward identities were previously obtained diagrammatically within the framework of the perturbation-theoretical approach without making use explicitly of the boson operators in terms of which the Tomonaga–Luttinger model is usually formulated. The method used in the present paper relies upon the algebra of these operators and allows us to see the role played in the physics of the system by the two underlying features of the model: the linear fermion spectrum and the conservation of the particle number for the fermions of each type. The two apparently distinct methods (bosonisation and diagrammatic analysis), both successfully used in studying the Tomonaga–Luttinger model, are thereby linked together.

References

Apostol M 1980 *Phys. Lett.* **78**A 91 (erratum **79**A 467)

—— 1981 *Solid State Commun.* **37** 257

Apostol M and Bârsan V 1981 *Rev. Roum. Phys.* **26** 375

Apostol M, Corciovei A and Stoica S 1981 *Phys. Status Solidi* b **103** 411

Bohr T 1981 preprint Nordita 81/4

Brout R and Carruthers P 1963 *Lectures on the Many-Electron Problem* (New York: Wiley) p 107

Dzyaloshinsky I E and Larkin A I 1973 *Zh. Eksp. Teor. Fiz.* **65** 411 (Engl. Transl. 1974 *Sov. Phys.–JETP* **38** 202)

Everts H U and Schulz H 1974 *Solid State Commun.* **15** 1413

Fogedby H C 1976 *J. Phys. C: Solid State Phys.* **9** 3757

Haldane F D M 1979 *J. Phys. C: Solid State Phys.* **12** 4791

—— 1981 *J. Phys. C: Solid State Phys.* **14** 2585

Luther A and Peschel I 1974 *Phys. Rev.* B **9** 2911

Luttinger J M 1963 *J. Math. Phys.* **4** 1154

Mattis D C 1974 *J. Math. Phys.* **15** 609

Mattis D C and Lieb E H 1965 *J. Math. Phys.* **6** 304

Sawada K 1957 *Phys. Rev.* **106** 372

Sólyom J 1978 *Proc. Int. Conf. Quasi-One Dimensional Conductors, Dubrovnik*, Vol II, ed. S Barisić, A Bjelis, J R Cooper and B Leontić (Berlin: Springer) p 100

—— 1979 *Adv. Phys.* **28** 201

Tomonaga S 1950 *Prog. Theor. Phys.* **5** 544