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On a non-linear diffusion equation describing clouds and wreaths of smoke

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Abstract

A non-linear diffusion equation is derived by taking into account the local variations in the solvent density, within a mechanism of diffusion driven by particle collisions. A particular class of radially-symmetric solutions is discussed, which are localized over finite ranges, centered either on the origin or on finite values of the radius. In the case of a thin layer of solute they look like disks or rings, and we suggest that they might be appropriate for describing the shapes of clouds, or wreaths of smoke. The latter case of ring-like solutions may correspond, for example, to the polluting puffs of smoke thrown into the atmosphere by chimneys and smokestacks. These patterns exhibit diffusion fronts propagating slower and slower, and growing in time at a constant rate of the total number of solute particles. © 1997 Elsevier Science B.V.

Recently [1], the ascent of warm, moist air in the Earth's atmosphere has been modelled with the Kardar-Parisi-Zhang (KPZ) equation [2], known from crystal growth on atomic surfaces. This equation has been solved numerically, [3,4] and the fractal aspects of the solutions has been emphasized. Leaving aside the noise term responsible for the fractal behaviour, the remaining part of this equation looks like a non-linear diffusion equation, though the non-linear term has originally been interpreted as describing growth. Taking into account the local variations in the solvent density, within a diffusion mechanism driven by particle collisions, we show here that the non-linear term of the KPZ equation may indeed arise in a diffusion process in non-equilibrium conditions, while at equilibrium, an identical term is obtained, only with opposite sign. The non-linear diffusion equation thereby obtained is solved for a particular class of radially-symmetric solutions. In the first case, it is found that the solute concentrations extend over finite ranges in space, having the shape of a disk in two dimensions. In the second case, solutions of annular shape are obtained in two dimensions. These patterns of the solute concentration exhibit diffusion fronts propagating slower and slower, and growing in time at a constant rate of the total number of solute particles. We suggest that they might be interpreted as describing clouds or wreaths of smoke, the latter, for example, as the polluting puffs thrown into the atmosphere by chimneys and smokestacks. Specifically, we discuss the two-dimensional case, but relevant features, wherever appropriate, are pointed out in one and three dimensions.

Suppose that we have a thin layer of fluid whose particles may jump from the position (x, y) at time t to the neighbouring positions $(x \pm a, y)$, $(x, y \pm a)$,

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where a is a generic, atomic length scale; suppose, further on, that these jumps proceed with an average frequency $\nu(x, y)$. Then, the time variation of the fluid concentration $n(x, y, t)$ is given by [5]

$$\frac{\partial n}{\partial t} = \nu(x+a, y)n(x+a, y) + \nu(x-a, y)n(x-a, y) - 2\nu n + \nu(x, y+a)n(x, y+a) + \nu(x, y-a)n(x, y-a) - 2\nu n. \quad (1)$$

Here we assume that n varies slowly over lapses of time much longer than the time scale ν^{-1}, and n and ν vary slowly in space too, over distances much larger than a. Then, by series expansion, the leading terms in Eq. (1) are given by

$$\frac{\partial n}{\partial t} = a^2 \nu \triangle n + 2a^2 \operatorname{grad} \nu \cdot \operatorname{grad} n. \quad (2)$$

Within a kinetic model of the fluid, the collision frequency ν is proportional to the density N of solvent particles, and during the diffusion process this density becomes either $N+n$, for a non-equilibrium process, or $N-n$, at equilibrium[2] $(n/N \ll 1)$. It is now easy to see that Eq. (2) becomes

$$\frac{1}{S}\frac{\partial n}{\partial t} = \triangle n + A(\operatorname{grad} n)^2 \quad (3)$$

in the former case, and

$$\frac{1}{S}\frac{\partial n}{\partial t} = \triangle n - A(\operatorname{grad} n)^2 \quad (4)$$

in the latter, where $S \sim a^2 N$ and $A \sim 1/N$. The first process is expected to occur more frequently in dilute gases, while the latter is more appropriate for dense gases or liquids. Eqs. (3) and (4) are non-linear diffusion equations, Eq. (3) being the KPZ equation [2] without noise. We note that the non-linear term in Eqs. (3) and (4) occurs as a result of taking into account the local variations of the solvent density, within a mechanism of diffusion driven by particle collisions. This is in contrast with the diffusion by tunnelling in an external potential, as, for example, in a (homogeneous) solid, where the jump frequency ν does not depend on the solute concentration n. Of course, if N is very large then we may neglect the non-linear term in Eqs. (3) and (4), and we get the usual, linear diffusion equation.

We note, first, that the non-linear diffusion equation given above by Eqs. (3) and (4) does not conserve the total number of solute particles, in contrast with the linear diffusion equation. Indeed, we have

$$\frac{\partial}{\partial t}\int n \, d\boldsymbol{r} = \pm SA \int (\operatorname{grad} n)^2 \, d\boldsymbol{r}, \quad (5)$$

and requiring the conservation of particle number would imply the trivial solution $n = \text{const}$. We shall look for radially-symmetric solutions (corresponding to radially-symmetric initial conditions), in which case Eq. (3) reads

$$\frac{1}{S}\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial r^2} + \frac{1}{r}\frac{\partial n}{\partial r} + A\left(\frac{\partial n}{\partial r}\right)^2; \quad (6)$$

further, we assume that n depends on $\xi = r/\sqrt{2St}$ only, which represents a particular class of solutions, corresponding to an increase of the total number of solute particles at a constant rate,

$$\int n(\xi) d\boldsymbol{r} = \text{const} \times t \quad (7)$$

($\sqrt{t}$ in one dimension, $t^{3/2}$ in three dimensions). Eq. (6) then reads

$$n'' + (1/\xi + \xi)n' + An'^2 = 0, \quad (8)$$

which is a Bernoulli-type equation. Introducing $f = 1/n'$ we get an equation of the type

$$f' + af + b = 0, \quad (9)$$

whose solution is [6]

$$f = e^{-F}\left(\text{const} - \int^{\xi} b e^{F} \, d\xi\right), \quad (10)$$

where $F = \int^{\xi} a \, d\xi$. In this way we obtain from Eq. (8)

$$n' = \frac{\exp(-\xi^2/2)}{\xi[\text{const} + A\sqrt{S/2}\,\text{Ei}(-\xi^2/2)]}, \quad (11)$$

[2] Here it is assumed that the solute and the solvent particles are of similar nature; for solute particles which differ from the solvent particles, as, for example, in the case of suspensions of smoke particles in air, the treatment is similar, only the particular expressions of the coefficients S and A will change.

where Ei is the exponential integral ($\mathrm{Ei}(x) = -\int_{-x}^{\infty}(\mathrm{e}^{-t}/t)\,\mathrm{d}t$). The exponential integral $\mathrm{Ei}(-\xi^2/2)$ is a monotonically increasing function which has the following asymptotic behaviour,

$$\mathrm{Ei}\left(-\xi^2/2\right) \sim \ln\left(\xi^2/2\right), \quad \xi^2/2 \ll 1,$$
$$\sim -\frac{\mathrm{e}^{-\xi^2/2}}{\xi^2/2}, \quad \xi^2/2 \gg 1. \tag{12}$$

It is now easy to see that for const $\geqslant 0$ in Eq. (11) we would get a negative, unphysical solution n, while for const < 0 we get a monotonically decreasing solution n whose asymptotics are

$$n \sim \frac{\ln\left|\ln\left(\xi^2/2\right)\right|}{A\sqrt{2S}}, \quad \xi^2/2 \ll 1, \tag{13}$$
$$\sim \frac{\mathrm{e}^{-\xi^2/2}}{\xi^2/2}, \quad \xi^2/2 \gg 1. \tag{14}$$

The small solute density is therefore given by the smooth function n in (14) for $\xi^2/2 \gg 1$, while it increases rapidly for $\xi^2/2 \ll 1$, as suggested by (13). There exists, consequently, a front of diffusion placed at $\xi^2/2 \sim 1$, i.e. at $r \sim 2\sqrt{St}$, which propagates slower and slower ($\mathrm{d}r/\mathrm{d}t \sim 1/\sqrt{t} \longrightarrow 0$ for $t \longrightarrow \infty$), encompassing an area which increases linearly with time, exactly as the total number of solute particles does. We may say that the solute particles placed initially over a certain, small area, propagate rather compactly (with a diffusion front), and slower and slower, while being fed continuously at a constant rate. This suggests that we may speak in this case of a disk-like cloud pattern, as of an atmospheric, or a smoke cloud.

We note that a similar solution exists also for the linear diffusion equation ($A = 0$), $n = \mathrm{const} \times \mathrm{Ei}(-\xi^2/2)$, but not for the number-conserving solution

$$n \sim \frac{1}{t}\exp(-\xi^2/2), \quad \xi^2/2 \gg 1, \tag{15}$$

which decreases uniformly to zero for $t \longrightarrow \infty$.

Similar solutions exist for the non-linear equation in three dimensions and in one dimension, but in the latter case the existence of the diffusion front is doubtful, since the solution n and its derivative are finite at the origin.

A similar analysis can be carried out straightforwardly for Eq. (4), which amounts to changing the sign of A in Eqs. (8) and (11). It is easy to see, in this case, that we would get an unphysical solution for const $\geqslant 0$ in (11), but a physical one for const < 0 (and $\xi^2/2 \gg 1$). However, this solution has a logarithmic singularity at certain, finite values ξ_0 (depending on this integration constant), where it looks like $n \sim -\ln\left|(\xi - \xi_0)/\sqrt{2}\right|$. For large values of the variable $\xi^2/2$ the solution behaves like the previous one, given by Eq. (14). One may say, therefore, that there exists in this case a finite value ξ_0 (and a finite value of the radius r_0), where the solute density is concentrated; it extends over a finite range of the order ξ_0, so that we may identify two diffusion fronts, one for each side of the singularity, propagating (in opposite directions) slower and slower, and encompassing an area which increases again linearly in time, in the same manner as the total number of solute particles. The pattern has an annular shape, suggesting again atmospheric clouds or wreaths of smoke. It may be appropriate for describing, for example, the polluting puffs of smoke thrown into the atmosphere by chimneys and smokestacks. We remark that such a solution might be expected for the equilibrium diffusion process, corresponding to Eq. (4). Indeed, a spot of solute particles cannot proceed to diffuse at equilibrium by removing the solvent particles only outwardly, they have to displace them inwardly too, acquiring thereby the ring-like shape of an annulus.

Similar conclusions may also be reached in one and three dimensions, though in the former case the solution may be peaked at the origin, too ($\xi_0 = 0$).

Finally, we mention that the smoothness conditions required for deriving Eq. (2) are met by our solutions given by (14) for distances Δr and lapses of time Δt much smaller than the distance r and, respectively, the time t of observation (but certainly much larger than a and, respectively, ν^{-1}); we note also that a low diffusion coefficient S favours the fulfillment of the asymptotic condition $\xi^2/2 \gg 1$ where our solutions hold.

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