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Euler's transform and a generalized Omori's law

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Abstract

The self-replication process of the statistical events generated by an original, main event and described by a finite distribution leads to a generalized Omori distribution singular at origin. The two distributions are related to each other by Euler's transform. The self-consistency of the generating process requires an exponential law for the finite distribution, which gives rise to the original Omori's law associated to the seismic activity accompanying a major earthquake.

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Singular distributions of power-law type $P(t) \sim 1/t^{\beta}$, for $t > 0$ and $\beta > 1$, seem to be ubiquitous [1]. Originally, they may have been introduced by Omori in 1894 [2,3] for describing the distribution of the seismic aftershocks with $\beta = 1^{+}$ and t denoting the time elapsed from the occurrence of the main seismic shock at $t = 0$. Such distributions, which may be called Omori-type singular distributions, are widely used in analyzing the seismic activity accompanying a major earthquake, both as aftershocks and foreshocks, as well as in a great variety of other situations [4–11]. The power-law bears also relevance on a critical-point theory for the accompanying seismic activity, and other similar phenomena, especially in connection with the self-organized criticality [12–14]. In view of their possible non-integrability, such power-laws are usually defined over a range $t_c < t < D$, as large as possible, where t_c is a lower-bound cutoff and $D \gg t_c$ is an upper-bound cutoff. The cutoff parameter t_c may be set zero (for $0 < \beta < 1$, for instance), and D may be extended to infinite (for $\beta > 1$, for instance).

It is shown here that such Omori-type singular distributions may arise from self-replicating events, originally produced by a main event according to a finite distribution. The two distributions are related to each other by Euler's transform, which provides a generalized form for Omori's law. The self-consistency of the production process implies a self-generating original distribution, which is given by an exponential law. The distribution of all the events produced in such a process, self-replication included, is the original Omori's law.

Let $p(t) = dN/dt$ be a finite distribution of N events over the range $t > 0$. The number $dN_0 = p_0\,dt$ of events placed at origin, where $p(0) = p_0$, can be viewed as the number of the main events, while the rest of events, distributed over $t > 0$, can be viewed as produced by the main events at a rate $r(t)$ given by

$$p(t) = p_0 r(t). \tag{1}$$

The self-replication process[1] implies a distribution $P(t)$ obeying the relationship

$$P(t) = p(t) + r(t)P(t) = p(t) + \frac{p(t)}{p_0}P(t). \tag{2}$$

[1] The referee kindly suggested that "self-replication" may be related to what in some other works is called "self-consistency", or "cascades", and to "self-excited processes".

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It follows that distribution $P(t)$ is given by

$$P(t) = \frac{p(t)}{1 - p(t)/p_0}, \tag{3}$$

which is Euler's transform between $p(t)/p_0$ and $-P(t)/p_0$. The distribution $P(t)$, as given by (3), corresponds to all the events generated in the process of producing accompanying events by the main events placed at $t = 0$. It is worth noting that $P(t)$ is singular at origin.

Distributions $P(t)$ as given by Euler's transform (3) can be considered for a general form of generating distributions $p(t)$, which amounts to including only the self-replication process for the accompanying events produced by $p(t) = p_0 r(t)$. For this general case, the series expansion $p(t) = p_0 - p_1 t \cdots$ can be considered in the neighbourhood of $t = 0$, leading to Omori's law $P(t) = p_0 t_0 / t$ for $t \ll t_0 = p_0/p_1$. Euler's transform (3) provides a general representation $P(t) = p_0/h(t)$ for such singular distributions, where $h(0) = 0$ and $h(\infty) \to \infty$ (such that, preferably, $P(t)$ is integrable at infinite). It implies $p(t) = p_0/(1+h) \simeq p_0(1-h)$ for $t \to 0$. Such a representation may be regarded as a generalized Omori-type distribution. For $h(t) \sim t^\beta$, $\beta > 0$, power-law distributions $P(t) \sim 1/t^\beta$ are obtained (an upper-bound cutoff D is necessary for $0 < \beta \leqslant 1$, as well as a lower-bound cutoff t_c for $1 \leqslant \beta$).

Since the accompanying events are produced by the main events at a rate $r(t) = p(t)/p_0$, and since the events are not differentiated otherwise except by their occurrence time t, it follows that the distribution p may also be produced for $t + \tau$ by its value for t multiplied by rate $r(\tau)$, i.e.

$$p(t + \tau) = p(t) r(\tau), \tag{4}$$

for any $t, \tau > 0$. This distribution may be viewed as a self-generating distribution, and Eq. (4) expresses a self-consistency character of the distribution $p(t)$. Eq. (4) can also be written as $p(t + \Delta t) = r(\Delta t) p(t)$, or $dp/dt = (-p_1/p_0) p(t)$, where $-p_1 = p'(0) < 0$ is the first derivative of $p(t)$ at origin. It follows immediately, from (1) and (4), that distribution $p(t)$ is given by an exponential law, $p(t) = p_0 e^{-p_1 t/p_0}$. It can be transformed into a normalized probability distribution $p(t) = p_0 e^{-p_0 t}$.

Inserting the exponential distribution $p(t) = p_0 e^{-p_0 t}$ in (3) the distribution

$$P(t) = \frac{p_0}{e^{p_0 t} - 1}, \tag{5}$$

is obtained, which is Omori's law $P(t) = 1/t$ for $p_0 t \ll 1$. It is customary to introduce a lower-bound cutoff t_c and to extend $1/t$ to infinite as $t_c^{\beta-1}/t^\beta$, where $\beta = 1^+$, such that

$$\int_{t_c}^{\infty} dt \, \frac{p_0}{e^{p_0 t} - 1} = \int_{t_c}^{\infty} dt \, (t_c^{\beta-1}/t^\beta). \tag{6}$$

Eq. (6) gives the exponent $\beta = 1 - 1/\ln(p_0 t_c) = 1^+$ in the limit $t_c \to 0$.

It might be noted that $P(t)$ as given by (5) is, formally, a Bose–Einstein-type occupation number (in two dimensions) for an exponential, Boltzmann-type, distribution $p(t)$. The self-replication equation (2), which describes a geometric series, has also a formal resemblance to Dyson's equation in the theory of interacting many-body ensembles. Eq. (5) can also be viewed as a generalized Omori's law.

In conclusion, it may be said that self-replication processes at a rate $r(t) = p(t)/p_0$ for a generating distribution $p(t)$ of events accompanying the main events placed at $t = 0$ lead to Omori-type singular distributions as given by Euler's transform (3). Such distributions include power-type distributions of the form $1/t^\beta$, where $\beta > 0$. The self-consistency of the generating process requires a self-generating distribution $p(t)$, which is given by an exponential law, and which leads to the original Omori's law $1/t^\beta$, with $\beta = 1^+$.

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