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A model of seismic focus and related statistical distributions of earthquakes

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Abstract

A growth model for accumulating seismic energy in a localized seismic focus is described, which introduces a fractional parameter r on geometrical grounds. The model is employed for deriving a power-type law for the statistical distribution in energy, where the parameter r contributes to the exponent, as well as corresponding time and magnitude distributions for earthquakes. The accompanying seismic activity of foreshocks and aftershocks is discussed in connection with this approach, as based on Omori distributions, and the rate of released energy is derived.

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1. Introduction

The physical mechanisms of the seismic sources are still unknown to a large extent, and the patterns exhibited by earthquakes in space and time are a matter of debate. The present Letter introduces a model of accumulating seismic energy in a localized critical focal zone, and derives statistical distributions of earthquakes in time, energy and magnitude which seem to enjoy a certain consensus. The focus model includes a fractional parameter r, derived on geometrical grounds, which turns out to be an Omori-type parameter in the power-law distribution of earthquakes with respect to energy. It affects also the distribution in magnitude, the Gutenberg–Richter recurrence law and the mean recurrence time of the earthquakes. The associated regime of seismic activity in the neighbourhood of a main, "regular" seismic shock is also discussed in connection with Omori distributions in time, magnitude and energy, and the rate of released energy is given.

It is widely agreed that the seismic energy E released in an earthquake is related to the earthquake's magnitude M by the Gutenberg–Richter-type relationship [1–3]

$$\ln E = a + bM. \tag{1}$$

Statistical analysis of moderate and strong earthquakes ($5.8 < M < 7$), which are probably most prone to represent a statistical ensemble, indicates values $a \simeq 10$ and $b \simeq 3.5$ (in decimal logarithms $a \simeq 4.4$ and $b \simeq 1.5$) for energy measured in J [4]. (The error in seismic energy may be up to a factor of 10.) These numerical values may be adopted for the present purpose,[1] although the considerations made herein do not depend critically on such numerical values. Parameter a in (1) indicates the existence of a threshold energy $E_0 = e^a$ ($E_0 \simeq 4.4 \times 10^4$ J), corresponding to $M = 0$, so that Eq. (1) can be recast as $E/E_0 = e^{bM}$.

It is customary to assign a region of characteristic length R to the seismic energy E, through $E \sim R^3$, and, similarly, a characteristic threshold length R_0 can be associated to the threshold

[1] There are various representations for energy E and magnitude M in the Gutenberg–Richter relationship (1), as depending on various practical conventions, the most used being related to the seismic moment. All of them obey a general relationship of the form given by (1), and their specific definitions are immaterial for the present purpose.

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energy $E_0 \sim R_0^3$, leading to

$$\ln(R/R_0) = \beta M, \tag{2}$$

where $\beta = b/3 = 1.17$. The two characteristic lengths R and R_0 have a twofold interpretation: On one side, they may be associated to the central core of the critical focal zone where the seismic energy accumulates, and, on the other, R may correspond to the characteristic length of the seismic region disrupted by the earthquake, R_0 being in this case a scale length. The empirical evidence [5,6] in the latter case seems to support an equation of type (2).

2. A model of seismic focus

It is assumed that the characteristic lengths R and R_0 correspond to a localized critical focal region where the seismic energy builds up by mechanical tension. It is also reasonable to assume that the process of accumulating energy in the seismic focus exhibits a uniform velocity $\mathbf{v}$, so that the accumulation of the seismic energy in focus obeys the continuity equation

$$\frac{\partial E}{\partial t} = -\mathbf{v}\,\mathrm{grad}\,E, \tag{3}$$

where t denotes the accumulation time. In Cartesian coordinates the rhs of Eq. (3) can be written as $\mathbf{v}\,\mathrm{grad}\,E = v_x(\partial E/\partial x) + v_y(\partial E/\partial y) + v_z(\partial E/\partial z)$, and we may assume that a localized process of uniformly accumulating energy is characterized by $v_x = v_y = v_z = v$ and $\partial E/\partial x = \partial E/\partial y = \partial E/\partial z = \partial E/\partial L$, where L is a generic coordinate. Under such circumstances we can write $\mathbf{v}\,\mathrm{grad}\,E = 3v(\partial E/\partial L)$. It is worth noting the occurrence of factor 3 in the rhs of this equation. However, if the process is not uniform, this factor changes, in general. For instance, if $v_x = v_y = v$ and $v_z = 0$, the resulting factor is 2, and, in general, it depends on the particular geometry of the critical focal zone and the particular mechanism of accumulating seismic energy. Therefore, in order to preserve the generality we may represent the rhs of Eq. (3) as $\mathbf{v}\,\mathrm{grad}\,E = (1/r)v(\partial E/\partial L)$, where r is a geometric factor accounting for such a non-uniform process of accumulating energy. In the former instances $r = 1/3$ and $r = 1/2$, respectively. Further, since the region of interest is localized and energy is very high ($E \gg E_0$), the spatial variation of energy can be represented as $\partial E/\partial L \simeq -E/L$, such that Eq. (3) becomes

$$\frac{dE}{dt} = \left(\frac{1}{r}\right) v \frac{E}{L}. \tag{4}$$

Eq. (4) leads also to consider the accumulation time $t = L/v$ (as well as a threshold time t_0), such that it can also be written as

$$\frac{dE}{dt} = \left(\frac{1}{r}\right) \frac{E}{t}. \tag{5}$$

A similar representation can be obtained directly from $E \sim R^3$, which leads to $dE = 3E\,dR/R = 3E\,dt/t$, i.e. Eq. (5) for $r = 1/3$. In this respect, parameter $1/r$ may be viewed as a "fractal dimension" [7].

Eq. (5) can be integrated straightforwardly, by making use of the cutoff parameters E_0 and t_0. We obtain

$$\frac{t}{t_0} = \left(\frac{E}{E_0}\right)^r, \tag{6}$$

which is the basic equation for the present model of seismic focus. Making use of the Gutenberg–Richter equation (1) we obtain also

$$t = t_0 e^{\beta M}, \tag{7}$$

where $\beta = br$. For $r = 1/3$ the parameter β acquires the value $\beta = b/3 = 1.17$ ($b = 3.5$).

According to Eq. (5), such a model looks like a growth model, with a typical power-law as given by (6).

3. Statistical distributions

Let N_0 be the number of earthquakes during a long time T, characterized by the average threshold time $t_0 = T/N_0$, where N_0 is very large. The cutoff parameter t_0 may be viewed as the inverse of the seismicity rate. Similarly, the frequency of N earthquakes in this series, characterized by time t can be written as $N/N_0 = (T/t)/(T/t_0) = t_0/t$. Hence, it follows the temporal probability distribution

$$P(t)\,dt = -d(t_0/t) = \frac{t_0}{t^2}dt. \tag{8}$$

Eqs. (6) and (8) give the probability distribution in energy

$$P(E)\,dE = \frac{rE_0^r}{E^{1+r}}dE, \tag{9}$$

providing the mean recurrence time is identified with the accumulation time. Similar power-law distributions in energy have been derived recently [8] by employing Tsallis entropy for the fragmentation of a dynamical fault-planes model. Such distributions are sometime called Omori-type distributions, where r is an Omori parameter.

Making use of the energy distribution (9) and the Gutenberg–Richter equation (1) the magnitude distribution

$$P(M)\,dM = \beta e^{-\beta M}\,dM \tag{10}$$

is obtained straightforwardly. The number ΔN of seisms with magnitude between M and $M + \Delta M$ is given by $\Delta N/N_0\Delta M = P(M)$, or

$$\lg\left(\frac{\Delta N}{T}\right) = A - BM, \tag{11}$$

where $A = \lg(\beta\Delta M/t_0)$ and $B = \beta/2.3$. Such a linear relationship, known as the Gutenberg–Richter law, has been checked for a large amount of earthquakes, and $A \simeq 4.6$ and $B \simeq 0.6$ were obtained, for instance, for $5.8 < M < 7.3$ (and $\Delta M = 0.1$) [4]. These values may be adopted here for the present purpose, though the numerical values of such quantities do not affect the results presented herein. Making use of $B = 0.6$ one obtains $\beta \simeq 1.38$, in fair agreement with $\beta = 1.17$ (corresponding to $r = 1/3$). Similarly, a global rate of seismicity $1/t_0 \sim 10^{5.5}$ per year is obtained from $A = 4.6$, which is consistent with estimations of cca 10^5–10^6 earthquakes per year, on average [4]. There are appreciable deviations from the Gutenberg–Richter linear relationship (11) for extreme values of the magnitude.[2] Such deviations may indicate either that the corresponding seismic events are not statistical events, or the deviations may be ascribed to a magnitude saturation phenomenon for large values of the magnitude.

It is also convenient to introduce the so-called recurrence law, or the exceedance rate, which gives the number N_{ex} of earthquakes with magnitude greater than M. The corresponding probability is readily obtained from (10) as $P_{ex} = e^{-\beta M}$, so the exceedance rate reads

$$\ln\left(\frac{N_{ex}}{T}\right) = -\ln t_0 - \beta M. \tag{12}$$

This relationship, also known as the Gutenberg–Richter law, is currently employed for analyzing the earthquake statistical distributions in magnitude. A recent analysis [9,10] seems to indicate a certain universality in the value of the β slope ($B = \beta/2.3 \simeq 0.6$). However, there is a great regional variability in the values of the coefficient B (and correspondingly in the values of the coefficient β and parameter r), as well as a variability associated to the data sets analyzed. For instance, $B = 1$, which seems to correspond to Southern California [11], leads to $\beta = 2.3$ and $r = 0.66$ (and a seismicity rate given by $-\ln t_0 = 17.25$ for t_0 measured in years). Statistical analysis for 1999 earthquakes with magnitude $M > 3$, recorded in the seismic region Vrancea (Romania) between 1974 and 2004 [12], indicates $B = 0.82$, $\beta = 1.89$ and $r = 0.54$ (and a seismicity rate given by $-\ln t_0 = 9.68$ for t_0 measured in years).

It is worth noting that Eq. (7) may be viewed as providing the mean recurrence time $t_r = t_0 e^{\beta M}$ for the occurrence of earthquakes of magnitude M. In fact, the mean recurrence time of earthquakes with magnitude in the range M to $M + \Delta M$ is of interest (for $\Delta M \ll M$). According to (10) the rate of such earthquakes is given by $\Delta N/T = (\beta \Delta M/t_0)e^{-\beta M}$, so the mean recurrence time can be obtained as

$$t_r = \left(\frac{t_0}{\beta \Delta M}\right) e^{\beta M}. \tag{13}$$

If the seismicity rate t_0 is known, this equation may be used to estimate the mean recurrence times. However, it must be noted that the relevance of such estimations is, in fact, very limited. Indeed, imposing a mean recurrence time t_r, the temporal distribution $(1/t_r)e^{-t/t_r}$ is obtained immediately from the maximum of the entropy, for instance (Poisson distribution). The root mean square is then $\langle t^2 \rangle^{1/2} = \sqrt{2} t_r$, which, compared with the mean recurrence time t_r, gives a relative deviation $\sqrt{2} - 1 = 41\%$, that may be taken as a measure for the uncertainty in estimating the recurrence time (the variability in the recurrence time).

[2] For instance, parameter B in (11) may double its value, becoming $B \sim 1$, for very strong earthquakes.

4. Accompanying seismic activity. Omori's law

The above description may be viewed as pertaining to "regular" earthquakes, characterized by a mean recurrence time. Such "regular" seismic events may be accompanied by an associated seismic activity, like foreshocks and aftershocks, in which case a "regular" earthquake is referred to as the main shock. Since 1894, when Omori suggested that seismic aftershocks are distributed according to $\sim 1/\tau^{\gamma}$, where $\gamma = 1^{+}$ and τ denotes the time elapsed from the main shock [13], the seismic activity accompanying a major earthquake is a matter of debate. One of the major difficulties in advancing knowledge in this subject is the lack of means for distinguishing between seismic events genuinely accompanying a main shock and other, "regular" seisms, superposed over the associated seismic activity, which may possibly belong to other "regular" time series of seismic activity, without any relationship with the main seismic shock. Statistical distributions of such events, both in time, magnitude and energy, may help in operating such a distinction, and it was precisely in this direction where progress has been recorded recently, especially in connection with the critical-point theory of foreshocks and aftershocks, as based on self-organized criticality [14–17].

It is assumed here that there may exist an associated seismic activity accompanying a main seismic event, as seismic foreshocks and aftershocks, and this whole "secondary" seismic activity forms a statistical ensemble, i.e. is described by probability distributions.

Let the main shock occurs at a critical time $t_c = 0$, and measure time τ of the accompanying seismic activity with respect to this initial moment of time. Time τ takes both positive values, for aftershocks, and negative values, for foreshocks. As this seismic activity corresponds to pairs of events separated by time τ, then the corresponding statistical distributions are functions of the absolute value $|\tau|$ of time τ, as pointed out in earlier studies [18]. It was shown recently [19] that the associated seismic activity proceeds by the self-replication of a generating distribution of accompanying events, the self-consistency of the process requiring an exponential form for the generating distribution. It amounts to viewing the accompanying seismic activity as a relaxation to equilibrium of the seismic zone, and the corresponding statistical distribution $p(\tau)$ can be obtained formally by using the principle of the maximal entropy $S = -\int d\tau \cdot p(\tau) \ln p(\tau)$. In order to fully characterize this associated seismic activity, a mean value t_c' of its duration may be introduced, where t_c' may be viewed as a characteristic scale time. By standard procedure the temporal probability distribution

$$p(\tau) = \alpha e^{-\alpha|\tau|}, \quad \alpha = 1/t_c' \tag{14}$$

is obtained straightforwardly, as the generating distribution for seisms accompanying a main shock. In general, the characteristic time t_c' may depend not only on the nature of the seismic source and the seismic zone, but also on the magnitude of the main shock. On the other hand, the distribution of the accompanying events can be obtained directly from (8) by expanding the temporal probability of the main shocks in powers of $|\tau|$ in the neighbourhood of a main shock with mean recurrence time t_r. It is easy to see that replacing $t = t_r$ by $t = t_r + |\tau|$ in (8), where $|\tau| \ll t_r$, the time distribution $p(\tau) \sim (1 + |\tau|/t_r)^{-2} \sim e^{-2|\tau|/t_r}$ can be extracted, as corresponding to the accompanying seismic activity. It follows that parameter α in (14) is given by $\alpha = 2/t_r$, and the characteristic time $t'_c = t_r/2$, where t_r is the mean recurrence time of the main shock, as given by (7) or (13). For large values of time t_r the distribution of the accompanying events has a long tail, but the corresponding time probability is very low. In contrast, the accompanying seismic activity ends quickly for small main shocks, characterized by a small value of the mean recurrence time t_r.

It was shown [19] that the self-replication process of the generating distribution given by (14) leads to the distribution $P(\tau) = \alpha/(e^{\alpha|\tau|} - 1)$ for the seismic events accompanying a major earthquake, which is Omori's law $P(\tau) = 1/|\tau|$ for $\alpha\tau \ll 1$. It may be extended to $\tau \to \infty$ as $P(\tau) = \tau_c^{\gamma-1}/|\tau|^{\gamma}$, where $\gamma = 1^+$ and τ_c is a lower-bound cutoff. This result is valid in general, for any finite generating distribution p, the two distribution p and P being inter-related by Euler's transform. This relationship provides also a generalized Omori's law [19]. According to Omori's law, the accompanying events are concentrated in the neighbourhood of the lower-bound cutoff τ_c. It might also be noted, according to Omori's law, that number n of associated seismic events goes like $dn/d\tau \sim 1/|\tau|$ [14,20].

A distribution similar to (14) holds also for the difference in magnitude of the associated seisms with respect to the main shock. Indeed, according to (10), the magnitude distribution can be written as $\sim e^{-\beta m} e^{-\beta M}$ for a main shock of magnitude M_0, where $m = M_0 - M$ is the difference in magnitude between the main shock and an accompanying seismic event of magnitude M. Negative values for the statistical variable $m = M - M_0$ must be allowed in such a distribution, which leads to $\beta e^{-\beta|m|}$ for the distribution in magnitude difference, as suggested previously [21]. It may also be noted that such a distribution can be obtained by the principle of the maximal entropy as $\beta' e^{-\beta'|m|}$, and, since this probability is equal to the probability of the main shock at $m = 0$, it follows that $\beta' = \beta$. Another remark might also be that associated earthquakes do follow the same exponential distribution in magnitude like the "regular" earthquakes.

It is worth noting that, by making use of the exponential distribution in magnitude difference and the temporal distribution given by (14), the time dependence $|m| = (\alpha/\beta)|\tau|$ is obtained, or $dm/d\tau = \alpha/\beta$, or, equivalently, the time dependence $M = M_0 - (\alpha/\beta)|\tau|$ of the magnitude of the accompanying seisms. It may be estimated that the associated seismic activity is extinct in time $\tau_0 = \beta M_0/\alpha = \beta M_0 t'_c$, though the long-tail values of the probability distributions of the accompanying seismic activity are very small. As described above, for small values of m ($|m| < 1/\beta$) the distribution in magnitude difference obeys the same Omori-type law $\sim dm/|m|$ (the lower bound corresponding to $m_c = (\alpha/\beta)\tau_c$). The mean difference in magnitude $\langle m \rangle$ vanishes for the distribution $\beta e^{-\beta|m|}$ ($\langle m \rangle = 0$), so it is reasonable to employ the standard deviation $\delta m = \langle m^2 \rangle^{1/2} = \sqrt{2}/\beta$ as a measure of the mean deviation in magnitudes of the accompanying seismic activity. Such an estimation is also consistent with the assumption that the associated seismic activity represents a relaxation regime of the seismic activity. Making use of $\beta \simeq 1.17$ the value $\delta m = \sqrt{2}/\beta \simeq 1.2$ is obtained, which is suggestive for the numerical value indicated by Bath's empirical law [22]. A similar analysis, though on a different conceptual basis, was made recently for the accompanying seismic activity [16,17,23]. It might be noted that the self-replication process is not included in estimating the standard deviation of the magnitude, and the variance $\delta m = \sqrt{2}/\beta$ occurs in time $\tau_B = (\beta/\alpha)\delta m = \sqrt{2} t'_c$.

The probability $\Pi(\Delta E)$ for two earthquakes separated in energy by ΔE is given by $P(E)\Pi(\Delta E) = P(E + \Delta E)$, where $P(E)$ is the probability (9). One obtains $\Pi(\Delta E) = E^{1+r}/(E + \Delta E)^{1+r} = (1 + \Delta E/E)^{-1-r}$, and for fixed ΔE we can see that the resulting decomposition indicates that the statistical variable corresponding to the energy for the accompanying seisms is actually $x = 1/E$. For small values of ΔE one may take $\Delta E = E_0$, so that the "energy" distribution $(1 + E_0/E)^{-1-r} = \exp[-(1 + r)\ln(1 + E_0/E)]$ can be written down for the associated seismic activity, or

$$p(x) \simeq E_0(1 + r)e^{-(1+r)E_0 x}, \quad x = \frac{1}{E}. \tag{15}$$

It may be noted that this distribution is similar to the exponential distributions in time, or magnitude, with a characteristic scale energy $(1 + r)E_0$. By comparing (15) and (14) the time dependence $E = (1 + r)E_0 t'_c/|\tau|$ of the released energy is obtained straightforwardly, which corresponds to the rate

$$dE/d|\tau| = -(1 + r)E_0 t'_c/\tau^2 \tag{16}$$

of the energy released in the accompanying seismic activity. Such an $\sim 1/\tau^2$-law seems to be supported by empirical data [14,20]. Similarly, the magnitude dependence $E = (1 + r)E_0/\beta|m|$ is obtained for the released energy, as well as an Omori-type law $\sim dx/x = -dE/E$.

In conclusion, a model is introduced here for the accumulation of the seismic energy in a localized focus, which implies a geometric parameter r, and statistical distributions in time, energy and magnitude are derived on this basis for regular earthquakes. Omori's distributions are discussed for the seismic activity accompanying a main seismic shock and the time dependence (16) is given for the released energy in an accompanying seismic activity.

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