WARD IDENTITY FOR NON-RELATIVISTIC FERMIONS

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The Ward identity, relating the three-legged vertex function to the Green function, is derived for non-relativistic fermions interacting through a two-body force by using the equation-of-motion technique.

The Feynman–Dyson perturbation theory for non-relativistic fermions can be formulated in terms of three fundamental quantities [1]: the one-particle Green function, the effective interaction and the three-legged vertex function. The three-legged vertex function represents all the diagrams with three external lines (one interaction line and two particle lines). The Green function and the effective interaction are related to the vertex function by the Dyson equations (via proper self-energy part and irreducible polarization, respectively) [‡1]. In order to get the third equation one can look for the Ward identity which, as known from quantum electrodynamics [2], relates the three-legged vertex function to the Green function. The Ward identity has been derived by diagrammatic methods [3] and the equation-of-motion technique [4] for the one-dimensional Fermi system, when the unperturbed one-fermion energy levels are linear in the wavevector. We obtain here the Ward identity for the general case of non-relativistic fermions moving in three-dimensional space and interacting through a two-body force.

The hamiltonian of the system is given by

$$H = H_0 + H_1 , \tag{1a}$$

$$H_0 = -\frac{1}{2m}\int \mathrm{d}\boldsymbol{x}\ \psi^+(\boldsymbol{x})\, \Delta\, \psi(\boldsymbol{x}) , \tag{1b}$$

$$H_1 = \frac{1}{2}\int \mathrm{d}\boldsymbol{x}\ \mathrm{d}\boldsymbol{y}\ V(|\boldsymbol{x}-\boldsymbol{y}|)\, \psi^+(\boldsymbol{x})\, \psi^+(\boldsymbol{y})\, \psi(\boldsymbol{y})\, \psi(\boldsymbol{x}) , \tag{1c}$$

[‡1] The zero-momentum component of the interaction is set equal to zero, so that the tadpole diagrams are excluded.

where $\psi(\boldsymbol{x})$ and $\psi^+(\boldsymbol{x})$ are the field operators of the fermions with mass m interacting through the potential $v(|\boldsymbol{x}-\boldsymbol{y}|)$ (spin index is omitted for simplicity). We use the plane wave representation for these operators,

$$\psi(\boldsymbol{x}) = \sum_{\boldsymbol{p}} c_{\boldsymbol{p}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{x}} , \quad \psi^+(\boldsymbol{x}) = \sum_{\boldsymbol{p}} c^+_{\boldsymbol{p}}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\boldsymbol{x}} , \tag{2}$$

$c_{\boldsymbol{p}}$ and $c^+_{\boldsymbol{p}}$ being the annihilation and creation operators of the one-fermion state labelled by the wavevector $\boldsymbol{p}$. Let us define the three-point vertex function

$$K(x_1,x_2,x_3) = -\langle 0|Tn(x_1)\, \psi(x_2)\, \psi^+(x_3)|0\rangle/\langle 0|0\rangle , \tag{3}$$

where the operators are written in the Heisenberg picture, the space–time coordinate x_i denotes the pair $(\boldsymbol{x}_i, t_i)$ $(i = 1, 2, 3)$, $n(x_1)$ is the particle density operator,

$$n(x_1) = \psi^+(x_1)\, \psi(x_1) , \tag{4}$$

$|0\rangle$ is the Heisenberg ground state of the system and T represents the time-ordering operator. Due to the space–time invariance of the system the function $K(x_1,x_2,x_3)$ depends on the difference of its variables only. Choosing $x_2 - x_3$ and $x_3 - x_1$ as independent variables this function can be Fourier transformed as

$$K(x_2 - x_3, x_3 - x_1) = (2\pi)^{-8} \times \int \mathrm{d}p\, \mathrm{d}k\, K(p,k) \exp[\mathrm{i}p(x_2 - x_3)] \exp[\mathrm{i}k(x_3 - x_1)] , \tag{5}$$

where the variables p and k denote, respectively, the wavevector-frequency pairs $(\boldsymbol{p}, \epsilon)$ and $(\boldsymbol{k}, \omega)$. The scalar product in eq. (5) is taken, as usually, $px = \boldsymbol{p}\boldsymbol{x} - \epsilon t$.

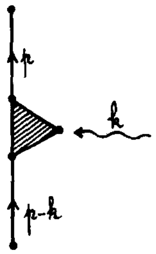

Fig. 1. The skeleton diagram of the three-point vertex function $K(p, k)$. The Green functions and the effective interaction are represented by full lines and wavy line, respectively. The dashed triangle represents the three-legged vertex function $\Gamma(p, k)$.

Using the interaction picture and the evolution operator of the perturbation theory it is easy to see that all the connected diagrams which contribute to the function $K(x_1, x_2, x_3)$ can be represented by the skeleton diagram shown in fig. 1 (in momentum space). The analytic expression of this diagram is

$$K(p,k) = G(p)\,G(p-k)\,\Gamma(p,k)\,, \tag{6}$$

where $G(p)$ and $G(p-k)$ are one-particle Green functions and $\Gamma(p,k)$ is the three-legged vertex function. Taking the time derivative of the three-point vertex function with respect to the variable t_1 we get

$$(\partial/\partial t_1)\,K(x_2-x_3, x_3-x_1) = \mathrm{i}(2\pi)^{-8} \times \int \mathrm{d}p\,\mathrm{d}k\,\omega K(p,k)\exp[\mathrm{i}p(x_2-x_3)]\exp[\mathrm{i}k(x_3-x_1)]\,. \tag{7}$$

The time-ordering operator in eq. (3) can be expressed with the step function $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t < 0$. Taking the time derivative of eq. (3) with respect to t_1 and using the fermion anticommutators of the equal-time field operators we get

$$(\partial/\partial t_1)\,K(x_1,x_2,x_3) = -\langle 0|T(\partial/\partial t_1)\,n(x_1)\,\psi(x_2)\,\psi^+(x_3)|0\rangle/\langle 0|0\rangle + \mathrm{i}\delta(x_1-x_2)\,G(x_1-x_3) - \mathrm{i}\delta(x_1-x_3)\,G(x_2-x_1)\,. \tag{8}$$

The time derivative of the particle density operator is given by the Heisenberg equation of motion

$$-\mathrm{i}(\partial/\partial t)\,n(x) = [H, n(x)] = [H_0, n(x)] = \psi^+(x)\left[\frac{1}{2m}\Delta\psi(x)\right] - \left[\frac{1}{2m}\Delta\psi^+(x)\right]\psi(x)\,. \tag{9}$$

Introducing this result in eq. (8) and Fourier transforming the right-hand-side of this equation we get, after straightforward calculation,

$$(\partial/\partial t_1)\,K(x_1,x_2,x_3) = \mathrm{i}(2\pi)^{-8} \times \int \mathrm{d}p\,\mathrm{d}k\,[(\epsilon_p - \epsilon_{p-k})\,G(p)\,G(p-k)\,\Gamma(p,k) - G(p) + G(p-k)]\exp[\mathrm{i}p(x_2-x_3)]\exp[\mathrm{i}k(x_3-x_1)]\,, \tag{10}$$

where $\epsilon_p = p^2/2m$ are the energy levels of the free fermions.

Comparing eq. (7) with eq. (10) we obtain the Ward identity

$$\Gamma(p,k) = \frac{[G^{-1}(p) - G^{-1}(p-k)]}{(\epsilon_{p-k} - \epsilon_p + \omega)}\,, \tag{11}$$

which is similar to that from quantum electrodynamics. In the case of spin-dependent interaction eq. (11) can be directly generalized to

$$G_{\rho\alpha}(p)\,\Gamma_{\mu\nu,\alpha\beta}(p,k)\,G_{\beta\lambda}(p-k) = \frac{\delta_{\rho\mu}G_{\nu\lambda}(p-k) - \delta_{\nu\lambda}G_{\rho\mu}(p)}{\epsilon_{p-k} - \epsilon_p + \omega}\,. \tag{12}$$

For one-dimensional Fermi systems the unperturbed spectrum ϵ_p becomes linear in the wavevector p and eq. (11) reduces to a result previously derived in refs. [3,4]. The usefulness of the relation (11) in the study of the one-dimensional Fermi systems will be discussed elsewhere.

References

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