ONE-ELECTRON GREEN FUNCTION FOR ELECTRONS
COUPLED WITH ACOUSTICAL PHONONS IN ONE DIMENSION

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The Ward identity for the one-dimensional many-electron system coupled with acoustical phonons is used to get the one-electron Green function. There are no single-electron excitations in this system and the electron momentum distribution is continuous at the Fermi points. Migdal's theorem is checked for this model.

The one-dimensional electron gas distinguishes itself by a linear Fermi sea which is a segment of straight line ending at $\pm k_F$, k_F being the Fermi momentum. As the low excited states of this system consist of electron–hole pairs excited in the neighbourhood of $\pm k_F$ we may use the linearized expression $\epsilon_p = \mu + |p| - k_F$ for the unperturbed one-electron levels, where μ is the chemical potential and the Fermi velocity v_F is set equal to unity ($v_F = 1$). The shortest wavelength of the acoustical phonons is assumed to be much larger than the average electron spacing so that the electron density fluctuations which couple to the phonons are well-defined collective waves. The electron–phonon coupling is described by the hamiltonian

$$H = \sum_p \epsilon_p a_p^+ a_p + \sum_q \omega_q b_q^+ b_q + g \sum_{p,q} (\omega_q/2)^{1/2} a_{p+q}^+ a_p (b_q + b_{-q}^+), \tag{1}$$

where a_p (a_p^+) and b_q (b_q^+) are electron and phonon annihilation (creation) operators, respectively, $\omega_q = c|q|$ is the bare phonon frequency (c being the unrenormalized sound velocity) and g is the bare electron–phonon coupling constant. The spin index is omitted but the spin contribution (a factor 2) will be counted in the number of electronic states.

It was shown [1] that the electron–phonon vertex function $\Gamma(p, \epsilon; q, \omega)$ is connected with the one-electron Green function $G(p, \epsilon)$ by the Ward identity [1]

$$\Gamma_\pm(p, \epsilon; q, \omega) = g[G_\pm^{-1}(p, \epsilon) - G_\pm^{-1}(p-q, \epsilon-\omega)]/(\omega \mp q) = g\{1 + [\Sigma_\pm(p-q, \epsilon-\omega) - \Sigma_\pm(p, \epsilon)]/(\omega \mp q)\}, \tag{2}$$

where $\Sigma(p, \epsilon)$ is the electron proper self-energy and the upper (lower) sign refers to the processes which take place near $+k_F$ $(-k_F)$. The Ward identity is an exact relation which is based upon the linearized electron spectrum and

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[1] The Ward identity has also been derived for the electron–electron interacting system with forward scattering in one dimension [2].

the conservation of the number of electrons. A straightforward diagrammatic analysis shows that the Ward identity leads to the exact cancellation of all the diagrams which contain loops with more than two electron lines. The cancellation of these diagrams among themselves (non-Tomonaga diagrams) has been checked [3] to high orders of perturbation theory.

Eq. (2) is particularly useful to calculate the irreducible polarization

$$\Pi_{\pm}(q, \omega) = -2ig(2\pi)^{-2} \int dp \ d\epsilon \ G_{\pm}(p, \epsilon) G_{\pm}(p - q, \epsilon - \omega) \Gamma_{\pm}(p, \epsilon; q, \omega) = [\lambda/(\omega \mp q)] \int dp \ (n_{p-q}^{\pm} - n_p^{\pm}), \quad (3)$$

where $\lambda = g^2/\pi$ is the dimensionless electron–phonon coupling constant and $n_p^{\pm}$ is the momentum distribution of electrons near $\pm k_F$. As the interaction leaves the electron number unchanged we may insert the free electron distribution in eq. (3) to get the total polarization

$$\Pi(q, \omega) = \Pi_+(q, \omega) + \Pi_-(q, \omega) = 2\lambda q^2/(\omega^2 - q^2). \quad (4)$$

Hence we obtain straightforwardly the phonon Green function

$$D(q, \omega) = \omega_q^2(\omega^2 - q^2)/(\omega^2 - v_1^2 q^2 + i0^+)(\omega^2 - v_2^2 q^2 + i0^+), \quad v_{1,2}^2 = \tfrac{1}{2}\{1 + c^2 \pm [(1 - c^2)^2 + 8c^2\lambda]^{1/2}\} \quad (5)$$

It results that two types of collective excitations are present in this system, the electron–hole pairs with velocity v_1 and the dressed phonons with velocity v_2. These elementary excitations have also been obtained [3] by using the bosonization technique and the summation of bubble diagrams. The sound velocity is proportional to the adiabatic parameter $(m/M)^{1/2}$, where m and M are the electronic and ionic mass, respectively. Since the usual values of this parameter are very small we may use the expressions $v_1 \approx (1 + 2\lambda c^2)^{1/2}$ and $v_2 \approx c(1 - 2\lambda)^{1/2}$ for the two group velocities.

The Dyson equation for the one-electron Green function can be written as

$$(\epsilon - p + k_F)G_+(p, \epsilon) = 1 + i(2\pi)^{-2} \int dq \ d\omega \ gD(q, \omega)\Gamma_+(p, \epsilon; q, \omega)G_+(p - q, \epsilon - \omega)G_+(p, \epsilon). \quad (6)$$

Hereafter we shall drop the + index and measure both the energy and momentum from the corresponding Fermi values. By inserting eqs. (2) and (5) in eq. (6) we get

$$(\epsilon - p)G(p, \epsilon) = 1 + i(2\pi)^{-2} \int dq \ d\omega \ g^2 \frac{c^2(\omega + q)}{v_1^2 - v_2^2} \left[\frac{1}{\omega^2 - v_1^2 q^2 + i0^+} - \frac{1}{\omega^2 - v_2^2 q^2 + i0^+} \right] G(p - q, \epsilon - \omega), \quad (7)$$

where the shift in the chemical potential was found to be zero by direct calculation. Though a bit more complicated than the corresponding equation for the electron–electron interaction [4] eq. (7) can be solved by Fourier transforming. An exponential cutoff factor $\exp(-|q|/\Lambda)$ is introduced in the integration over momentum to account for the electronic states in the neighbourhood of the Fermi point. After performing the integration the differential equation which is obtained may easily be solved for the one-electron Green function in the coordinate representation

$$G(x, t) = (2\pi)^{-1}\Lambda^{-2(\beta_1 - \beta_2)}\{\exp[ik_F(x - t)]/[x - t - i0^+\text{sgn}(t)]\}[x - t + i\Lambda^{-1}(t)]^{-\alpha_1 + \alpha_2 - \beta_1 + \beta_2}$$

$$\times [x - v_1 t + i\Lambda^{-1}(t)]^{\alpha_1}[x - v_2 t + i\Lambda^{-1}(t)]^{-\alpha_2}[x + v_1 t - i\Lambda^{-1}(t)]^{-\beta_1}[x + v_2 t - i\Lambda^{-1}(t)]^{\beta_2}, \quad (8)$$

where

$$\alpha_{1,2} = [\lambda c^2/4(v_1^2 - v_2^2)](1 + v_{1,2}^{-1})/(1 - v_{1,2}), \quad \beta_{1,2} = [\lambda c^2/4(v_1^2 - v_2^2)](1 - v_{1,2}^{-1})/(1 + v_{1,2}), \quad (9)$$

and $\Lambda(t) = \Lambda \ \text{sgn}(t)$. Due to the cutoff factor over momentum this expression is valid in the limits $x, t \gg \Lambda^{-1}$ and $x, t \ll \Lambda^{-1}$. It may easily be checked that eq. (8) satisfies the condition $G(x, 0^+) - G(x, 0^-) = -i\delta(x)$ and $G(x, t)$ is an analytic function in the quadrants Re $t > 0$, Im $t < 0$ and Re $t < 0$, Im $t > 0$. Since eq. (8) contains the sound

velocity c as a small parameter we may keep the terms linear in c only, thus arriving at

$$G(x, t) = (2\pi)^{-1} \{ \exp[ik_F(x - t)] / [x - t - i0^+ \mathrm{sgn}(t)] \} (1 + \Lambda^2 x^2)^{-\alpha}, \quad \alpha = \lambda c/4(1 - 2\lambda)^{1/2}, \tag{10}$$

the Fourier transform of which is [5]

$$G(p, \epsilon) = \tfrac{1}{2} \Lambda^{-2} (\epsilon - p)(\alpha - 1)^{-1} \,_1F_2(1; 2 - \alpha, \tfrac{3}{2}; |\epsilon - p|^2/4\Lambda^2)$$

$$+ \tfrac{1}{2} \Lambda^{-1} \mathrm{sgn}(\epsilon - p) \sqrt{\pi} \, (|\epsilon - p|/2\Lambda)^{2\alpha - 1} [\Gamma(1 - \alpha)/\Gamma(\tfrac{1}{2} + \alpha)] \,_1F_2(\alpha; \alpha + \tfrac{1}{2}, \alpha; |\epsilon - p|^2/4\Lambda^2)$$

$$- i\Lambda^{-1} \mathrm{sgn}(\epsilon)(|\epsilon - p|/2\Lambda)^{\alpha - 1/2} [\sqrt{\pi}/\Gamma(\alpha)] K_{\alpha - 1/2}(|\epsilon - p|/\Lambda), \tag{11}$$

$_1F_2$ being the generalized hypergeometric series and $K_{\alpha - 1/2}$ the cylindrical function of imaginary argument. We may conclude by looking at eqs. (8) and (11) that, irrespective of the coupling strength λ $(0 < \lambda < \tfrac{1}{2})$ and small α approximation, $G(p, \epsilon)$ has no simple poles in the complex energy plane, that is, there are no single electron excitations. Thus, as in the case of the one-dimensional many-electron interacting system with forward scattering [6], the basic assumption of Landau's Fermi-liquid theory is not fulfilled. Consequently, the only excitations which are present in this system are those given by the poles appearing in eq. (5). This is why the hamiltonian given by eq. (1) could have been diagonalized by using the Tomonaga bosonization technique [3]. It is noteworthy that our expression for $G(p, \epsilon)$ given by eq. (11) fulfills the sum rule

$$\int\limits_0^\infty d\epsilon \, [A(p, \epsilon) + B(p, \epsilon)] = 1, \quad A(p, \epsilon) = -\pi^{-1} \mathrm{Im} \, G(p, \epsilon), \quad B(p, \epsilon) = \pi^{-1} \mathrm{Im} \, G(p, -\epsilon).$$

Indeed, we have

$$\int\limits_0^\infty d\epsilon \, [A(p, \epsilon) + B(p, \epsilon)] = [2^{\alpha - 1/2} \Gamma(\alpha) \sqrt{\pi}]^{-1} \left[\int\limits_{-|p|/\Lambda}^\infty dx \, |x|^{\alpha - 1/2} K_{\alpha - 1/2}(|x|) + \int\limits_{|p|/\Lambda}^\infty dx \, x^{\alpha - 1/2} K_{\alpha - 1/2}(x) \right], \tag{12}$$

which in the limit $|p| \ll \Lambda$ gives 1 [7]. The momentum distribution of electrons near $+k_F$,

$$n_p = -(i/2\pi) \int\limits_{-\infty}^{+\infty} dx \, e^{-ipx} G(x, 0^-) = \tfrac{1}{2} - (2\pi)^{-1} \int\limits_{-\infty}^{+\infty} dx \, x^{-1} \sin px \, (1 + \Lambda^2 x^2)^{-\beta_1 + \beta_2}, \tag{13}$$

can easily be calculated [5]. For $c \ll 1$ and $|p| \ll \Lambda$ we get

$$n_p = \tfrac{1}{2} - \pi^{-1} \mathrm{sgn}(p) g(|p|/\Delta), \tag{14}$$

where

$$g(x) = \tfrac{1}{2}\sqrt{\pi} \, [\Gamma(\alpha - \tfrac{1}{2})/\Gamma(\alpha)] \, x \qquad \text{for } \tfrac{1}{2} - c^2/32 < \lambda < \tfrac{1}{2},$$

$$= \tfrac{1}{2}\sqrt{\pi} \, [\Gamma(\tfrac{1}{2} - \alpha)/\Gamma(1 + \alpha)] (x/2)^{2\alpha} \quad \text{for } 0 < \lambda < \tfrac{1}{2} - c^2/32. \tag{15}$$

One can conclude that there is no jump in the momentum distribution at the Fermi points and the number of electrons is conserved ($n_p + n_{-p} = 1$). This result is similar to that corresponding to the electron–electron interaction in one dimension [4].

By expanding the Green function given by eq. (11) in powers of α we obtain the leading contribution to the electron self-energy

$$\Sigma(p, \epsilon) = \alpha 2\Lambda z \{ 2z^2 f_1(z) + 2 \ln |z| [2z^2 f_2(z) - 1] - f_3(z) + i\pi \, \mathrm{sgn}(\epsilon z) \, e^{-2|z|} \}, \tag{16}$$

where

$$z = (\epsilon - p)/2\Lambda, \quad f_1(z) = \sum_{k=0}^{\infty} [z^{2k}/(k+2)!] [\psi(k+3) + C], \quad f_2(z) = {}_1F_2(1;2,\tfrac{3}{2};z^2),$$

$$f_3(z) = \sum_{k=0}^{\infty} [(2z)^{2k}/k!] [\psi(2k+2) + \psi(\tfrac{1}{2})],$$

(17)

ψ being the logarithmic derivative of the gamma function and C Euler's constant. This expression is valid for $|\alpha \ln|z|| \ll 1$, which means $\exp(-\text{const} \times \sqrt{M/m}) \ll |\epsilon - p|/2\Lambda \ll \exp(\text{const} \times \sqrt{M/m})$, and for small values of the adiabatic parameter $(m/M)^{1/2}$. It results that for $z \neq 0$ and in the limit $m/M \ll 1$ the electron self-energy is $O(\sqrt{m/M})$. Before making use of this result in discussing eq. (2) let us first remark that we are not interested in the vertex function itself but rather in other quantities which are expressed in terms of Γ by integrals. There are three types of singularities which may arise in the integrands of these quantities: (i) $\epsilon = p$, (ii) $\omega = q$, (iii) $\omega = q \rightarrow 0$. The singularity of the type (i) is removable due to the analytic structure of the one-electron Green function (singularity weaker than a pole). The same is true for the singularity of type (ii) since the poles of the phonon Green function are never of the form $\omega = q$ (these being virtual phonons). In the case (iii) we have from eqs. (2) and (16) $[\Sigma(p-q, \epsilon-\omega) - \Sigma(p, \epsilon)]/(\omega - q) \rightarrow \partial\Sigma(p, \epsilon)/\partial p = O(\sqrt{m/M})$. Therefore, we may conclude that in treating the electron–phonon interaction in one dimension, the vertex function may satisfactorily be approximated by $\Gamma \cong g[1 + O(\sqrt{m/M})]$. This is what Migdal's theorem [8] claims for the three-dimensional case. We have thereby checked this theorem for the present exactly soluble one-dimensional model.

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