JORDAN'S BOSON REPRESENTATION FOR THE ONE-DIMENSIONAL TWO-FERMION MODEL

M. APOSTOL
Department of Fundamental Physics, Institute for Physics and Nuclear Engineering, MG-6, Magurele, Bucharest, Romania

Received 18 March 1982
Revised manuscript received 5 June 1982

It is shown that the Jordan commutator has been overlooked so far in the theory of the one-dimensional two-fermion model. The Jordan's boson representation is generalized to this model and the proper use of the α cut-off parameter is introduced.

The boson representation for the one-dimensional two-fermion model (Tomonaga–Luttinger model without interaction) previously proposed by Luther and Peschel [1] and Mattis [2] was recently discussed by Haldane [3]. A major lack of the boson representation given in refs. [1,2] is the contribution of the zero-modes associated with the particle-number operators. These zero-modes were consistently taken into account by Haldane [3] who derived also the complete form of the bosonized kinetic hamiltonian of the system. The boson representation of the fermion fields given in ref. [3] looks very much the same as that encountered in the field-theoretical literature [4] and, in fact, it was derived many years ago by Jordan [5] for a single fermion field in one dimension.

An important feature of the boson representation of the fermion fields in one dimension is the prescription of introducing the cut-off parameter α. As the particle density is infinitely large a proper cut-off procedure is required which should ensure the correct commutation relations and provide a practical way of handling with the divergencies. The boson representation given by Luther and Peschel [1] is not normal ordered in boson operators. When normal ordering is attempted factors appear which contain divergent summations over an infinite range of wavevectors. The cut-off parameter α was introduced by Luther and Peschel [1] in the boson representation in such a way as to simply ensure the convergence of these sums. However, this cut-off procedure was proved to lead to some inconsistencies in treating the one-dimensional two-fermion model [6]. Haldane [3] used a normal ordered boson representation, as did Mattis [2], so that the cut-off parameter α is no longer needed in the bosonized expression of the fermion fields. However, products of two or more fermion fields have to be calculated (to evaluate the anticommutators, correlation functions, etc.) and therefore the normal ordering problem arises again. In order to make finite the summations over wavevectors appearing in the problems of this type Haldane [3] pointed out an essentially similar cut-off procedure as that given in ref. [1], although the interpretation of the parameter α in ref. [3] differs from that given by Luther and Peschel [1]. The absence of the cut-off parameter α in the bosonized expression of the fermion fields given by Haldane [3] removes the aforementioned inconsistencies of the two-fermion model. However there is a quantity not explicitly discussed in these previous works which was pointed out long ago by Jordan [5] and which will be hereafter referred to as Jordan's commutator. This commutator plays the role of a supplementary condition which has to be satisfied by the boson representation and its importance is directly connected to a consistent way of renormalizing the infinitely large density of fermions. One can easily see that the cut-off procedures proposed in refs. [1,3] do not make the boson representation satisfy the Jordan commutator. The proper cut-off procedure was suggested by Jordan [5]. The aim of the present letter is to draw attention to Jordan's commutator and to introduce the proper cut-off procedure as well as to generalize the Jordan theory to the two-fermion model.

0 031-9163/82/0000–0000/$02.75

Let us suppose that the fermion field in one dimension $\psi(x) = \Sigma_p\, e^{ipx} a_p$, $p = 2\pi \times$ integer, is governed by the kinetic hamiltonian

$$H_0 = \sum_{p>0} p a_p^+ a_p - \sum_{p\leqslant 0} p a_p a_p^+ = \sum_{p>0} p a_p^+ a_p + \sum_{p\leqslant 0} p(a_p^+ a_p - 1)\,. \tag{1}$$

The ground-state of this system is infinitely filled with particles from $p = -\infty$ to $p = k_F$, k_F being the Fermi momentum. Instead of working with the particle-number operator $\Sigma_p\, a_p^+ a_p$ which has infinite values when acting upon the states of the system, Jordan [5] introduced the charge operator

$$B = \sum_{p>0} a_p^+ a_p - \sum_{p\leqslant 0} a_p a_p^+ = \sum_{p>0} a_p^+ a_p + \sum_{p\leqslant 0} (a_p^+ a_p - 1)\,, \tag{2}$$

which counts the particles with $p > 0$ minus the holes with $p \leqslant 0$. Owing to the infinite filling of the states the particle-density operator $\psi^+(x)\psi(x)$ has infinite values:

$$\psi^+(x)\psi(x) = \sum_{p\leqslant 0} 1 + B + (2\pi)^{-1}[F(x) + F^+(x)]\,, \quad \psi(x)\psi^+(x) = \sum_{p>0} 1 - B - (2\pi)^{-1}[F(x) + F^+(x)]\,, \tag{3}$$

where $F(x) = 2\pi\Sigma_{k>0}\, e^{ikx}\rho(-k)$, $\rho(-k) = \rho^+(k) = \Sigma_p\, a_p^+ a_{p+k}$ are the Fourier components of the particle-density operator. In order to extract these infinities as c-numbers Jordan [5] introduced the cut-off parameter $\alpha > 0$ by

$$\psi^+(x)\psi(y) = \lim_{\alpha\to 0}\, [\psi(x - i\alpha/2)]^+ \psi(y - i\alpha/2), \quad \psi(x)\psi^+(y) = \lim_{\alpha\to 0}\, \psi(x + i\alpha/2)[\psi(y + i\alpha/2)]^+\,, \tag{4}$$

and found

$$[\psi(x - i\alpha/2)]^+\psi(x - i\alpha/2) = (2\pi\alpha)^{-1} + B + (2\pi)^{-1}[F(x) + F^+(x)] + O(\alpha)\,,$$

$$\psi(x + i\alpha/2)[\psi(x + i\alpha/2)]^+ = (2\pi\alpha)^{-1} - B - (2\pi)^{-1}[F(x) + F^+(x)] + O(\alpha)\,. \tag{5}$$

One can see from eq. (5) that α^{-1} may be interpreted as a bandwidth cut-off. For $\alpha \to 0$ we get from eq. (5) the Jordan commutator

$$[\psi^+(x), \psi(x)] = 2B + \pi^{-1}[F(x) + F^+(x)]\,. \tag{6}$$

Jordan [5] proved that $\rho(-k)$ satisfy rigorously boson-like commutation relations $[\rho(-k), \rho^+(-k')] = (2\pi)^{-1} k\delta_{kk'}$ and established the boson representation

$$\psi(x) = cS^{-1}\exp[i2\pi(B - 1/2)x]\, \exp\left(-2\pi \sum_{k>0} k^{-1} e^{-ikx}\rho^+(-k)\right) \exp\left(2\pi \sum_{k>0} [k^{-1} e^{ikx}\rho(-k)]\right), \tag{7}$$

where c is a constant with $|c| = 1$ and S a unitary operator defined by $Sa_pS^{-1} = a_{p+2\pi}$, $Sa_p^+S^{-1} = a_{p+2\pi}^+$ with the properties

$$SBS^{-1} = B - 1\,, \quad SH_0S^{-1} = H_0 - 2\pi(B - 1/2)\,. \tag{8}$$

All the properties of the fermion field $\psi(x)$ (commutators with $\rho(-k)$, B and S, equation of motion, anticommutation relations), Jordan's commutator (6) included, are satisfied by the boson representation (7) provided that Jordan's prescription (4) is followed. Working with the boson representation (7) we encounter sums of the type $f(z) = 2\pi\Sigma_{k>0}\, k^{-1}e^{-kz}$, Re $z \geqslant 0$, $z \neq 0$ which are to be approximated by $f(z) \approx -\ln(2\pi z) + \pi z$. This approximation is valid for $L^{-1}|z| \ll 1$, L being the length of the box the system is confined to (we put $L = 1$ in writing down our equations for simplicity). The complete form of the bosonized kinetic hamiltonian is derived as follows:

$$\sum_{p>0} e^{p\alpha} p a_p^+ a_p - \sum_{p\leqslant 0} e^{p\alpha} p(a_p a_p^+ - 1) = \frac{\partial}{\partial\alpha}\left(\sum_{p>0} e^{p\alpha} a_p^+ a_p - \sum_{p\leqslant 0} e^{p\alpha}(a_p a_p^+ - 1)\right)$$

$$= \frac{\partial}{\partial\alpha}\int dx\, [\psi(x - i\alpha/2)]^+ \psi(x - i\alpha/2)\,, \tag{9}$$

where we simply used the Fourier representation of the $\psi(x)$. By using eq. (7) we obtain

$$[\psi(x - i\alpha/2)]^+ \psi(x - i\alpha/2) = (2\pi\alpha)^{-1} + B + (2\pi)^{-1}[F(x) + F^+(x)] + \pi\alpha :\{B + (2\pi)^{-1}[F(x) + F^+(x)]\}^2: + O(\alpha^2)\,, \tag{10}$$

where : : means normal ordering, so that

$$\int dx\, [\psi(x - i\alpha/2)]^+ \psi(x - i\alpha/2) = (2\pi\alpha)^{-1} + B + \pi\alpha\left(B^2 + 2\sum_{k>0} \rho^+(-k)\rho(-k)\right) + O(\alpha^2)\,, \tag{11}$$

and comparing with eq. (9) we get

$$H_0 = \lim_{\alpha\to 0}\left(\sum_{p>0} e^{p\alpha} p a_p^+ a_p - \sum_{p\leqslant 0} e^{p\alpha} p a_p a_p^+\right) = \pi B^2 + 2\pi \sum_{k>0} \rho^+(-k)\rho(-k)\,. \tag{12}$$

The boson representation (7) and the bosonized hamiltonian (12) are the same as those derived by Haldane [3] by an entirely different method. However it is a matter of a simple algebra to show that the cut-off procedure used in refs. [1,3] does not make the fermion field (7) satisfy Jordan's commutator (6) [and eqs. (5)]. The proper cut-off procedure is that given by eq. (4).

The generalization of the Jordan theory to the two-fermion model is straightforward. This model is described in terms of the fermion fields $\psi_{js}(x) = \Sigma_p e^{ipx} a_{jps}$ ($j = 1, 2$ and $s = \pm 1$ is the spin index) and the kinetic hamiltonian

$$H_0 = \sum_{s,p>0} p a_{1ps}^+ a_{1ps} + \sum_{s,p\leqslant 0} p(a_{1ps}^+ a_{1ps} - 1) - \sum_{s,p<0} p a_{2ps}^+ a_{2ps} - \sum_{s,p\geqslant 0} p(a_{2ps}^+ a_{2ps} - 1)\,, \tag{13}$$

the ground state being filled with $j = 1$ particles from $p = -\infty$ to $p = +k_F$ and with $j = 2$ particles from $p = -k_F$ to $p = +\infty$. The charge operators are

$$B_{1s} = \sum_{p>0} a_{1ps}^+ a_{1ps} + \sum_{p\leqslant 0} (a_{1ps}^+ a_{1ps} - 1)\,,\quad B_{2s} = \sum_{p<0} a_{2ps}^+ a_{2ps} + \sum_{p\geqslant 0} (a_{2ps}^+ a_{2ps} - 1)\,, \tag{14}$$

and the operators S_{js} have the properties

$$S_{js} B_{j's'} S_{js}^{-1} = \delta_{jj'}\delta_{ss'}(B_{js} \mp 1) + (1 - \delta_{jj'}\delta_{ss'})B_{j's'}\,,\quad S_{js} H_0 S_{js}^{-1} = H_0 \mp 2\pi(B_{js} \mp 1/2)\,, \tag{15}$$

the upper (lower) sign corresponding to $j = 1$ (2). The boson representation is

$$\psi_{js}(x) = c_{js} S_{js}^{\mp 1} \exp[\pm i 2\pi(B_{js} - 1/2)x] \exp\left(-2\pi \sum_{k>0} k^{-1} e^{\mp ikx} \rho_{js}^+(\mp k)\right) \exp\left(2\pi \sum_{k>0} k^{-1} e^{\pm ikx} \rho_{js}(\mp k)\right), \tag{16}$$

where c_{js} should satisfy the conditions

$$c_{js}^+ c_{js} = c_{js} c_{js}^+ = 1\,,\quad \{c_{js}, c_{j's'}\} = \{c_{js}, c_{j's'}^+\} = 0\,,\quad (js) \neq (j's')\,, \tag{17}$$

so that the Dirac matrices may be taken as a matrix realization of c_{js}. The representation given by eq. (16) should be used with the following prescription of introducing the cut-off parameter α:

$$\psi_{js}^+(x)\psi_{js}(y) = \lim_{\alpha\to 0}\, [\psi_{js}(x \mp i\alpha/2)]^+ \psi_{js}(y \mp i\alpha/2)\,,\quad \psi_{js}(x)\psi_{js}^+(y) = \lim_{\alpha\to 0}\, \psi_{js}(x \pm i\alpha/2)[\psi_{js}(y \pm i\alpha/2)]^+\,. \tag{18}$$

The bosonized form of H_0 given by eq. (13) is

$$H_0 = \pi \sum_{js} B_{js}^2 + 2\pi \sum_{js, k>0} \rho_{js}^{+}(\mp k)\rho_{js}(\mp k) . \tag{19}$$

All the properties of the field operators listed below

$$[\psi_{js}(x), \rho_{j's'}(\mp k)] = \delta_{jj'}\delta_{ss'}e^{\mp ikx}\psi_{js}(x), \quad [\psi_{js}(x), B_{j's'}] = \delta_{jj'}\delta_{ss'}\psi_{js}(x),$$

$$S_{js}\psi_{j's'}(x)S_{js}^{-1} = \delta_{jj'}\delta_{ss'}e^{-i2\pi x}\psi_{js}(x) + (1-\delta_{jj'}\delta_{ss'})\psi_{j's'}(x), \quad [\psi_{js}(x), H_0] = \mp i\partial\psi_{js}(x)/\partial x ,$$

$$\{\psi_{js}^{+}(x), \psi_{j's'}(y)\} = \delta_{jj'}\delta_{ss'}\delta(x-y), \quad \{\psi_{js}(x), \psi_{j's'}(y)\} = 0 , \quad [\psi_{js}^{+}(x), \psi_{js}(x)] = 2B_{js} + \pi^{-1}[F_{js}(x) + F_{js}^{+}(x)] , \tag{20}$$

where $F_{js}(x) = 2\pi \sum_{k>0} e^{\pm ikx}\rho_{js}(\mp k)$, are satisfied by the boson representation (16) and the cut-off procedure (18). The last line in eq. (20) is the Jordan commutator for the two-fermion model. The aforementioned inconsistencies of the two-fermion model are also removed by the cut-off procedure proposed here, although this procedure differs from that given in ref. [3]. These results will be published in a forthcoming paper.

References

[1] A. Luther and I. Peschel, Phys. Rev. B9 (1974) 2911.
[2] D.C. Mattis, J. Math. Phys. 15 (1974) 609.
[3] F.D.M. Haldane, J. Phys. C12 (1979) 4791; 14 (1981) 2585.
[4] R. Heidenreich, B. Schroer, R. Seiler and D. Uhlenbrok, Phys. Lett. 54A (1975) 119.
[5] P. Jordan, Z. Phys. 93 (1935) 464; 99 (1936) 109; 102 (1936) 243; 105 (1937) 114, 229.
[6] A. Theumann, Phys. Rev. B15 (1977) 4524;
G.S. Grest, Phys. Rev. B14 (1976) 5114;
J. Sólyom, Adv. Phys. 28 (1979) 201.