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Entropic "sound" in the atmosphere

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Abstract

It is shown that small, local disturbances of entropy in the atmosphere may give rise to "sound" waves propagating with a velocity which depends on the amplitude ratio of the local relative variations of temperature and volume. This velocity is much smaller than the mean molecular velocity and the usual, adiabatic sound velocity.

It is well-known[1]$^{-}$[4] that the balance of entropy in the atmosphere is described by the equation

$$\frac{\partial\left(\rho s\right)}{\partial t}+div\left(\rho s\mathbf{c}\right)=f \quad , \tag{1}$$

where ρ is the density of the air, s is the entropy per unit mass, $\mathbf{c}$ is the transport velocity and f is a source term, including the heating rate due to the radiation, diffusion, evaporation and condensation, as well as the internal friction. The detailed content of this source term (which can be found, for example, in Ref.5) is not relevant for the present discussion, since we shall focus ourselves on the homogeneous part of (1). Using the mass continuity the homogeneous part of the equation above becomes

$$\frac{\partial s}{\partial t}+\mathbf{c}grads=0 \quad , \tag{2}$$

which expresses the conservation of entropy. As it is well-known,[6] the entropy of a closed thermodynamic system may only increase as a result of the intermolecular collisions (the H theorem). For an open system however, as the small regions in the atmosphere, the entropy may vary locally by the transfer of the entropy from one region to another, by the free motion of the molecules. This variation is expressed in equation (2) above. This mechanism of local transfer of entropy requires the transport velocity $|\mathbf{c}|$ be much smaller than the mean molecular velocity. For a constant velocity $\mathbf{c}=const$ equation (2) admits plane waves for the general solution.[7] Indeed, with $\mathbf{c}$ oriented along the x-axis (2) amounts to the wave equation

$$\frac{\partial^2 s}{\partial t^2}-c^2\frac{\partial^2 s}{\partial x^2}=0 \quad , \tag{3}$$

whose general solution is a superposition of plane waves with frequency $\omega = cq$, where q is the wavevector. This suggests that small, local perturbations of entropy may propagate in atmosphere with a velocity c. We show below that these entropic waves are actually an entropic "sound".

Indeed, suppose that we have such an entropic wave $s\exp\left[i\left(qx-\omega t\right)\right]$, with the amplitude much smaller than unity; we may also assume that the system is in thermodynamical equilibrium, so that the first principle of thermodynamics (with usual notations) reads

$$dE = -pdV + Tse^{i(qx-\omega t)} \quad . \tag{4}$$

Further on, we assume an ideal gas model for the atmosphere, such that the energy is given by $E = nc_vT + const$, and the equation of state is $pV = nT$, where n is the number of molecules per unit mass, and c_v is the heat capacity (per molecule) at constant volume. We remark that the number of molecules per unit mass is constant, $n = 1/m$, where m is the mass of a molecule. Under this assumption (4) becomes

$$c_v\frac{dT}{T} + \frac{dV}{V} = s_0e^{i(qx-\omega t)} \quad , \tag{5}$$

where $s_0 = s/n = sm$ is the variation of entropy per molecule. Equation (5) implies that the relative variations of temperature and volume may be represented as

$$\frac{dT}{T} = \frac{A}{c_v}e^{i(qx-\omega t)} \quad , \tag{6}$$

$$\frac{dV}{V} = Be^{i(qx-\omega t)} \quad , \tag{7}$$

with

$$A + B = s_0 \quad , \tag{8}$$

the coefficients A and B being otherwise undetermined. Similarly, we get the relative variation of pressure

$$\frac{dp}{p} = \left(A/c_v - B\right)e^{i(qx-\omega t)} \quad . \tag{9}$$

Equations (6), (7) and (9) indicate that an entropic "sound", *i.e.* small, local variations of volume and pressure (and of temperature, as well), is produced by the entropy disturbances, propagating with the velocity c.

On the other hand, it is well-known[8] that such disturbances propagate in a fluid with the velocity v given by

$$v^2 = 1/\kappa\rho \quad , \tag{10}$$

where $\kappa = -(1/V)\partial V/\partial p$ is the fluid compressibility. For $A/c_v - B \neq 0$ we get from (7) and (9)

$$\kappa = \frac{1}{p}\cdot\frac{B}{B - A/c_v} \quad . \tag{11}$$

The (actual) sound proceeds by adiabatic, local compressions and dilations, which correspond to putting $s_0 = 0$ in (8); this leads to $A = -B$ and, we can check that we obtain from (11) the adiabatic compressibility $\kappa_{ad} = 1/p\gamma$, where $\gamma = c_p/c_v$ is the adiabatic exponent and $c_p = c_v + 1$ is the heat capacity at constant pressure. We get also from (10) the well-known sound velocity c_s given by $c_s^2 = p\gamma/\rho$. We can also check that we get from (11) the isothermal compressibility

$\kappa_{is} = 1/p$ for $A = 0$, which is indeed, according to (6), the condition for an isothermal process. Making use of (10) and (11) we may express, therefore, the velocity of the entropic "sound" as

$$\frac{p}{\rho} \cdot \frac{B - A/c_v}{B} = c^2 \ , \tag{12}$$

whence one can see that the velocity c is determined by the amplitude ratio A/B of the local relative variations of temperature and volume. Assuming c known equations (8) and (12) can be solved for the coefficients A and B, and we obtain

$$\begin{aligned} \frac{dT}{T} &= \frac{c_s^2 - \gamma c^2}{c_s^2 - c^2} \cdot \frac{s_0}{c_p} e^{i(qx - \omega t)} \ , \\ \frac{dV}{V} &= \frac{c_s^2}{c_s^2 - c^2} \cdot \frac{s_0}{c_p} e^{i(qx - \omega t)} \ , \\ \frac{dp}{p} &= -\frac{\gamma c^2}{c_s^2 - c^2} \cdot \frac{s_0}{c_p} e^{i(qx - \omega t)} \ . \end{aligned} \tag{13}$$

Since c is much smaller than the mean molecular velocity we have also $c \ll c_s$, so that these perturbations do propagate indeed as an entropic "sound" with the velocity c, *i.e.* the entropic "sound" does indeed exist. For c approaching c_s we see from (12) that B approaches $-A$, *i.e.*, according to (8), the entropy variations vanish ($s_0 \to 0$), and the entropic "sound" becomes the usual, adiabatic sound.

Equations (8) and (12), which were solved for the coefficients A and B above, i.e. for the small amplitudes of the relative variations of the temperature and, respectively, volume (the relative variations of the pressure are obtained from the equation of state), may also be viewed in another way. We may either consider that the velocity c is given, and then we obtain the amplitudes A and B, or consider these temperature and volume variations as being fixed (such as to satisfy the first law of thermodynamics), by various, undetermined circumstances which caused the initial entropy disturbance, and then we get the velocity c. One can see that the velocity c is therefore not fixed, but depends on the relative magnitude of the original local disturbances of temperature and volume (or pressure). In addition, we may also remark that if the masses of air in the atmosphere are in motion with an additional transport velocity u, *i.e.* with a wind velocity u, then the entropy waves may be written as $s \exp\left[i\left(q(x+ut) - \omega t\right)\right]$, such that the frequency is given by $\omega = (c+u)q$, in agreement with the Galilei principle of translational symmetry. As is well-known the wind velocity u itself is usually much smaller than the mean molecular velocity (and the adiabatic sound velocity).

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[6] See, for example, L. Landau and E. Lifshitz, *Physical Kinetics*, Pergamon Press, Oxford (1981).

[7] Small variations of the velocity, governed by Euler's equation, bring only a second-order contribution to the entropy continuity equation (2); we can solve the linearized mass continuity equation and Euler's equation for these velocity variations under non-adiabatic conditions, and find that they propagate with the velocity c given by the well-known expression (10) of the sound velocity.

[8] See, for example, H. Lamb, *Hydrodynamics*, Dover, NY (1945).