Journal of Theoretical Physics

Founded and Edited by M. Apostol **163** (2007)

ISSN 1453-4428

Elastic waves on a surface with irregularities

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Abstract

Small superficial irregularities are considered for the surface waves propagating in an isotropic elastic body. For a statistical distribution of irregularities the only effect is a slight change in the amplitudes of the surface waves.

We consider the elastic waves equation

$$\ddot{\mathbf{u}} = c_t^2\Delta\mathbf{u} + (c_l^2 - c_t^2)grad \cdot div\mathbf{u} \tag{1}$$

where $\mathbf{u}$ is the displacement field,

$$c_t = \sqrt{E/2\rho(1+\sigma)} \ , \ c_l = \sqrt{E(1-\sigma)/\rho(1+\sigma)(1-2\sigma)} \tag{2}$$

are the velocities of the transverse and, respectively, longitudinal waves, E is the Young's modulus, σ is the Poisson ratio ($0 < \sigma < 1/2$) and ρ is the density of the isotropic body. As it is well known, the elastic field can be written as $\mathbf{u} = \mathbf{u}_t + \mathbf{u}_l$, with $div\mathbf{u}_t = 0$ and $curl\mathbf{u}_l = 0$. This splitting leads to $\ddot{\mathbf{u}}_t = c_t^2\Delta\mathbf{u}_t$ and $\ddot{\mathbf{u}}_l = c_l^2\Delta\mathbf{u}_l$. We consider surface waves of the form $e^{-i\omega t}e^{i\mathbf{kr}}e^{\kappa z}$ which are vanishing for $z \rightarrow -\infty$. The frequencies are given by $\omega_t^2 = c_t^2(k^2 - \kappa_t^2)$ and $\omega_l^2 = c_l^2(k^2 - \kappa_l^2)$. The above conditions of transversality and irotationality leads to the following representation for these waves:

$$\begin{gathered}\mathbf{u}_l = A(k_x, k_y, -i\kappa_l)e^{-i\omega t}e^{i\mathbf{kr}}e^{\kappa_l z} \ , \\ \mathbf{u}_t = (B\kappa_t, C\kappa_t, -i(Bk_x + Ck_y))e^{-i\omega t}e^{i\mathbf{kr}}e^{\kappa_t z}\end{gathered} \tag{3}$$

where A, B, C are coefficients which depend on the wavevector $\mathbf{k}$ (for a given frequency). The elastic field is a superposition over $\mathbf{k}'s$ of $\mathbf{u}_l + \mathbf{u}_t$ given by (3).

We consider a free surface with a normal vector $\mathbf{n}$ and impose the free-force boundary condition $\sigma_{ij}n_j = 0$, where

$$\sigma_{ij} = \frac{E}{1+\sigma}(u_{ij} + \frac{\sigma}{1-2\sigma}u_{ll}\delta_{ij}) \tag{4}$$

is the stress tensor and $u_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$ is the strain tensor. We give here the strain tensor as computed from (3):

$$\begin{gathered}u_{xx} = \sum_{\mathbf{k}} i\left(k_x^2 A_{\mathbf{k}}e^{\kappa_l z} + k_x\kappa_t B_{\mathbf{k}}e^{\kappa_t z}\right)e^{i\mathbf{kr}} \\ u_{yy} = \sum_{\mathbf{k}} i\left(k_y^2 A_{\mathbf{k}}e^{\kappa_l z} + k_y\kappa_t C_{\mathbf{k}}e^{\kappa_t z}\right)e^{i\mathbf{kr}} \\ u_{zz} = -\sum_{\mathbf{k}} i\left[\kappa_l^2 A_{\mathbf{k}}e^{\kappa_l z} + \kappa_t\left(k_x B_{\mathbf{k}} + k_y C_{\mathbf{k}}\right)e^{\kappa_t z}\right]e^{i\mathbf{kr}}\end{gathered} \tag{5}$$

and

$$u_{xy} = \sum_{\mathbf{k}} i\left[k_x k_y A_{\mathbf{k}} e^{\kappa_l z} + \tfrac{1}{2}\kappa_t\left(k_y B_{\mathbf{k}} + k_x C_{\mathbf{k}}\right) e^{\kappa_t z}\right] e^{i\mathbf{kr}}$$

$$u_{xz} = \sum_{\mathbf{k}} \left[\kappa_l k_x A_{\mathbf{k}} e^{\kappa_l z} + \tfrac{1}{2}\left((k_x^2+\kappa_t^2)B_{\mathbf{k}} + k_x k_y C_{\mathbf{k}}\right) e^{\kappa_t z}\right] e^{i\mathbf{kr}} \tag{6}$$

$$u_{yz} = \sum_{\mathbf{k}} \left[\kappa_l k_y A_{\mathbf{k}} e^{\kappa_l z} + \tfrac{1}{2}\left(k_x k_y B_{\mathbf{k}} + (k_y^2+\kappa_t^2)C_{\mathbf{k}}\right) e^{\kappa_t z}\right] e^{i\mathbf{kr}} \ .$$

In addition

$$u_{ll} = \sum_{\mathbf{k}} i(k^2-\kappa_l^2)A_{\mathbf{k}} e^{\kappa_l z} e^{i\mathbf{kr}} \tag{7}$$

We consider the free surface described by equation $z = f(x, y)$. Its normal vector $\mathbf{n}$ is proportional to $(f_1, f_2, -1)$, where $f_1 = \partial f/\partial x$ and $f_2 = \partial f/\partial y$. We assume that f_1 and f_2 are small quantities and limit ourselves to the second-order expansion in these quantities; we get the normal vector $\mathbf{n} = (f_1, f_2, -(1 - f_1^2/2 - f_2^2/2))$. The boundary condition $\sigma_{ij}n_j = 0$ reads

$$(u_{xx} + \tfrac{\sigma}{1-2\sigma}u_{ll})f_1 + u_{xy}f_2 = u_{xz}(1 - f_1^2/2 - f_2^2/2)$$

$$u_{xy}f_1 + (u_{yy} + \tfrac{\sigma}{1-2\sigma}u_{ll})f_2 = u_{yz}(1 - f_1^2/2 - f_2^2/2) \tag{8}$$

$$u_{xz}f_1 + u_{yz}f_2 = (u_{zz} + \tfrac{\sigma}{1-2\sigma}u_{ll})(1 - f_1^2/2 - f_2^2/2) \ .$$

We assume further that $\kappa_{l,t}f \ll 1$, and write $e^{\kappa_{l,t}z} \simeq 1 + \kappa_{l,t}f + \kappa_{l,t}^2 f^2/2$. We use then the Fourier decomposition $f = \sum_{\mathbf{q}} f_{\mathbf{q}} e^{i\mathbf{qr}}$. The system of equations (8) can then be solved, in principle. It is of the form $A_{ij}X_j = \lambda_{ij}X_j$, where λ denotes a small parameter, originating in function f. For a flat surface $\lambda = 0$ ($z = 0$) and we get the Rayleigh waves. For a surface defined by function f the above system leads to $det(A - \lambda) = 0$ which induces small changes in the $\mathbf{k}$-dependence of $\kappa_{l,t}$; and leads also to small changes in coefficients A, B, C. Essentially, the effect of the irregular surface is to introduce a superposition of waves instead of a pure $\mathbf{k}$-Rayleigh plane wave. The actual superposition depends on the particular form of the function f.

We adopt a statistical view and average over function f and its derivatives in (8). We assume $\bar{f}_{1,2} = 0$, $\overline{ff}_{1,2} = 0$, $\bar{f}^2 = 2f^2$ and $\bar{f}_{1,2}^2 = q^2 f^2$, where f is now a parameter and q^{-1} is a measure of the scale of the variations of the function f. Then, the wavevectors are not coupled anymore in equations (8), so we may restrict ourselves to one $\mathbf{k}$-plane wave; in addition, we may assume $k_y = 0$. The *lhs* of equations (8) is now vanishing, and the remaining *rhs* gives

$$2\kappa_l kA(1 + \kappa_l^2 f^2 - q^2 f^2) + (k^2 + \kappa_t^2)B(1 + \kappa_t^2 f^2 - q^2 f^2) = 0$$

$$\kappa_l^2 A(1 + \kappa_l^2 f^2 - q^2 f^2) - \kappa_t kB(1 + \kappa_t^2 f^2 - q^2 f^2) - \tfrac{\sigma}{1-2\sigma}(k^2 - \kappa_l^2)A(1 + \kappa_l^2 f^2 - q^2 f^2) = 0 \tag{9}$$

and $C = 0$. We introduce new variables $A' = A(1 + \kappa_l^2 f^2 - q^2 f^2)$ and $B' = B(1 + \kappa_t^2 f^2 - q^2 f^2)$ and the above equations become

$$2\kappa_l kA' + (k^2 + \kappa_t^2)B' = 0$$

$$2\kappa_t kB' + (k^2 + \kappa_t^2)A' = 0 \tag{10}$$

The second equation (10) is obtained by using the dispersion relations (frequency ω *vs* wavevector $\mathbf{k}$) and the definition of the waves velocities as functions of σ.

Equations (10) define the Rayleigh waves. With $\omega = c_l k\xi$ and $\kappa_l^2 = (1 - c_t^2\xi^2/c_l^2)k^2$, $\kappa_t^2 = (1-\xi^2)k^2$ they lead to

$$\xi^6 - 8\xi^4 + 8\frac{2-\sigma}{1-\sigma}\xi^2 - \frac{8}{1-\sigma} = 0 \tag{11}$$

which has a solution close to unity. The amplitude ratio is guven by

$$A'/B' = -2\frac{\sqrt{1-\xi^2}}{2-\xi^2} \ . \tag{12}$$

The statistical effect of small superficial irregularities on the surface waves is the small change in amplitudes according to $A' = A(1 + \kappa_l^2 f^2 - q^2 f^2)$ and $B' = B(1 + \kappa_t^2 f^2 - q^2 f^2)$.