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On the theory of war between "vicious" civilizations

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Abstract

A recently proposed model of war between "vicious" civilizations (I. Ispolatov, P. L. Krapivsky and S. Redner, Phys. Rev. **E54** 1274 (1996)) is analyzed in the limit of the infinite number of degrees of freedom. An approximate solution is obtained in the long-wavelength limit for the "mean-field" model of "fair" war in the "heterogeneous" case, which shows that such a "free world" in a "global" war undergoes an extinction $\sim t^{-1}$, where t is proportional to the time. A discussion is given of the approximate solution obtained here with respect to the scaling solution reported in the work cited above.

A model of warfare between "vicious" civilizations has recently been formulated.[1] We analyze here the model from the perspective of the statistical physics and the physical kinetics, and give, in the long-wavelength limit, an approximate solution to the "mean-field" model of "fair" war, in the so-called "heterogeneous" case. Beforehand, we give, however, a brief description of the model, following Refs.1,2.

Suppose that we have a collection of civilizations of size (population) k, $k = 1, 2, ...$, and concentration c_k. In a peacefully flourishing world (a "free world") each civilization grows at a constant rate λ, as described by the equation

$$\dot{c}_k = \lambda \left(c_{k-1} - c_k\right) \quad , \tag{1}$$

where $c_0 = 0$. The approach to the equilibrium ($c_k \to 0$) is made by slow "spatial" variations, so that we may replace k in (1) by a continuous variable x, and expand the function $c(x-1)$ in a Taylor series around x, for large x. Doing so, equation (1) becomes

$$\dot{c}\,(x,t) = -\lambda \frac{\partial c}{\partial x} \tag{2}$$

in the first approximation, whose solution $c(x,t) = \delta\left(x - \lambda t\right)$ propagates freely with the velocity λ; it corresponds to the "Adam" initial condition $c(x,0) = \delta(x)$. This solution has a fast variation near the propagating singular front $x \sim \lambda t$, so that, in order to improve the accuracy, we retain the second-order derivative too, in (1),

$$\dot{c}\,(x,t) = \lambda \left(-\frac{\partial c}{\partial x} + \frac{1}{2}\frac{\partial^2 c}{\partial x^2}\right) \quad . \tag{3}$$

The solution is now the well-known gaussian

$$c(x,t) = \frac{1}{\sqrt{2\pi\lambda t}} e^{-\frac{(x-\lambda t)^2}{2\lambda t}} \quad , \tag{4}$$

which has a propagating front at $x = \lambda t$, a dispersion $\sim \sqrt{\lambda t}$, and is a slowly varying function everywhere in the limit of the long times; it corresponds to the long-wavelength limit of equation (3).

However, the world is made of "vicious" civilizations, which means that whenever two civilizations meet they engage in war. In a war event the smaller civilization is annihilated, while the larger civilization is reduced in size by a fraction ε of the casualties of the loser; this war event can be described symbollically by the scheme $(i,k) \rightarrow (i - \varepsilon k, 0)$, $i \geq k$. A "fair" war is that war for which $\varepsilon = 1$, while a completely "unfair' war would correspond to $\varepsilon = 0$. Here we shall discuss only the "fair" wars. The rate equation for a "fair" war is given by

$$\dot{c}_k = \sum_i M(i, i+k) c_i c_{i+k} - c_k \sum_i M(i,k) c_i + \lambda \left(c_{k-1} - c_k \right) \tag{5}$$

where M is the war rate. The first term in (5) describes the gain of k-civilizations from a war between the i- and $i+k$-civilizations, while the second term in (5) corresponds to the loss in k-civilizations because of war between these civilizations and any other civilization. It is worth nothing here the self-destruction contribution $-c_k^2$ included in the second term in (5). In addition to the war terms equation (5) contains also the free growth at the rate λ. We are interested in the asymptotic solutions of (5) at long times and over large distances (*i.e.* large values of k), either for a given initial distribution of concentrations ("heterogeneous" case), or for a continuous, uniform input of small civilizations ("homogeneous" case). In this latter case one can imagine that the input is provided by the remnants of the destroyed civilizations. Here we limit ourselves to the "heterogeneous" case only.

The model of warfare as formulated above seems to have been suggested by studies in the kinetics of reacting systems, including coagulation,aggregation, annihilation, etc; it can also be viewed as an attempt to formulate quantitative models of competing biological species.[2]$^-$[4] Though rather naive and drastic in certain respects (as, for example, in assuming the complete annihilation of the loser), it still incorporates the essential features of growth and casualties, and may exhibit a rich dynamics. For example, the model is expected to illustrate the extinction and the persistence of the civilizations, the egalitarian competition between civilizations of equal size, the ultimate domination of a few superpowers, as well as other aspects of social history.[1]

The civilizations in the model presented above can also be viewed as domains in the physical space. In this case a critical space dimension can appear, and the fluctuations become a matter of importance, as they dominate in lower dimensions.[5] The usual "mean-field" approximation $M(i,k) = const$, which might be termed the "global" war approximation, could have drastic implications in this case. We shall disregard the spatial structure of the model and limit ourselves to the "mean-field" approximation. By a suitable choice of dimensionless variables the model is described by the rate equation

$$\dot{c}_k = \sum_i c_i c_{i+k} - c_k \sum_i c_i + \lambda \left(c_{k-1} - c_k \right) \quad . \tag{6}$$

Both numerical simulations and analytical results are available for certain cases, in various particular contexts.[6] In general, a coarsening has been suggested in the "heterogeneous" case, even for a "fair" war ($\varepsilon = 1$), while a few superpowers may ultimately dominate in an "unfair" war ($0 < \varepsilon < 1/2$) in the "homogeneous" case. In this latter case, however, an egalitarian competition is expected between civilizations of equal size (steady state) in a relatively "fair" war

$1/2 < \varepsilon < 1$. In what follows we shall focus ourselves on the "mean-field" model of "fair" war in the "heterogeneous" case, as described by (6), in the limit of the infinite number of degrees of freedom.

By using the notations

$$S = \sum_k c_k \quad , \quad \rho^2 = \sum_k c_k^2 \tag{7}$$

it is easy to establish the extremely useful relation

$$\sum_{ik} c_i c_{i+k} = \frac{1}{2} S^2 - \frac{1}{2} \rho^2 \quad , \tag{8}$$

as pointed out in Ref.2. Summing up the rate equations (6) we obtain

$$\dot{S} = -\frac{1}{2} S^2 - \frac{1}{2} \rho^2 \quad , \tag{9}$$

an equation which contains only the war terms. Let us denote by N, $N \to \infty$, the number of degrees of freedom $k = 1, 2, ...N$. The sum S can be viewed as the scalar product of the vector $\mathbf{c} = (c_1, c_2, ...c_N)$ by the unit vector,

$$S = \rho \sqrt{N} \cos \varphi \quad , \tag{10}$$

where φ is the angle between the two vectors. In order to have a finite S in the limit $N \to \infty$, either ρ, or $\cos \varphi$, or both, must vanish. However, it is easy to see that a vanishing $\cos \varphi$ leads to a contradiction; indeed, if $\sqrt{N} \cos \varphi \to A(t)$ for $N \to \infty$, where $A(t)$ is a function of t, then N is finite, $N = A^2(t)$, and all c_k are equal, which contradicts our hypothesis. In fact, $\cos \varphi \to 0$ means nothing but the principle of statistical (or thermodynamical) equilibrium, where all kinetics is lost. Indeed, $\cos \varphi \to 0$ for $N \to \infty$ would imply $\varphi \to \pi/2$; however, φ depends on time, in principle, and $\varphi \to \pi/2$ would mean that we have taken the limit of the infinite time $t \to \infty$, where the system reached its equilibrium and all c_k vanished. It follows that, in order to have a non-trivial kinetics, we must have $\rho \to 0$, which corresponds indeed to the physical notion of "thermalization"; or, in other words, we must have, in the limit of infinite N, a vanishing mean-square deviation of the kinetical variables, *i.e.* vanishing fluctuations, as corresponding to the approach to equilibrium. We remark that this conclusion is also consistent with the "mean-field" approximation. Moreover, c_k must, therefore, be slowly varying functions in space in the limit of the long times, which means that we can restrict ourselves to the long-wavelength limit. From these considerations it follows that we may neglect ρ^2 in (9) in comparison with S^2, and solve (9) for

$$S(t) = \frac{2}{2+t} \quad , \tag{11}$$

where $S(0) = 1$. We emphasize that this approximation is only valid in the limit of the infinite number of degrees of freedom, $N \to \infty$.

We return now to the rate equation (6), and write it down in the continuum limit $k \to x$ as

$$\dot{c}\,(x,t) = F(x) - S(t) c(x) + \lambda \left(-\frac{\partial c}{\partial x} + \frac{1}{2} \frac{\partial^2 c}{\partial x^2} \right) \quad , \tag{12}$$

where

$$F(x) = \int_0^\infty dy \cdot c(y) c(x+y) \quad . \tag{13}$$

By using the Fourier transform (on the half space $x>0$) (15) becomes

$$\overset{\bullet}{c}(q,t)=F(q)-S(t)c(q)+\lambda\left(-iq-\frac{1}{2}q^2\right)c(q)\quad , \tag{14}$$

where $F(q)$ is the Forier transform of $F(x)$. The λ-term in (14) represents the long-wavelength contribution to the "free growth" equation (3). Since the function $S(t)$ given by (11) has been derived by taking the contribution of the war terms in the same long-wavelength limit we use the same approximation here, *i.e.* we keep only the $q=0$-contribution to the war terms in (14). Therefore, we put[7]

$$F(q)\cong F(q=0)=\frac{1}{2}S^2(t)\quad , \tag{15}$$

according to (8), where ρ^2 is neglected in comparison with S^2, and $c(q)\cong c(q=0)=S(t)$. This long-wavelength approximation amounts to neglecting the local effects of the war; in this sense, we can speak of a "global" war, as corresponding to the "mean-field" approximation. Equation (14) becomes then

$$\overset{\bullet}{c}(q,t)\cong-\frac{1}{2}S^2(t)+\lambda\left(-iq-\frac{1}{2}q^2\right)c(q)\quad , \tag{16}$$

whose solution can be written as

$$c(q,t)\cong\left\{\int_0^t dt'\left[-\frac{1}{2}S^2(t')\right]e^{-\lambda\left(-iq-\frac{1}{2}q^2\right)t'}+1\right\}\cdot e^{\lambda\left(-iq-\frac{1}{2}q^2\right)t}\quad , \tag{17}$$

for $c(q,0)=1$, corresponding to the "Adam" initial solution. At long times we may take again the limit $q\to 0$ in the integral in (17), and obtain

$$c(x,t)\cong\frac{2}{2+t}\cdot\frac{1}{\sqrt{2\pi\lambda t}}e^{-\frac{(x-\lambda t)^2}{2\lambda t}}\quad , \tag{18}$$

i.e. the "free world" solution diminished drastically by the war effects contained in the prefactor $S(t)=2/(2+t)$. One can see that the effect of a "global" war on a "free world" is the extinction of this world as the power t^{-1} of the time. From the technical point of view we mention here that any attempt of improving upon the approximate solution given by (18) would require to accounting for the fluctuations effect, which amounts to include the contribution of the ρ^2-term to equation (9).

The approximate solution obtained in this paper, as given by (18), can be viewed as a counterpart to the scaling solution given in Ref.1. Indeed, an asymmetrical solution is obtained in Ref.1 by assuming the scaling variable x/t, an approach which may give correctly the propagating front at $x\sim t$, while the distribution width is also proportional with t. This may overestimate the role of fluctuations, since the solution given here by (18), while locating indeed the front at $x\sim t$, indicates however a narrower distribution, whose width is proportional with $t^{1/2}$. The present solution corresponds to the complete disregard of the role played by the fluctuations, in agreement with the evolution of the system close to the equilibrium. It is also worth pointing out that numerical simulations reported in Ref.1 for the one-dimensional model indicate a more symmetric distribution of an approximate triangular shape, whose width is also proportional with t.

A different asymptotic behaviour (with temporal logarithmic corrections) has been reported in Ref.2 for a finite system with war-terms only; we emphasize that the problem is distinct from the present approach, as (18) corresponds to an infinite system ($N\to\infty$) where ρ^2 may be neglected with respect to S^2, which is not the case for a finite system. One can check that (18) does indeed satisfy $\rho^2\ll S^2$, while for the finite-system solution given in Ref.2 ρ^2 is comparable with S^2. Among many open problems associated with this model we may raise the question whether a finite system would not exhibit a chaotic behaviour, as generated by its non-linearities.

References

[1] I. Ispolatov, P. L. Krapivsky and S. Redner, Phys. Rev. **E54** 1274 (1996).

[2] S. Redner, D. Ben-Avraham and B. Kahng, J. Phys. **A20** 1231 (1987).

[3] E. Ben-Naim and P. L. Krapivsky, Phys. Rev **E52** 6066 (1995).

[4] P. Meakin, Reps. Progr. Phys. **55** 157 (1992); J. Stavans, Reps. Progr. Phys. **56** 733 (1993); A. J. Bray, Adv. Phys **43** 357 (1994).

[5] See, for example, M. V. Smoluchovski, Z. Phys. Chem. **92** 215 (1917); M. H. Ernst, in *Fundamental Problems in Statistical Physics* VI, ed. E. G. D. Cohen, Elsevier, NY (1985); Y. B. Zeldovich and A. S. Mikhailov, Usp. Fiz. Nauk 153 469 (1987) [Sov. Phys. Usp. **30** 977 (1988)]; a critical space dimension $d_c = 2$ has been suggested in this case, see, for example, Ref.2.

[6] See, for instance, Refs.1,2.

[7] In replacing $F(q=0)$ by $\frac{1}{2}S^2(t)$ caution should be exercised to account for the Gibbs phenomenon of the Fourier transform at $x=0$, *i.e.* $F(q=0) = \int dxdy \cdot c(y)c(x+y) = \sum_{ik} c_i c_{i+k} = \frac{1}{2}S^2 - \frac{1}{2}\rho^2 \cong \frac{1}{2}S^2$, according to (8) and the subsequent discussion.