Journal of Theoretical Physics

Founded and Edited by M. Apostol **80** (2002)

ISSN 1453-4428

A critical-point theory of the earthquakes

B.-F. Apostol and M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox Mg-35, Romania
email: apoma@theory.nipne.ro

Abstract

The scaling hypothesis underlying the theory of the critical point is briefly presented, and its application to earthquakes is outlined. The relevance of the "self-organized criticality" is emphasized for the critical behaviour of the probabilistic seismic events.

The main characteristic of the earthquakes is probably their large-scale behaviour. Indeed, the famous Gutenberg-Richter law[1] reads

$$\ln E \cong 11 + 3.5M \quad , \tag{1}$$

where E (in J) is the released energy and M denotes the magnitude of the earthquake; the latter may go up to $7-8$, which implies an enormous range of energies. This only observation suffices to suggest a phase-transition nature at least for those earthquake focuses that display a rich distribution in magnitudes, as those associated with a seismic fault network, for instance. Such a standpoint is taken in the present paper.

As it is known, the phase transitions are understood by means of the so-called theory of the critical point. This theory has two basic ingredients: the renormalization group[2] and the scaling hypothesis.[3] The former gives critical exponents (at least in principle), as based on the general statistical principles of the phase transitions; this part has a limited applicability to earthquakes, as long as the adequacy of such general principles to earthquakes is not yet fully known. The latter part however, the scaling hypothesis that underlines the phase transitions, is sufficiently general as to make it a reasonable hypothesis for the critical behaviour of a very large class of complex systems, earthquakes included.

The probability for a randomly-distributed earthquake to occur within the energy range dE with an energy release E is proportional to dE/E. Making use of (1) it is easy to see that such a probability implies a uniform distribution in magnitudes, $dE/E \sim dM$. Except for the background of small seismic events, such a uniform distribution in magnitudes seems to be a reasonable empirical knowledge. Consequently, we adopt here such type of probability distribution. Since a seismic event occurs in time dt at a moment t of time, the same probability would be $\sim dt/t$, so that we might write $dE/E = Adt/t$, where A is a constant originating in the normalization of the probabilities. However, the critical behaviour involves an accelerating evolution towards the critical time t_c, and an accelerated seismicity, which might be a representative of what is called a "self-organized criticality".[4] Consequently, the probability is not uniformly distributed in time. It is given in fact by $h(t)dt$, where $h(t_c) \to \infty$.[5] For an earthquake t_c may be considered the failure time. Using the notation $\tau = t_c - t$ we can write the rate h of probability as a function of time τ, $h(\tau) = h(t_c - t)$, where $-\infty < t < t_c$ and $h(0) \to \infty$. Therefore, we may write down

$$dE/E = Ah(\tau)dt \ . \tag{2}$$

The "self-organized criticality" of the new phase occurring at the critical point suggests a general scaling hypothesis

$$h(\alpha\tau) = \beta h(\tau) \ , \tag{3}$$

where α and β are positive-valued parameters ($\neq 1$) characterizing the transition. The general solution to (3) is[6]

$$h(\tau) = \tau^{\ln\beta/\ln\alpha} f(\ln\tau/\ln\alpha) \ , \tag{4}$$

where f is a periodic function with period 1. Consequently, this latter function has a Fourier expansion

$$f(\ln\tau/\ln\alpha) = \sum_n a_n e^{i2\pi(\ln\tau/\ln\alpha)n} \ , \tag{5}$$

where n are integers. We may say that function h is characterized by a set of complex exponents $\ln\beta/\ln\alpha + i(2\pi/\ln\alpha)n$. The scaling hypothesis (3) requires a divergent $h(0)$. Therefore, we introduce the critical exponent $m = -\ln\beta/\ln\alpha$, which must be positive for a critical point, and limit ourselves to the first terms in expansion (4), corresponding to slow oscillations. The rate of probability h can then be represented as

$$h(\tau) = h_0\tau^{-m}[1 + a\cos(2\pi\ln\tau/\ln\alpha + \varphi)] \ , \tag{6}$$

where $a \ll 1$. Such a function exhibits characteristic log-oscillations in time-to-failure τ. Moreover, leaving aside the log-oscillations, we get $t_c = t + (h_0/h)^{1/m}$ from (6); the derivatives of t_c with respect to time t must vanish, hence one obtains $0 < m < 1$ for the critical exponent m. The small log-oscillations preserve this limitation imposed upon the critical exponent.

If the largest earthquake expected from a given focus in a certain epoch is viewed as the critical point of a phase transition occurring in a critical region surrounding that focus, then the probability $dP = hdt$ must be normalized,

$$\int_{t_0}^{t_c} hdt = 1 \ , \tag{7}$$

over the duration $\tau_0 = t_c - t_0$, where t_0 is the time of occurrence of the preceding largest earthquake produced by the same focus. We obtain $h_0 = (1-m)/\tau_0^{1-m}$ from (7) (leaving aside the log-oscillations), where τ_0 is the duration from the preceding largest earthquake up to the next largest earthquake. Consequently, the probability of occurrence of a critical seismic event is given by

$$dP = h(\tau)dt = \frac{1-m}{\tau_0}(\tau/\tau_0)^{-m}dt \ . \tag{8}$$

It is worth noting that immediately after the largest preceding seismic shock ($\tau = \tau_0$) the density of probability of another shock is finite, though extremely low ($(1-m)/\tau_0$).[7]

The energy release E of the earthquake occurring at time t can be obtained from (2) by integration,

$$\int_{E_0}^{E} dE'/E' = A\int_{t_0}^{t} hdt \ , \tag{9}$$

where E_0 is a scale energy. Making use of 8 we obtain

$$\ln(E/E_0) = A[1-(\tau/\tau_0)^{1-m}] \ . \tag{10}$$

The scale energy E_0 can be associated with the occurrence of an "earthquake" of magnitude zero at time $\tau = \tau_0$, *i.e.* $\ln E_0 \cong 11$ according to (1). Similarly, the constant A can be associated with the largest earthquake occurring at time $\tau = 0$, *i.e.* $A = \ln(E_{max}/E_0) = 3.5M_{max}$. Therefore, making use of the Gutenberg-Richter law (1), we obtain the time-dependence of the magnitudes

$$M = M_{max}[1-(\tau/\tau_0)^{1-m}] \ . \tag{11}$$

Equation (11) must be viewed as a fit to the recorded magnitudes M, the fit parameters being the critical exponent m and the duration τ_0 up to the next largest seismic event. For times t close to the momentum t_0 of the preceding largest event we have $\tau = t_c - t = \tau_0 - (t-t_0)$, and we may expand (11) in powers of $t-t_0$; we obtain $M \simeq M_{max}(1-m)(t-t_0)/\tau_0$, which may fix the ratio $(1-m)/\tau_0$ from the temporal slope of M/M_{max}; however, this slope is very small, and it may not be very useful for analyzing the data.

A phase transition is a sudden change over a very large spatial range. Consequently it is characterized by a length R, which, for earthquakes, may be taken as the radius of the critical seismic region. The energy E released in a seismic event is proportional to the volume of the critical region. Consequently, we may write $E = E_0(R/R_0)^3$, where R_0 is the minimum value of the critical region associated with "earthquakes" of magnitude zero (and energy E_0). It is worth emphasizing that the scale energy $E_0 = e^{11}$J corresponding to magnitude $M = 0$ is very small in comparison with the energy released in usual earthquakes, and, correspondingly, the associated scale length R_0 is very small in comparison with usual critical radii,[8] in agreement with the assumptions of the critical-point theory. Introducing this R^3-dependence of the energy into (10) we obtain the critical temporal behaviour

$$\ln(R/R_0) = 1.17M_{max}[1-(\tau/\tau_0)^{1-m}] \tag{12}$$

for the critical radius, and, by making use of (11), the linear law

$$\ln(R/R_0) = 1.17M \tag{13}$$

which seems to be supported by the recorded data.[9]

In conclusion, we may say that an earthquake of magnitude M may appear at time τ as given by (11) with the probability dP given by (8), having a critical radius R given by (13) (or (12)). It is worth noting that the scale length R_0 may be determined by analyzing the experimental critical radius R as function of magnitude, as given by (13). It is also worth noting that the probability of occurrence of an earthquake of magnitude M is $(1/A)dE/E$ according to (2), which leads to dM/M_{max}; one can see that the earthquakes distribution is uniform in magnitudes, as expected, the difference being made by the moment of their occurrence. Similarly, the probability of having an earthquake with magnitude greater than M is given by

$$P_{>M} = \frac{1-m}{\tau_0}\int_{t}^{t_c} (\tau/\tau_0)^{-m}dt = (\tau/\tau_0)^{1-m} = 1 - M/M_{max} \ ; \tag{14}$$

it appears after time $t = t_c - \tau_0(1 - M/M_{max})^{1/(1-m)}$, and affects a critical region of radius larger than $R_0 e^{1.17M}$, according to (13); this probability may be viewed as the hazard, or the seismic risk, of a certain seismic area. More exactly, the hazard, or seismic rate, is $h(\tau)/P_{>M} = (1-m)/\tau = (1-m)(1-M/M_{max})^{-1/(1-m)}/\tau_0$. It is interesting that this rate of precursory events is similar with Omori's law of the aftershocks ($\sim 1/\tau$), suggesting a time-reversal invariance of the critical regime. Making use of the probability (8) and the magnitudes distribution (11) we can also compute various probabilistic momenta of the magnitudes; for instance, the average magnitude is $\overline{M} = M_{max}(1-m)^2/(2-m)(3-2m)$.

Finally, we note that the above formulae have been derived without the *log*-oscillatory contribution to (6). Including this contribution we obtain the normalization condition

$$\int_{t_0}^{t_c} h dt = h_0 \frac{\tau_0^{1-m}}{1-m}[1 + a(1-m)\cos(\frac{2\pi}{\ln\alpha}\ln\tau_0 + \varphi - \psi)] = 1 \tag{15}$$

for the probability, where $\tan\psi = 2\pi/(1-m)\ln\alpha$. As we can see from (15) the *log*-oscillations contribute little to the normalization condition, so that we may use approximately

$$dP = h(\tau)dt \simeq \frac{1-m}{\tau_0}(\tau/\tau_0)^{-m}[1 + a\cos(2\pi\ln\tau/\ln\alpha + \varphi)]dt \tag{16}$$

for the probability given by (6). Under this approximation the critical radius R given by (12) and the magnitudes given by (11) are left unchanged. However, the *log*-oscillations must be included for a more accurate fitting to the recorded data .

References

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[2] See, for instance, K. G. Wilson, Revs. Mod. Phys. **55** 583 (1983) and references therein, and K. G. Wilson and J. Kogut, Phys. Rep. **C12** 75 (1974).

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[4] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett. **59** 381 (1987); Phys. Rev. **A38** 364 (1988).

[5] Such a critical-point theory has recently been suggested for the crashes of the financial markets, see, for instance, A. Johansen et al, xxx.lanl.gov, cond-mat/9810071 (1998).

[6] D. Sornette and C. G. Sammis, J. Physique I **5** 607 (1995); D. Sornette, Phys. Reps. **297** 239 (1998).

[7] This prediction does not refer to aftershocks.

[8] See, for instance, K. E. Bullen, *An Introduction to the Theory of Seismology*, Cambridge (1963).

[9] See, for instance, C. G. Bufe and D. J. Varnes, J. Geophys. Res. **98** 9871 (1993) and D. D. Bowman, G. Ouillon, C. G. Sammis, A. Sornette and D. Sornette, J. Geophys. Res. **103** 24 359 (1998) and references therein; sometimes the Benioff strain $(E/E_0)^{1/2} = R/R_0$ is used for the critical radius, which, however, does not change the critical behaviour (12); .