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On the cubic anharmonic oscillator

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Abstract

The exact solution is derived for the classical cubic anharmonic oscillator, and the first-orders terms are computed in the perturbation series of the anharmonic correction.

There is a huge literature on anharmonic oscillators, both quantal and classical.[1] Exact solutions are known for classical cubic and quartic anharmonic oscillators with and without dissipation,[2, 3] and detailed studies have been performed for forced classical oscillator with higher-order anharmonicities.[4] We present here a simple derivation of the exact solution for the classical cubic oscillator, and the first-orders terms in the corresponding series expansion in powers of the anharmonicity.

Let $T = m\dot{u}^2/2$ be the kinetic energy and

$$U = \frac{1}{2}m\omega^2 u^2 + \frac{1}{3}m\omega^2 a u^3 \tag{1}$$

the potential energy of a cubic anharmonic oscillator of mass m, frequency ω and anharmonicity parameter $a > 0$. The energy conservation gives

$$\dot{u}^2 = \frac{2}{m}(E - U) = \omega^2(x^2 - u^2 - \frac{2}{3}au^3) \ , \tag{2}$$

for this oscillator, where $E = m\omega^2 x^2/2 > 0$ is the energy. For $x^2 > 1/3a^2$ the velocity in (2) vanishes for $u_1 > 0$ and the motion is infinite for $u < u_1$. For $x^2 < 1/3a^2$ the velocity in (2) vanishes for $u_3 < u_2 < u_1$ and the motion is infinite for $u < u_3$ and finite for $u_2 < u < u_1$. For this finite motion (2) can also be written as $\dot{u}^2 = (2a\omega^2/3)(u_1 - u)(u - u_2)(u - u_3)$, and the integral of motion reads

$$\int_{u_2}^{u} \frac{dy}{\sqrt{(u_1 - u)(u - u_2)(u - u_3)}} = \sqrt{2a/3}\omega t \ , \tag{3}$$

for $u_2 < u < u_1$ and the initial conditions $u = u_2$, $\dot{u} = 0$ for $t = 0$. The integral in (3) can be expressed by means of the elliptic function of the first kind $F(\varphi, k)$ by introducing $\sin\alpha = [(u_1 - u_3)(y - u_2)/(u_1 - u_2)(y - u_3)]^{1/2}$.[5] (p. 219, 3.131(5)) We obtain

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2\alpha}} = \tau \ , \tag{4}$$

where

$$\sin\varphi = \sqrt{\frac{u_1 - u_3}{u_1 - u_2}}\sqrt{\frac{u - u_2}{u - u_3}} \ , \tag{5}$$

the modulus of the elliptic function is given by

$$k^2 = \frac{u_1 - u_2}{u_1 - u_3} \ , \tag{6}$$

and the dimensionless time τ is given by

$$\tau = \frac{1}{2}\sqrt{u_1 - u_3}\sqrt{2a/3}\omega t \ . \tag{7}$$

From (5) we obtain the solution

$$u = \frac{u_2 - k^2 u_3 \sin^2\varphi}{1 - k^2\sin^2\varphi} \ , \tag{8}$$

or, making use of the Jacobi sine-amplitude $snF = sn\tau = sin\varphi$,[5] (p. 910) we get

$$u = \frac{u_2 - k^2 u_3 sn^2\tau}{1 - k^2 sn^2\tau} \ . \tag{9}$$

This is the exact solution of the cubic anharmonic oscillator. It oscillates between u_2 for $\varphi = n\pi$, and u_1 for $\varphi = (2n+1)\pi/2$, n being an integer. The period T of the motion is given by

$$K = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2\sin^2\alpha}} = \frac{1}{4}\sqrt{2a(u_1 - u_3)/3}\omega T \ , \tag{10}$$

where K is the complete elliptic function. A similar exact solution can also be obtained for the quartic anharmonic oscillator.[2]

It is worth noting that the infinite motion proceeds in a finite time. Indeed, let $u_1 > 0$ and $u_{2,3} = A \pm iB$ for $x^2 > 1/3a^2$. Then, the integral (3) becomes $F(\varphi, k) = \sqrt{2aD/3}\omega t$, where $k^2 = [1 + (u_1 - A)/D]/2$, $D = [(u_1 - A)^2 + B^2]^{1/2}$ and $u = u_1 - D\tan^2(\varphi/2)$. One can see that $u \to -\infty$ for $\varphi \to \pi$, which means that motion goes to infinite in a finite time T_1 given by $2K = \sqrt{2aD/3}\omega T_1$.

It is often useful to have the solution of the cubic oscillator in the limit of the weak anharmonicity. In order to get this limit we need the approximate roots $u_{1,2,3}$ of the equation $x^2 - u^2 - \frac{2}{3}au^3 = 0$ in this limit. Introducing $z = 2au/3$ this equation becomes $z^3 + z^2 - \varepsilon^2 = 0$, where the perturbational parameter is $\varepsilon = 2ax/3$. It is easy now to solve perturbationally this equation; its solutions are given by $z_{1,2} = \pm\varepsilon(1 \mp \varepsilon/2 + \varepsilon^2/4)$ and $z_3 = -1 + \varepsilon^2$, or

$$u_1 = x(1 - \varepsilon/2 + \varepsilon^2/4) \ , \ u_2 = -x(1 + \varepsilon/2 + \varepsilon^2/4) \ , \ u_3 = -\frac{x}{\varepsilon}(1 - \varepsilon^2) \ . \tag{11}$$

Making use of these expansions in powers of ε we obtain $k^2 = 2\varepsilon(1 - \varepsilon + 11\varepsilon^2/4)$ and $K = \pi(1 + \varepsilon/2 + \varepsilon^2/16)/2$. Using the same expansions in (10) we get the well-known second-order shift

$$\Omega = 2\pi/T = \omega(1 - 15\varepsilon^2/16) = \omega(1 - 5a^2x^2/12) \tag{12}$$

in frequency. Similarly, the angle φ is obtained from (4) as

$$\varphi = \frac{1}{2}\Omega t + \frac{\varepsilon}{4}\sin\Omega t + \frac{\varepsilon^2}{64}\sin 2\Omega t \ , \tag{13}$$

and the oscilaltor coordinate

$$u = -x\cos\Omega t - \frac{x\varepsilon}{4}(3 - \cos 2\Omega t) - \frac{x\varepsilon^2}{2}(2 - \frac{17}{8}\cos\Omega t + 2\cos 2\Omega t - \frac{11}{8}\cos 3\Omega t) \ . \tag{14}$$

It is worth noting that the renormalized frequency Ω appears in these expansions, instead of the original frequency ω. All these expansions in powers of ε can also be obtained directly by solving perturbationally the equation of motion $\ddot{u} = -\omega^2(u + au^2)$, including the frequency renormalization.

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