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### On linear anharmonic oscillators and self-consistent harmonic approximation

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Through the years, anharmonic oscillators generated a great deal of technical work, both classically and quantally. In fact, they are reducible to harmonic oscillators, with a good approximation.

Let  $T = m\dot{u}^2/2$  be the kinetic energy of a linear oscillator of mass  $m$ , and  $U = (m\omega^2/2)(u^2 + 2au^3/3)$  its potential energy with cubic anharmonicities, where  $a$  is a parameter. Let  $u^3$  be approximated by

$$u^3 = \frac{3}{2}(Au + Bu^2) , \quad (1)$$

where  $A = \bar{u}^2$ ,  $B = \bar{u}$ , the averages being taken over the motion and the coefficients  $3/2$  in (1) being chosen such as  $\bar{u}^3 = 3\bar{u}\bar{u}^2$ . It is easy to see that the oscillator becomes then a displaced one, with the frequency  $\Omega = \omega(1 + aB)^{1/2}$ ; the solution is  $u = u_0 \cos \Omega t - C$ , where  $u_0$  is an amplitude and  $C = aA/2(1 + aB)$ . The condition  $\bar{u}^3 = 3\bar{u}\bar{u}^2$  is fulfilled only for small values of  $C$ , as expected ( $\bar{u} = -C$ ,  $\bar{u}^2 = u_0^2/2 + C^2$ ,  $\bar{u}^3 = -3u_0^2C/2$ ). It follows  $C \cong au_0^2/4$  and  $A \cong u_0^2/2$ ,  $B = -C \cong -au_0^2/4$ . The frequency shift is then given by

$$\Omega = \omega(1 + aB)^{1/2} \cong \omega(1 - a^2u_0^2/8) , \quad (2)$$

which compares rather satisfactorily with the exact result[1]  $\Omega = \omega(1 - 5a^2u_0^2/12)$ .

A similar decomposition  $u^4 = 3Au^2/2$  holds for the quartic anharmonicity in the potential energy  $U = (m\omega^2/2)(u^2 + bu^4/2)$ , where  $A = \bar{u}^2$  and  $b$  is the anharmonic parameter. The condition  $\bar{u}^4 = 3(\bar{u}^2)^2/2$  is then fulfilled exactly ( $u^2 = A = u_0^2/2$ ,  $\bar{u}^4 = 3u_0^4/8$  for solution  $u = u_0 \cos \Omega t$  and frequency  $\Omega = \omega(1 + 3Ab/4)^{1/2}$ ). It follows the frequency shift given by

$$\Omega = \omega(1 + 3Ab/4)^{1/2} \cong \omega(1 + 3bu_0^2/16) , \quad (3)$$

for small  $b$ , which again compares well with the exact result[1]  $\Omega = \omega(1 + 3bu_0^2/8)$ . It is worth noting that the frequency shift is quadratic in amplitude for cubic anharmonicities, and linear for quartic anharmonicities.

Similar approximations can be used approximately for higher-order anharmonicities, without any loss of qualitative behaviour, and a satisfactory representation of the quantitative results. They are called generically the self-consistent harmonic approximation.

## References

- [1] L. Landau and E. Lifshitz, *Mecanique*, Mir, Moscow (1965).