Few-optical-cycle solitons: Modified Korteweg–de Vries sine-Gordon equation versus other non–slowly-varying-envelope-approximation models

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We put forward through both analytical and numerical methods the advantage of using non–slowly-varying envelope-approximation model equations for describing the propagation of few-optical-cycle pulses in transparent media. It is proven that the dynamical model based on the generic modified Korteweg–de Vries sine-Gordon equation retrieves the results reported so far in the literature, and so demonstrating its remarkable mathematical capabilities in describing the physics of few-cycle-pulse optical solitons.

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I. INTRODUCTION

In past years, intense ultrashort light pulses comprising merely a few-optical cycles became routinely available; for a review of the various techniques of their production and measurement as well as relevant theoretical methods used to model their unique features, see Ref. [1]. These intense ultrashort optical pulses have various applications in the field of light-matter interactions, high-order harmonic generation, extreme nonlinear optics [2], and attosecond physics [3].

Several theoretical approaches have been considered thus far to describe the physics of few-cycle-pulse (FCP) optical solitons; chiefly we have three classes of governing models: (i) the full quantum approach [4–7], (ii) the refinements of envelope equations of nonlinear Schrödinger (NLS) type, in the framework of the slowly-varying envelope approximation (SVEA) [8–10], and (iii) the non–SVEA models. In the present work we will concentrate on non–SVEA model equations for describing the propagation of FCP optical solitons in transparent nonlinear media.

The propagation of FCPs in Kerr media can be described beyond the SVEA by using the modified Korteweg–de Vries (mKdV) [11], sine-Gordon (sG) [12,13], or mKdV-sG equations [14,15]. The mKdV and sG equations are completely integrable by means of the inverse scattering transform (IST) method [16,17], whereas the mKdV-sG equation is completely integrable only if some condition between its coefficients is satisfied [18]. All these equations admit breather solutions, which can realistically describe the few-optical-cycle solitons. In $(2+1)$ dimensions, the mKdV model is replaced by the (nonintegrable) generalized Kadomtsev-Petviashvili equation, which accounts for two-dimensional FCP soliton propagation [19,20].

Other non–SVEA models [21–23], especially the so-called short-pulse equation (SPE) [24], have been proposed. The aim of this paper is to show that the mKdV-sG model is the most general of all approximate non–SVEA models for FCPs, and in fact contains all of them. Here we restrict ourselves to the one-dimensional scalar situation; however, the results of the present paper can be easily generalized to higher dimensions.

The generic mKdV-sG model and the physical hypotheses it involves are described in Sec. II. We show in Sec. III that both the SPE, and another model put forward in Refs. [21–23,25] can be considered as approximate versions of mKdV-sG equation (6). However, the FCP solitons obtained in [25] differ from the already published breathers of mKdV-sG equation [14]. Indeed, in the work [14] it was assumed as a self-focusing-type mKdV equation, whereas in Ref. [25], it was assumed as a self-defocusing one. The self-defocusing-type mKdV equation cannot support any breather solitons (see Sec. IV A) but we show by approximate analytics and by numerical computation that the mKdV-sG equation containing a self-defocusing mKdV term is, within some approximation, equivalent to a pure sG equation, and therefore supports breather-type solitons very close to the sG ones. First, the case when the mKdV-type dispersion term is negligible is investigated in Sec. IV B. Then in Sec. IV C we get within the SVEA the linear dispersion relation $k=k(\omega)$, the group-velocity dispersion (GVD), and a rough approximation of the pulse shape for which self-focusing occurs. Numerical computations confirm the qualitative conclusions, and also that the FCP propagation strongly differs from that one predicted by the SVEA. We then show that both the FCP solitons given in Ref. [25] as well as other soliton solutions can be described by generic mKdV-sG equation (6). The paper is concluded in Sec. V.

II. MKDV, SG, AND MKDV-SG MODELS

As is well known, a soliton is a pulse that propagates in a dispersive medium in such a way that a nonlinear effect compensates dispersion, and the pulse remains unchanged during propagation. This implies two essential features: (i) the medium is lossless, and (ii) the order of magnitude of propagation distance, wave amplitude, wavelength, dispersion, and nonlinear characteristics of the medium are such that neither the dispersion nor the nonlinearity is negligible, and that both have effects comparable in magnitude. We insist on the fact that such assumptions are unavoidable as soon as any kind of soliton is considered (except obviously the so-called "dissipative solitons" but the latter requires gain and loss).

Hence, soliton propagation implies that damping can be neglected. In dielectric media, this occurs far from any resonance frequency. Let us first consider a two-level model with characteristic frequency Ω, and denote by ω a frequency characteristic for the FCP soliton under consideration. The transparency condition implies that either $\omega \ll \Omega$ or $\Omega \ll \omega$. The former case ($\omega \ll \Omega$) corresponds to the long-wave approximation. Assuming further that the wave amplitude is such that the nonlinear and dispersive effects are comparable [hypothesis (ii) above], the perturbative reduction method [26] allows the derivation of the following approximate evolution equation, which is the mKdV one [11,12]:

$$\partial_z E = \frac{1}{6}\frac{d^3 k}{d\omega^3}\partial_t^3 E - \frac{6\pi}{nc}\chi^{(3)}_{xxxx}(\omega;\omega,\omega,-\omega)\partial_t E^3 = 0. \quad (1)$$

Here $n=n(\omega)$ is the refractive index, c is the light velocity in vacuum, and the derivative of the dispersion relation $k=k(\omega)$ and the third-order nonlinear susceptibility are formally taken at the limit $\omega \to 0$, according to the assumption that $\omega \ll \Omega$.

It is important to notice that, since Eq. (1) is not an envelope equation, the third-order time derivative does not represent a third-order dispersion. Within the considered derivation, it is a second-order one: the dispersion coefficient is $(1/6)3/cn''$ in this limit. However, as shown in Ref. [27], it may also include in addition a contribution of the third-order dispersion.

If, on the contrary, the characteristic frequency of the pulse is well above the resonance line ($\omega \gg \Omega$), the short wave approximation [26,28] allows the derivation of the approximate equations [12],

$$\partial_z E = \frac{4i\pi\Omega N}{c}p, \quad (2)$$

$$\partial_z p = \frac{-i}{\hbar}|\mu|^2 E w, \quad (3)$$

$$\partial_t w = \frac{-4i}{\hbar}Ep, \quad (4)$$

where w is the population inversion ($-1 < w < 1$), μ is the dipolar momentum matrix element, and N is the density of atoms.

Equations (2)–(4) coincide with the equations of the self-induced transparency (SIT) [29] although the physical situation is quite different since in the case of SIT one assumes that the wave frequency is close to the resonance, in contrast with the present assumption. Furthermore, the quantities E and w describe here the electric field and population inversion themselves, and not amplitudes modulating some carrier. Here, they are real quantities, and not complex ones as in the case of the SIT. Furthermore, p is not the polarization density but is proportional to its t derivative. Another difference is the absence of a factor of 1/2 in the right-hand side of Eq. (2).

It must be noticed that exact solutions of the Maxwell-Bloch equations (without envelope approximation, nor short- or long-wave one), have been derived by Bullough *et al.* as early as 1971 [30]. A cnoidal solution accounting for a continuous wave and a single-oscillation solution, with determined duration, was also found. These two exact solutions are similar with our approach in that they are not envelope solutions; however, they correspond to the SIT situation where the damping exactly vanishes, close to the resonance, due to a nonlinear effect. Although mathematical expressions may show some analogy, this is quite different from the situation we consider in the present work, in which damping is small because we are far from the resonance.

However, as they are formally identical to the SIT equations; Eqs. (2)–(4) also reduce to the sG equation, which is written as [16,17]

$$\partial_{zt} v = \sin v. \quad (5)$$

Consider now a two-component medium, in which each component is described by a two-level model. There are two resonance frequencies, say $\Omega_1 < \Omega_2$. An appreciable change with respect to the previous situation arises if the transparency domain lies between Ω_1 and Ω_2. In this case we can assume that $\Omega_1 \ll \omega \ll \Omega_2$, and show that the propagation of FCPs can be described by a mKdV-sG equation, of the form [14]

$$u_z + c_1 \sin\left(\int^t u\right) + c_2(u^3)_t + c_3 u_{ttt} = 0. \quad (6)$$

It must be noticed that, in accordance with Eqs. (2)–(4), the coefficient c_1 of the sG-type term in Eq. (6) is proportional to the population inversion. Especially, c_1 is usually positive but becomes negative when a population inversion is realized, and vanishes if the two levels are equally populated.

The approximation used is quite realistic in the general setting. Indeed, in order to get a soliton, the entire pulse spectrum must belong to the transparency domain. Hence, all transitions of the medium separate into two groups: one well below ω, and the other one well above ω. If each of these two sets of resonance frequencies is approximated by a single transition, we exactly get the assumptions under which the model has been derived. In the general case, it is reasonable to consider that the various lines will cumulate together to reconstruct the same terms with more complicated coefficients. This has been proven by means of the reductive perturbation method in the special case of two independent transitions in the long-wave approximation. A rigorous proof in the general case remains to be performed. Notice that the quite general model equation [Eq. (6)] was also derived and studied in Refs. [31,32].

It is well known that the mKdV and sG equations are integrable by means of the IST method [17,33]. Furthermore, they admit breather solutions, which are known to describe FCPs [12], and have both spectrum and field profiles analogous to the ones that can be obtained either experimentally or using other models. Moreover, from the established mathematical properties of these completely integrable equations, any Gaussian input is expected to evolve into a FCP soliton [34]; hence, breathers can be considered as the fundamental solutions of the equations as soon as the input is symmetrical with respect to a change in the sign of the field. mKdV-sG equation (6) is also integrable if $c_3=2c_2$ [18]. For other values of parameters, it has been shown numerically that FCP solitons (or breathers) still exist, and their robustness has been investigated, as well [14].

III. SPE IS A SPECIAL CASE OF MKDV-SG EQUATION

A. Short-pulse equation

An alternative model equation for FCPs is the so-called SPE

$$U_{zt} = U + \frac{1}{6}(U^3)_{tt}. \tag{7}$$

It was first introduced in [24] to describe FCP propagation in silica fibers. The derivation was based on a parabolic approximation of the dispersion relation $\chi=\chi^{(1)}(\lambda)$, valid in silica glass for 1.6 μm$\leq\lambda\leq 3$ μm, and a purely cubic instantaneous nonlinear polarization. The reduction of the two-directional Maxwell equations to a one-directional one was performed by means of a short wave approximation. The mathematical validity of the SPE as an asymptotics to Maxwell equations has been justified in Ref. [35].

The SPE is integrable by means of the IST method [36], and soliton solutions have been given in Ref. [37]. Vectorial versions of the SPE have also been proposed and their soliton solutions have been investigated, as well [38–40].

A third model, which is in fact the SPE with an additional dispersion term, has been first derived in Ref. [21] long before the introduction of the SPE model. It was considered a set of four classical oscillators with frequencies Ω_v, Ω_i, Ω_e, and Ω_{e1}, governing the contribution to polarization of electronic (e) and ionic (i) components, and the electronic ($e1$) and electronic-vibrational (v) nonlinearities. By means of the approximation $\Omega_v,\Omega_i \ll \omega \ll \Omega_e,\Omega_{e1}$, it was derived as a nonlinear evolution equation that reads, in its scalar and normalized form, as

$$U_{zt} + U - \mu U_{tttt} + (U^3)_{tt} = 0. \tag{8}$$

The above assumption is formally identical to the one used in [14] but has a slightly different physical interpretation.

A multidimensional version of Eq. (8) was given in [22] and the self-focusing and pulse compression has been demonstrated in Ref. [23]. However, it has been recently considered again in a vectorial version, which has shown pulse self-compression and FCP soliton propagation [25].

Concerning the validity of the model, it was shown in Refs. [23,25] that, from the general Kramers-Kronig or Sellmeier formula, the general form of the $\varepsilon=\varepsilon(\omega)$ dependence can always be approximated, in the transparency domain, by the relationship

$$\varepsilon(\omega) = \varepsilon_0 - \frac{a}{\omega^2} + b\omega^2, \tag{9}$$

with a and b as some constants, from which the short-pulse equation (8) was straightforwardly deduced, which includes an additional dispersion term; see the third term in Eq. (8).

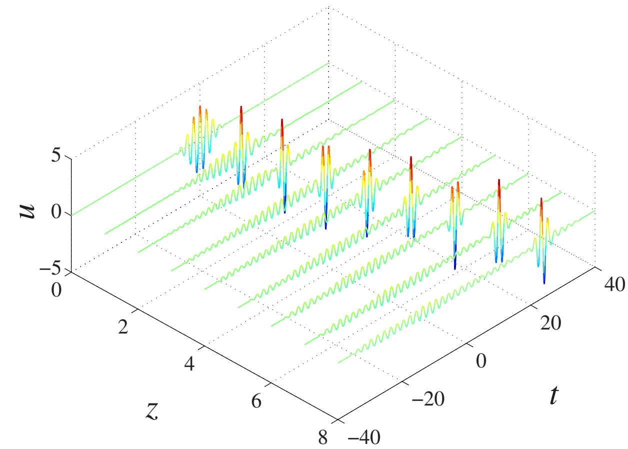

FIG. 1. (Color online) Compression of a Gaussian input pulse to a FCP soliton, as described by the mKdV-sG equation with parameters $c_1=50$, $c_2=0.5$, and $c_3=1$.

B. Reduction of the mKdV-sG equation to the SPE

Obviously, mKdV-sG model (6) reduces to mKdV one [Eq. (10)] if $c_1=0$ and to the sG equation (5) if $c_2=c_3=0$. In the same way, Eq. (8) reduces to SPE (7) if $\mu=0$.

It is easy to derive SPE equation (7) from mKdV-sG one (6): a small amplitude approximation yields $\sin(\int^t u)\simeq\int^t u$, the mKdV-type dispersion term is neglected ($c_3=0$); then setting $c_1=-1$, $c_2=-1/6$, Eq. (6) becomes, after derivation with respect to t, identical to Eq. (7). The same transform but with $c_1=c_2=1$ and $c_3=-\mu$ gives alternative model equation (8).

The mKdV-sG model is able to predict pulse compression, as shown in Fig. 1, which is similar to the result presented in Ref. [25].

However, the concrete situation considered in Ref. [25] involves $\mu>0$, i.e., $c_3<0$. If we disregard the term u coming from the sG equation or the "resonant" part of the equation, this corresponds to a defocusing mKdV equation. It is worthy to mention that, as has been shown in Ref. [14], the mKdV-sG equation supports FCP solitons but only the case of focusing mKdV equation was considered in that work. However, the defocusing mKdV equation does not support FCP solitons, as is detailed in Sec. IV A below.

On the other hand, the pure sG equation admits breather solutions that allow the describing of the FCP solitons [12]. It is worthy to notice that the soliton put forward in Ref. [25] is nothing else than a soliton of the mKdV-sG equation by using the dispersion term of the sG equation and the nonlinearity term of the mKdV equation, whose relative signs correspond to the self-focusing case. This statement will be proven by the analysis below.

IV. SOLITON PROPAGATION IN THE CASE OF A DEFOCUSING KERR TERM

A. Defocusing mKdV equation

For the two-level model, as considered in [12], the group-velocity dispersion is normal ($n''>0$) and the third-order nonlinear susceptibility $\chi^{(3)}$ is negative; hence, mKdV equa-

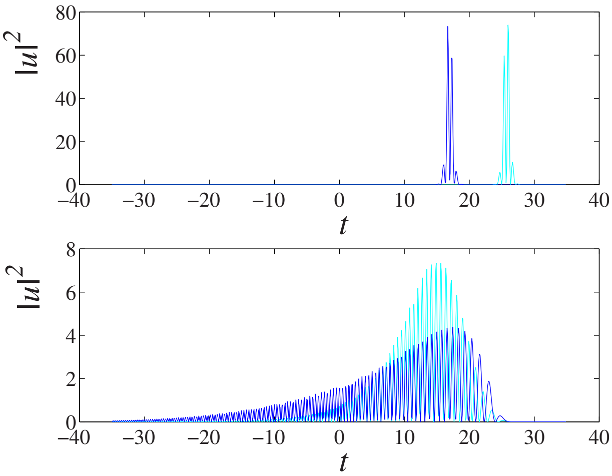

FIG. 2. (Color online) Top: the intensity $|u|^2$ of the input FCP [light blue (gray) curve] and its evolution at $z=0.2$ according to the focusing mKdV equation [dark blue (black) curve] for $c_2=+1/3$. Bottom: the linear dispersion (for $c_2=0$) of the same input at the same propagation distance [light blue (gray)], and its nonlinear dispersion according to the defocusing mKdV equation [dark blue (black) curve] for $c_2=-1/3$. The input is a breather of the focusing mKdV equation, with angular frequency of four and inverse pulse duration of two.

tion (1) is of self-focusing type, and can be written as

$$\partial_z u + u^2 u_t + u_{ttt} = 0, \tag{10}$$

where $u \propto E$, in normalized form.

However, we can consider a general situation of a defocusing optical nonlinearity, with either a normal dispersion ($n''>0$) and a positive $\chi^{(3)}$ nonlinearity, or with an anomalous dispersion ($n''<0$) and a negative $\chi^{(3)}$ nonlinearity. Therefore, the obtained mKdV equation is of defocusing type and can be written in normalized units as

$$\partial_\zeta u - u^2 u_\xi + u_{\xi\xi\xi} = 0. \tag{11}$$

It is well known that Eq. (11) does not admit any soliton solution [17]. An incident wave packet is spread out by dispersion and nonlinearity, as shown by numerical computation using the so-called "exponential time differencing method" [41] along with absorbing boundary conditions introduced to avoid numerical instability of the background, see Fig. 2. The results displayed in this figure clearly demonstrate that the nonlinear dispersion is stronger than the linear one.

B. Defocusing nonresonant interaction and focusing resonant one

We start from the mKdV-sG Eq. (6) and by setting $\int^t u = v$, we get the equation

$$v_{zt} + c_1 \sin v + c_2 (v_t^3)_t + c_3 v_{tttt} = 0. \tag{12}$$

For low amplitudes, the sG term in the above equation can be expanded in a power series of v to yield

$$v_{zt} + c_1 \left(v - \frac{v^3}{6} \right) + c_2 (v_t^3)_t + c_3 v_{tttt} = 0. \tag{13}$$

Due to the peculiar form of the third term in Eq. (13) containing the partial derivative $(v_t^3)_t$, the two nonlinear terms in Eq. (13) differ and cannot be straightforwardly compared. However, let us perform a slowly-varying envelope approximation, according to $v=A(t)e^{i\omega t}+$c.c.. (where "c.c." holds for "complex conjugate"), with $\omega \gg 1$. It yields

$$(v_t^3)_t \simeq 3\omega^4 A^3 e^{3i\omega t} - 3\omega^4 A |A|^2 e^{i\omega t} + \text{c.c.} \tag{14}$$

On the other hand

$$v^3 = A^3 e^{3i\omega t} + 3A|A|^2 e^{i\omega t} + \text{c.c.} \tag{15}$$

For large values of ω, the third harmonic terms do not couple appreciably with the fundamental one; hence they can be neglected and

$$(v_t^3)_t \simeq -\omega^4 v^3. \tag{16}$$

Let us now compare Eq. (12) with Eq. (8). Setting $v_t = pU$ in Eq. (13) we get

$$U_{zt} + c_1 U + p^2 c_2 (U^3)_{tt} - p^2 \frac{c_1}{6} \left(\int^t U \right)_t^3 + c_3 U_{tttt} = 0. \tag{17}$$

Using Eq. (16) this yields

$$U_{zt} + c_1 U + p^2 \left(c_2 + \frac{c_1}{6\omega^4} \right) (U^3)_{tt} + c_3 U_{tttt} = 0. \tag{18}$$

Equations (8) and (18) coincide if $c_1=1$, $c_3=-\mu$, and

$$p^2 \left(c_2 + \frac{c_1}{6\omega^4} \right) = 1. \tag{19}$$

Especially, if $c_2=c_3=0$, Eq. (12) is exactly the sG equation. If we assume $c_1=1$, the approximate transformation above holds in this case if $p=\omega^2\sqrt{6}$. Hence, if u_0 is some breather solution of mKdV-sG equation (6) (which is a sG equation in this case), with a large enough carrier frequency ω, the field $U_0=u_0/(\omega^2\sqrt{6})$ satisfies approximately Eq. (8) with $\mu=0$. In short, the term $\sin u \simeq u - u^3/6$ contains contributions from both dispersion and nonlinearity. Therefore, within the slowly-varying envelope approximation, both cubic nonlinearities, involving time derivatives or not, are equivalent.

These analytical estimates were checked numerically, as shown in Fig. 3: it is clearly seen in Fig. 3 that an initial adequately rescaled sG breather propagates almost without deformation according to the mKdV-sG equation (6) with $c_1=c_2=1$, $c_3 \simeq 0$, which is the case considered in Ref. [25]; see also the short-pulse equation (4) and Eq. (5) (c_3 is set to a small nonzero value because our numerical code does not allow $c_3=0$ exactly. The exact value of c_3 is thus of no significance).

C. Pulse compression with a defocusing dispersion term

We intend now to show that mKdV-sG model (6) with a true defocusing mKdV part, that is, when $c_3<0$, also pos-

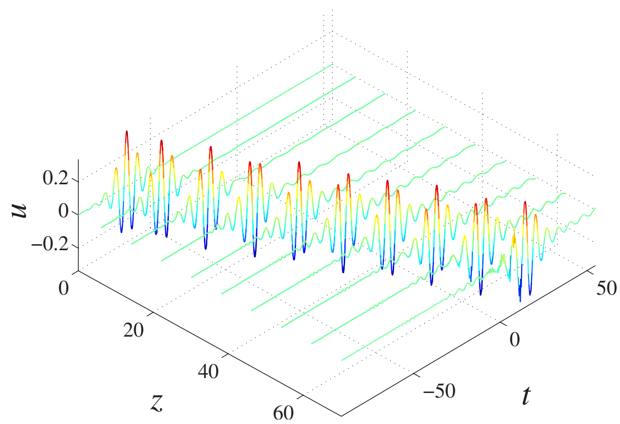

FIG. 3. (Color online) Propagation of a FCP soliton in the frame of the mKdV-sG model (arbitrary units, $c_1=c_2=1$, $c_3=0.0001$). The input is an adequately rescaled sG breather.

sesses FCP soliton solutions and accounts for pulse self-compression. The "focusing" or "defocusing" character of the equation can be determined in the SVEA limit. In fact, one can doubt that this concept makes sense out of the SVEA. The SVEA limit can be derived by means of the reductive perturbation method [26]. Thus in order to treat all equations at the same time, we consider a more general equation

$$v_{zt}+c_1 v-\frac{c_1'}{6}v^3+c_2(v_t^3)_t+c_3 v_{tttt}=0, \tag{20}$$

which becomes mKdV-sG equation (9) if $c_1'=c_1$, and reduces to Eq. (8) if $c_1'=0$. We expand the field v as

$$v=\varepsilon(Ae^{i\varphi}+\text{c.c.})+\sum_{n,p\geq 0}\varepsilon^n v_n^p e^{ip\varphi}, \tag{21}$$

in which the fundamental phase $\varphi=\omega t-kz$, and the profiles A and v_n^p depend on the slow variables $\tau=\varepsilon(-z/V)$ and $\zeta=\varepsilon^3 t$. If series expansion (21) is inserted into Eq. (20), in the leading order in ε, we get the linear dispersion relation

$$k=\frac{(-c_1)}{\omega}-c_3\omega^3. \tag{22}$$

Then in the order ε^2 for the fundamental frequency we get the velocity V, which satisfies $V=d\omega/dk$ as expected. In the order ε^3 and for $p=1$, we obtain the form of the governing nonlinear evolution equation, which is, as expected, the NLS one, written as

$$A_\zeta+\frac{\beta_2}{2}A_{\tau\tau}+\gamma A|A|^2=0. \tag{23}$$

One additional important result of this reductive perturbation approach is the ω dependence of the coefficients of NLS equation (23), and the corresponding relationships between the coefficients c_1, c_2, and c_3 of the original mKdV-sG equation and of those of the NLS equation:

$$\beta_2=\frac{d^2k}{d\omega^2}=\frac{(-2c_1)}{\omega^3}-6c_3\omega, \tag{24}$$

$$\gamma=\frac{(-c_1)}{2\omega}-3c_2\omega^2. \tag{25}$$

It is well known that NLS equation (23) is of focusing type and admits soliton solutions if the product $(\beta_2\gamma)$ is positive (the Lighthill criterion). In the following we analyze the implications of the above results, which were obtained by using the reductive perturbation method, on the characteristic features of the five main equations describing the dynamics of FCP solitons [see Eqs. (1)–(8)].

1. sG equation

The general Eq. (20) becomes a sG equation (in the low-amplitude limit involved by the SVEA) if $c_2=c_3=0$ and $c_1'=c_1$. Then $\beta_2=-2c_1/\omega^3$, $\gamma=-c_1/(2\omega)$, and hence $\beta_2\gamma$ is always positive. Hence the sG equation is of focusing type, regardless of the frequency.

2. mKdV equation

The general equation (20) becomes a mKdV equation if $c_1'=c_1=0$. Then $\beta_2=-6c_3\omega$, $\gamma=-3c_2\omega^2$, and hence $\beta_2\gamma$ has the same sign as c_2c_3. Therefore the sign of the product c_2c_3, as it is already known, makes the distinction between a focusing-type and a defocusing-type mKdV equation. As in the case of the sG equation, the Lighthill criterion holds regardless of the frequency.

3. Short-pulse equation.

If $c_1'=c_3=0$ then Eq. (20) becomes the SPE, and has normalized form (7) if, additionally, $c_1=-1$ and $c_2=-1/6$. We get $\beta_2=-2c_1/\omega^3$, $\gamma=-3c_2\omega^2$, and hence the product $\beta_2\gamma$ has the same sign as the product c_1c_2. Thus we get both a focusing-type and a defocusing-type SPE, normalized form (7) being a focusing-type SPE. Here again, the Lighthill criterion holds regardless of frequency.

4. Evolution equation (8)

If we put $c_1'=0$, $c_1=c_2=1$, and $c_3=-\mu$ in Eq. (20) we get normalized version (8). The nonlinear coefficient γ has the same expression as for the SPE but β_2 is general.

Here the Lighthill criterion $\beta_2\gamma>0$ writes

$$c_2\left(c_3+\frac{c_1}{3\omega^4}\right)>0,\quad \text{or}\quad \mu>\frac{1}{3\omega^4} \tag{26}$$

in its normalized form.

5. mKdV-sG equation.

In this case $c_1'=c_1$, and the Lighthill criterion for the existence of NLS solitons becomes $(c_1+6c_2\omega^3)[c_3+c_1/(3\omega^4)]>0$. It involves both the sign of coefficients c_i and the frequency ω. Let us restrict the discussion to those coefficients for which mKdV-sG equation reduces to Eq. (8), i.e., for c_1, $c_2>0$. Then the condition for mKdV-sG equation to be of focusing-type is the same as Lighthill condition (26) for Eq. (8).

From the single soliton solution of NLS equation, we get the following approximate solution to mKdV-sG equation:

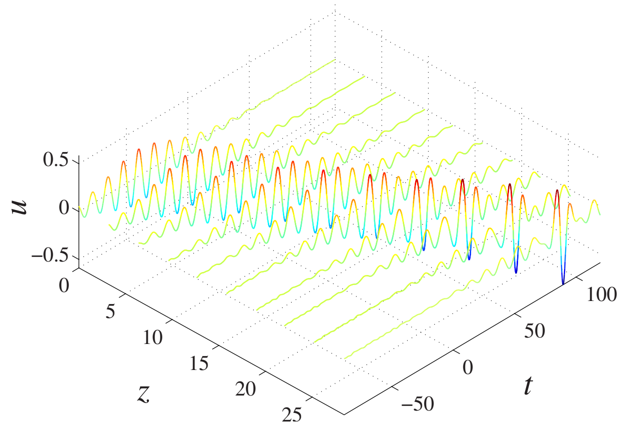

FIG. 4. (Color online) Pulse compression described by mKdV-sG equation (12) with defocusing mKdV part. Input is a pulse with hyperbolic secant envelope solution to the SVEA limit of mKdV-sG, with pulse length $\tau_0=25$ and angular frequency $\omega=0.5$. Parameters are $c_1=c_2=1$, $c_3=-0.5$.

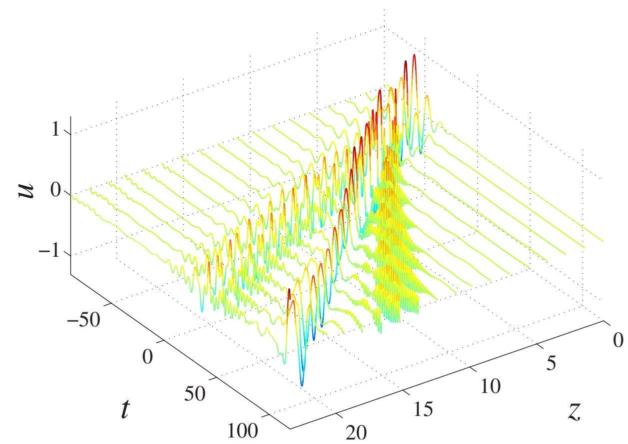

FIG. 5. (Color online) FCP soliton propagation according to mKdV-sG equation (12) with defocusing mKdV part. From the sech-shaped input pulse with length $\tau_0=8$ and angular frequency $\omega=0.5$, two FCP solitons, one larger than the other, plus dispersing waves, are formed. Parameters are $c_1=c_2=1$, and $c_3=-0.5$.

$$v=\frac{1}{\sqrt{2|\gamma|}}\frac{2pe^{ip^2z}}{\cosh[p\sqrt{2/|\beta_2|}(t-z/V)]}e^{i(\omega t-kz)}+\text{c.c.},\qquad(27)$$

p being an arbitrary real parameter.

Obviously, Eq. (23) coincides with the NLS equation derived in the standard way within the SVEA, and pulse (27) is not a FCP but is valid only within SVEA (for small values of the soliton parameter p). However, the existence of a soliton in the SVEA limit is a good heuristic argument for the existence of a FCP soliton; therefore, the expression of the NLS soliton can be taken as initial data for the numerical resolution of mKdV-sG equation. It is quite difficult to check numerically with accuracy how the separation line between focusing and defocusing behaves as the pulse becomes shorter and the SVEA fails. Indeed, the separatrix corresponds to zero group-velocity dispersion, and the discrimination between pulse reshaping and mere spreading out requires computation of the evolution on many dispersion lengths. It would thus require not only lengthy computation but also compensation of the pulse velocity, which is difficult to determine for the several splinters that result from the reshaping of a FCP (see below). Hence this point is left for further investigation. It is seen that, even for rather long FCPs, the discrepancy between SVEA approximation (27) and a FCP soliton solution to mKdV-sG equation can be appreciable. Especially, pulse compression can be clearly seen on the example displayed in Fig. 4. In this case, the pulse length $\tau_0=\sqrt{|\beta_2|/2}/p$ is 25 and the angular frequency is $\omega=0.5$. The velocity V_p of the pulse is computed from the numerical data, as $V_p\simeq 0.222$. It is close to, but significantly differs from, the value of group velocity $d\omega/dk\simeq 0.229$ predicted by the SVEA according to Eq. (22).

For shorter pulses the numerical solution to the mKdV-sG equation goes further away from the SVEA-NLS approximation [see approximate solution (27) to mKdV-sG equation] but the FCP soliton propagation still occurs, as can be seen in Fig. 5; here two FCP solitons, one taking the major part of the energy, and the other much smaller, are formed while a non-negligible part of energy is radiated as dispersing waves. The pulse velocity computed from the numerical data pertaining to Fig. 5 is $V_p\simeq 0.134$, while $d\omega/dk\simeq 0.229$ as above. The discrepancy is large, thus confirming the fact already evidenced in Ref. [14] that the usual expression of the group velocity is not valid any more for a short FCP soliton.

V. CONCLUSION

In conclusion, we have revisited five main dynamical models existing in the literature, which describe the propagation of few-optical-cycle pulses in transparent media. We have proven that the dynamical model based on the modified Korteweg–de Vries sine-Gordon partial differential equation was able to retrieve the results reported so far in the literature, and so demonstrating its remarkable mathematical capabilities in describing the physics of few-cycle-pulse optical solitons. To this aim we have shown that the generic modified Korteweg–de Vries sine-Gordon equation contains all non–slowly-varying envelope-approximation model equations that have been earlier proposed for the description of (1+1)-dimensional few-cycle-pulse soliton propagation. The results obtained in this work can be relatively easily generalized to (2+1)-dimensional few-cycle-pulse propagation models.

[1] T. Brabec and F. Krausz, Rev. Mod. Phys. **72**, 545 (2000).
[2] M. Wegener, *Extreme Nonlinear Optics* (Springer-Verlag, Berlin, 2005).
[3] A. Scrinzi, M. Yu Ivanov, R. Kienberger, and D. M. Villeneuve, J. Phys. B **39**, R1 (2006).
[4] X. Tan, X. Fan, Y. Yang, and D. Tong, J. Mod. Opt. **55**, 2439 (2008).
[5] N. N. Rosanov, V. E. Semenov, and N. V. Vyssotina, Laser Phys. **17**, 1311 (2007).
[6] N. N. Rosanov, V. E. Semenov, and N. V. Vysotina, Quantum Electron. **38**, 137 (2008).
[7] A. Nazarkin, Phys. Rev. Lett. **97**, 163904 (2006).
[8] T. Brabec and F. Krausz, Phys. Rev. Lett. **78**, 3282 (1997).
[9] M. V. Tognetti and H. M. Crespo, J. Opt. Soc. Am. B **24**, 1410 (2007).
[10] A. A. Voronin and A. M. Zheltikov, Phys. Rev. A **78**, 063834 (2008).
[11] I. V. Mel'nikov, D. Mihalache, F. Moldoveanu, and N.-C. Panoiu, Phys. Rev. A **56**, 1569 (1997).
[12] H. Leblond and F. Sanchez, Phys. Rev. A **67**, 013804 (2003).
[13] I. V. Mel'nikov, H. Leblond, F. Sanchez, and D. Mihalache, IEEE J. Sel. Top. Quantum Electron. **10**, 870 (2004).
[14] H. Leblond, S. V. Sazonov, I. V. Mel'nikov, D. Mihalache, and F. Sanchez, Phys. Rev. A **74**, 063815 (2006).
[15] H. Leblond, I. V. Mel'nikov, and D. Mihalache, Phys. Rev. A **78**, 043802 (2008).
[16] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equations* (Academic Press, London, 1982).
[17] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
[18] K. Konno, W. Kameyama, and H. Sanuki, J. Phys. Soc. Jpn. **37**, 171 (1974).
[19] I. V. Mel'nikov, D. Mihalache, and N.-C. Panoiu, Opt. Commun. **181**, 345 (2000).
[20] H. Leblond, F. Sanchez, I. V. Mel'nikov, and D. Mihalache, Math. Comput. Simul. **69**, 378 (2005).
[21] S. A. Kozlov and S. V. Sazonov, J. Exp. Theor. Phys. **84**, 221 (1997).
[22] V. G. Bespalov, S. A. Kozlov, Y. A. Shpolyansky, and I. A. Walmsley, Phys. Rev. A **66**, 013811 (2002).
[23] A. N. Berkovsky, S. A. Kozlov, and Yu. A. Shpolyanskiy, Phys. Rev. A **72**, 043821 (2005).
[24] T. Schäfer and C. E. Wayne, Physica D **196**, 90 (2004).
[25] S. A. Skobelev, D. V. Kartashov, and A. V. Kim, Phys. Rev. Lett. **99**, 203902 (2007).
[26] H. Leblond, J. Phys. B **41**, 043001 (2008).
[27] Y. S. Kivshar, Opt. Lett. **16**, 892 (1991).
[28] M. A. Manna and V. Merle, Phys. Rev. E **57**, 6206 (1998).
[29] R. K. Bullough, P. J. Caudrey, J. C. Eilbeck, and J. D. Gibbon, Opto-electronics (London) **6**, 121 (1974).
[30] R. K. Bullough and F. Ahmad, Phys. Rev. Lett. **27**, 330 (1971).
[31] S. V. Sazonov, J. Exp. Theor. Phys. **92**, 361 (2001).
[32] A. N. Bugay and S. V. Sazonov, J. Opt. B: Quantum Semiclassical Opt. **6**, 328 (2004).
[33] M. Wadati, J. Phys. Soc. Jpn. **34**, 1289 (1973).
[34] S. Clarke, R. Grimshaw, P. Miller, E. Pelinovsky, and T. Talipova, Chaos **10**, 383 (2000).
[35] Y. Chung, C. K. R. T. Jones, T. Schäfer, and C. E. Wayne, Nonlinearity **18**, 1351 (2005).
[36] A. Sakovich and S. Sakovich, J. Phys. Soc. Jpn. **74**, 239 (2005).
[37] A. Sakovich and S. Sakovich, J. Phys. A **39**, L361 (2006).
[38] M. Pietrzyk, I. Kanattšikov, and U. Bandelow, J. Nonlinear Math. Phys. **15**, 162 (2008).
[39] Sh. Amiranashvili, A. G. Vladimirov, and U. Bandelow, Phys. Rev. A **77**, 063821 (2008).
[40] A. V. Kim, S. A. Skobelev, D. Anderson, T. Hansson, and M. Lisak, Phys. Rev. A **77**, 043823 (2008).
[41] S. M. Cox and P. C. Matthews, J. Comput. Phys. **176**, 430 (2002).