FACULTATEA DE FIZICĂ
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NONLINEAR PROCESSES IN FIELD THEORY AND ASTROPHYSICS

Nonlinear Mathematical Methods and Dusty Plasma Physics

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- Bucureşti, 2010 -

The author is very much indebted to Professor Mihai Vişinescu for his permanent interest and guidance during the whole period in which this thesis was elaborated.
Also he wants to thank to his colleagues Dr. Anca Vişinescu, Dr. Adrian Ştefan Cârstea and Dr. Dan Grecu with whom he collaborated in obtaining most of the results presented in this paper.
All the helpful discussions and support from the colleagues in the Department of Theoretical Physics through-out the last seven years, is kindly acknowledged.

Contents

Chapter I: Introduction

In a nonlinear system "*the whole is larger than the sum of its parts*". This is achieved through the emergence of new structures that are spatially or temporally coherent, and which do not exist in a linear approximation. These emergent structures can be considered objects with their own features, lifetimes and are interacting with each other specifically. Since these interactions are also nonlinear, the dynamics of these structures generates new emergent structures at higher order of description. Thus the molecules in organic chemistry appear from nonlinear interactions between elements, providing a structural basis for the proteins and ribonucleic acids of biochemistry, and so on up to the many levels of activity of cells in living organisms. In all these hierarchical stages nonlinearity is a basic *ingredient.*

Usually the behavior of nonlinear systems is described in the language of nonlinear partial, differential equations. Contrary to the case of linear partial differential equations where the superposition principle works, in the nonlinear case such a principle is no longer valid, and no general methods exist to construct exact (closed-form) solutions. This assertion is correct in general, except for the special situation of completely integrable systems, where, since 1960s, general methods, known under the name "*inverse scattering transform*" (IST) methods, were developed to solve these equations when the initial conditions are given. Fortunately these systems are not simple mathematical *games*, but they describe quite general processes, and are encountered in various domains of physics (hydrodynamics, plasma physics, nonlinear optics, quasi-one-dimensional molecular systems) and not only. The subject is discussed in many textbooks, of which we mention [12, 13, 29, 38, 44, 45, 47, 50], and it isn't the place to further insist on it.

Two of these completely integrable partial differential equations will be permanently present in the present work, namely the Korteweg-de Vries and the nonlinear Schrödinger equation. They describe general phenomena evolution and therefore they appear frequently in various branches of physics. For instance the KdV equation gives the evolution of a wave in a weak dispersive medium with a small nonlinearity where a subtle balance is attained between the effect of the dispersion to spread out the energy of the pulse and that of the nonlinearity which draws it back together. The solitary wave, thus emerged, is an independent dynamic entity maintaining the balance between dispersion and nonlinearity. On the other hand, the NLS equation describes the evolution of the amplitude of a quasi-monochromatic wave propagating in a weakly nonlinear medium. The history of these equations almost confounds with the history of what, nowadays, is known as the field of completely integrable systems. Some stages of the evolution of this field of mathematical physics will be briefly reviewed below.

It all started in 1834 when John Scott Russell, a Scottish scientist and engineer, observed an unexpected phenomenon - a novel type of wave on the surface of a narrow channel provoked by the heavily loaded boat when it suddenly stopped – "*not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel without change of form or diminution of speed*". He "*followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot and a half in height*" until the wave became lost in the windings of the channel. He called this wave "*the great wave of translation*" and found a very simple relation between the velocity v of the wave, the channel depth h, the height u_m of the wave and the gravitational acceleration g

$$v = \sqrt{g(h + u_m)}$$

Furthermore he demonstrated that a sufficiently large initial mass of water would produce two or more solitary waves, when suddenly set in motion, and that the solitary waves collide and cross each other without change of any kind. The phenomenon, soon confirmed by experiments on the Canal of Bourgogne, near Dijon (1865), generated a long debate between eminent mathematicians of the nineteenth century. This debate finished in 1895 with the paper of Diederik Johannes Korteweg and his PhD student Gustav de Vries [30], based on earlier work of Joseph Boussinesq, where they obtained the nonlinear partial differential equation which bears their name

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \varepsilon\frac{\partial^3 u}{\partial x^3} + \gamma u\frac{\partial u}{\partial x} = 0$$

and which came to play a key role in future theoretical developments. Here $c = \sqrt{gh}$ is the speed of the small amplitude waves, $\varepsilon = c\left(h^2/6 - T/2\rho g\right)$ is the dispersive parameter, $\gamma = 3c/2h$ the nonlinear one, T the surface tension and ρ the density of the water. They showed that the equation has exact traveling wave solutions of the form $u(x,t) \sim \operatorname{sech}^2(\kappa(x - vt))$, similar to the wave shape found experimentally.

This episode in the history of the KdV equation was followed by a "silence" of more than half a century. A striking and very unexpected result was obtained in a totally unrelated problem by Fermi, Ulam and Pasta [20] in the mid forties at Los Alamos. They considered a chain with 64 atoms, with linear and nonlinear nearest neighbor interaction and with fixed conditions at its ends. The linear model has 64 normal modes. Injecting the energy in the first lowest normal modes, they expected that the nonlinear system will evolve toward an equilibrium state characterized by the equipartition of the energy on all the normal modes. Although initially the energy had the tendency to spread over the other normal modes, after a certain time it came back to the initial modes. It is the first numerical experiment which contradicted a largely accepted opinion that a system with nonlinear interaction between its components evolves toward thermodynamic equilibrium.

When the Los Alamos papers were declassified this result puzzled many researchers who repeated the experiment with similar results. A step further was made by Kruskal and Zabusky in 1965 [51]. Instead of working in the normal mode space, they used the real space representation with periodic boundary conditions. They found that an initial perturbation splits into a train of several pulses, the highest traveling with higher velocity. Due to the periodic boundary conditions the higher pulses were catching up the smaller ones, they collide and separate from each other without changing their form. They called these entities SOLITONS. Also they showed that in the continuum limit the Fermi-Pasta-Ulam problem is connected with the KdV equation.

Two years later Gardner, Green, Kruskal and Miura [21] solved the KdV equation with given initial condition using the so-called "*inverse scattering transform*" (IST) method. The method is schematically presented in the diagram I.1. Given the initial condition $u(x, 0)$ the solution $u(x, t)$ at an

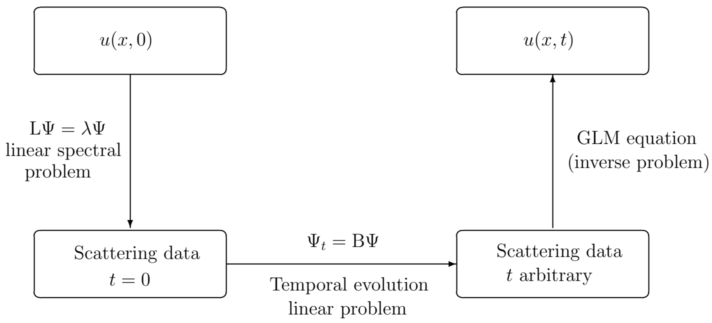

Figure I.1: The scheme of Inverse Scattering Transform method used to solve completely integrable partial differential equations.

arbitrary time t is obtained by applying the following algorithm:

- One associates a linear spatial problem

$$\mathrm{L}\Psi = \lambda\Psi,$$

  where L is a linear operator in which $u(x, 0)$ appears as a potential. This eigen-equation allows the determination of the scattering data (bound eigenvalues, the reflection coefficient in continuum). For the KdV equation L is the well known one-dimensional Schrödinger equation of quantum mechanics

$$\mathrm{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x, 0).$$

- One associates a second linear problem which allows the study of the time evolution of the scattering data

$$\Psi_t = \mathrm{B}\Psi,$$

where the lower index indicates partial derivative with respect to the specified variable t. For the KdV equation B is a skew-symmetric operator

$$\mathrm{B} = -4\frac{\partial^3}{\partial x^3} + 3\left(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u\right).$$

These two operators are not independent but they satisfy the Lax equation [31]

$$\Psi_t + [\mathrm{L},\ \mathrm{B}]\,\Psi = 0$$

and using the previous expressions for L and B, it transforms into nothing else but the KdV equation.

- Having the form of the scattering data at any time t the last step in deriving $u(x,t)$ is the reconstruction (inverse problem) of the potential corresponding to these data. This is done by solving a linear integral problem, the so-called Gelfand-Levitan-Marchenko equation

$$K(x,y) + F(x,y) + \int_x^\infty F(x+z)K(x,z)\mathrm{d}z = 0$$
$$u(x,t) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x)$$

with the kernel $F(x,y)$ being constructed from the scattering data.

The essential step in this scheme is finding the right spectral problem.

Soon, Zakharov and Fadeev [52] proved the complete integrability of the KdV system, and probably more important Zakharov and Shabat [53] showed that the IST method can be applied to solve also the NLS equation. Later in 1974, Ablowitz, Kaup, Newell and Segur [2] extended the IST method to a larger class of nonlinear evolution equations [49] comprising the modified KdV equation and the sine-Gordon equation, and thus, a new branch of (applied) mathematics was born.

Even before the invention of the IST method, in the mid 1960s, the KdV equation appeared in plasma physics describing the evolution of nonlinear ionic excitations. In their paper [49] Washimi and Taniuti started from hydrodynamic equations of a plasma (electrons and positive ions) and found the KdV equation as the equation governing the evolution of the local electrostatic potential using an appropriate asymptotic method called *the multiple scales method.* This method (borrowed from mathematics) takes into account that the effect of a weak nonlinearity is cumulative and manifests at large time and space scales which require the introduction of *stretched variables*

$$\xi = \varepsilon^{1/2}(x - ct), \quad \tau = \varepsilon^{3/2}t.$$

Here, ε is a small parameter and all the important physical quantities of the problem like the densities, velocities and electrostatic potential expand in power series of it. Further details about this technique can be found in chapter 4 of the present paper dedicated to the study of dusty plasma.

The second equation on which the present paper will concentrate is the (cubic) NLS equation

$$i\frac{\partial A}{\partial t} + \beta\frac{\partial^2 A}{\partial x^2} + \gamma|A|^2 A = 0$$

It plays a significant role in the theory of propagation of the envelope of wave trains in many stable dispersive physical systems in which no dissipation occurs. The evolution of many nonlinear systems can be given by harmonic wave train (solutions) $\Phi = A\exp[i(kx - \omega t)]$ with an amplitude dependent dispersion relation $\omega = \omega(k, |A|^2)$. A Taylor expansion around some suitable wave number k_0 and frequency ω_0 of the main carrier wave gives [26]

$$\omega - \omega_0 = \left(\frac{\partial \omega}{\partial k}\right)_{k_0}(k - k_0) + \frac{1}{2}\left(\frac{\partial^2 \omega}{\partial k^2}\right)_{k_0}(k - k_0)^2 + \left(\frac{\partial \omega}{\partial |A|^2}\right)_{k_0}|A|^2 + \ldots$$

This is the Fourier transform of an operator equation acting on the amplitude A ($\omega - \omega_0 \to -i\frac{\partial}{\partial t}$, $k - k_0 \to i\frac{\partial}{\partial x}$)

$$i\left(\frac{\partial}{\partial t} + \left(\frac{\partial \omega}{\partial k}\right)_{k_0}\frac{\partial}{\partial x}\right)A - \frac{1}{2}\left(\frac{\partial \omega}{\partial k}\right)_{k_0}\frac{\partial^2 A}{\partial x^2} + \left(\frac{\partial \omega}{\partial |A|^2}\right)_{k_0}|A|^2 A = 0$$

which is the NLS equation. This heuristic derivation shows how the effect of the nonlinearity can be modeled by thinking of a system as having an amplitude dependent dispersion relation. But it says nothing about the coefficients of the equation, in particular about the coefficient of the nonlinear term, and one shall see that the sign of this term is rather important.

The equation has arisen in practice to describe nonlinear envelope waves in hydrodynamics [9], nonlinear optics [29,38] , especially in the propagation of nonlinear optical pulses in optical fibers [5,27,28], and plasma physics [26].

One of the most puzzling phenomenon occurring out at sea in deep waters of the oceans is the generation of "freak waves" (also referred to as rogue or giant waves), which can arise from a relatively calm sea. It is an exciting problem of modern oceanography with a yet unclear explanation. There are different mechanisms leading to the formation of such events. In a linear theory the constructing interference between two or more waves of different lengths with equal phases (linear superposition of Fourier modes) can lead to the formation of such waves. Other mechanisms could be: wave-current interaction, refraction around shoals, diffraction around islands. Recently, an increasingly popular opinion is that the nonlinear effect of water waves propagating in deep waters is a basic ingredient to produce these giant waves [40, 41, 43, 48] (background informations with a short history of the phenomenon can be consulted even in free on-line encyclopaedia [1]). The NLS equation is the first candidate to explain their formation and the Benjamin-Feir instability [8] and the nonlinear focusing are believed to play a significant role in the process. A higher order equation was obtained by Dysthe [15]. It couples the mean flow with the slowly modulated amplitude of the surface elevation

$$\begin{aligned} i\frac{\partial B}{\partial t} + \hat{\mathrm{L}}B &= \frac{1}{2}|B|^2 - \frac{3i}{2}|B|^2\frac{\partial B}{\partial x} - \frac{i}{4}B^2\frac{\partial B^*}{\partial x} + \frac{\partial\bar{\phi}}{\partial x}B \\ \frac{\partial\bar{\phi}}{\partial z} &= \frac{1}{2}\frac{\partial}{\partial x}|B|^2, \qquad z = 0 \\ \nabla^2\bar{\phi} &= 0, \qquad z < 0 \end{aligned}$$

Here B is the slowly modulated amplitude of the surface elevation, $\bar{\phi}$ is the slowly varying mean velocity, $\hat{\mathrm{L}}$ is the spatial linear differential operator, and convenient dimensionless quantities are used. Numerical simulation based on Dysthe's system are in better agreement with the experimental data.

Another possible mechanism for the formation of freak waves could be the interaction of two-wave systems in deep water with two different directions of propagation [42]. Such wave systems, characterized by two different spectral peaks, also known as crossing sea states, have become of particular interest as they occur when a wind sea (a wave system that is produced by the local wind) and a swell (a system of waves generated elsewhere, that have propagated far from the area where they were generated and are not affected by the local wind) coexist. Using the hypothesis that both sea systems are narrow banded, a system of coupled NLS equations is found [25,42]

$$\begin{aligned} i\frac{\partial A}{\partial t} + \alpha\frac{\partial^2 A}{\partial x^2} - \left(\xi|A|^2 + 2\zeta|B|^2\right)A = 0 \\ i\frac{\partial B}{\partial t} + \alpha\frac{\partial^2 B}{\partial x^2} - \left(\xi|B|^2 + 2\zeta|A|^2\right)B = 0 \end{aligned}$$

For $\xi = 2\zeta$ the system is completely integrable [32]. The modulational instability of this system was discussed in [42] from a deterministic point of view and in [46] from a statistical one with the general result of an increase of the instability growth rate and an enlargement of the instability region. This coupled NLS system is relevant also for studying the propagation of two waves with different polarizations in a nonlinear optical medium [33].

Another problem of physics that contributed to the large popularity of NLS equation was related to the soliton propagation in nonlinear optical fibers. It was pointed out by Hasegawa and Tappert [28] that in a single-mode optical fiber the nonlinearity of the refractive index could be used to compensate the pulse broadening due to dispersion. The basic equation describing the pulse propagation is the NLS equation and it was suggested that the special properties of its soliton solutions make them ideal *bits* for long-distance communication transmissions. Seven years later Mollenauer and his co-workers [34] described for the first time the experimental production of temporal optical solitons. Since then the field exploded exponentially both from theoretical an experimental point of view (see [5, 6, 27] and references therein). Some experimental data showing these developments are: in 1991 a Bell Labs research team succeeded to transmit solitons over more than 14,000 km of optical fiber using erbium optical fiber amplifiers; in 1998 a team at France Telecom demonstrated a data transmission of one Terabit per second (1 Tbps); in 2001 the practical use of solitons became a reality when Algety Telecom deployed submarine telecommunications equipment in Europe carrying real traffic.

A significant moment in the history of NLS equation was the research workshop organized in the summer of 1972 by Alan Newell [36]. It was a seminal meeting for soliton research in the English speaking world, with participants ranged over a wide spectrum of ages and background interest. One of the most significant contribution arrived by mail from Soviet Union from Zakharov and Shabat [53] who have shown that even NLS equation can be solved by an IST method and it represents a completely integrable system. At the end of the conference many participants left with the conviction that the most fundamental nonlinear equations (Korteweg-de Vries, Nonlinear Schrödinger, Sine-Gordon, Toda model) displayed solitary wave behavior and can be solved if the initial conditions are given. The next two years had seen many important developments: Lax operators for IST method, discovery of Bäcklund transformation for KdV and its equivalence with the IST method [3, 14, 37, 39].

These developments were not singular. At the same time (second half of the last century) various other fields of science registered developments in which the nonlinearity played a fundamental rôle. It is worth to mention the advances in biophysics and chemistry – nerve impulse propagation, autocatalytic chemical reactions, biochemical solitons as the agents for the dynamics of enzymes and DNA, the problem of tumor growth – see [4, 7, 12, 35, 44, 45] for details, where discrete and continuum nonlinear equations were fruitfully used. Although not directly related to the subject of the present thesis, the author and his collaborators also obtained some results on these directions mostly in the last two years [10, 11, 22–24].

The present thesis is centred around these two equations, Korteweg-de Vries and Nonlinear Schrödinger, and several extensions of them. The present introductory section is followed by three main chapters. The second chapter is devoted to the analysis of the modulational instability of a plane wave solution propagating in a weakly nonlinear medium. Known for a long time (Lighthill 1965, Bespalov and Talanov 1966, Benjamin and Feir 1967) it is the initial stage of the generation of solitary wave solutions of many nonlinear wave equations. Two main approaches are possible. The first, the deterministic approach (D.A.M.I.), is the most used and known and it implies considering a perturbed Stokes wave solution of the nonlinear equation (plane wave solution with an amplitude dependent dispersion relation), then solving the corresponding linearized equation for the small perturbation. Usually the instability appears in the long wave length limit and certain relations between the coefficients of the nonlinear equation have to be satisfied (focusing case of the NLS equation). The second approach is of statistical nature (S.A.M.I.) and it takes into account the statistical properties of the medium and their influence on the instability development (Alber 1978). The main contributions of the author refer to this statistical approach method applied to several NLS-type equations (NLS equation with cubic nonlinearity, derivative NLS equations, cylindrical and spherical NLS equations). In the chapter three, following Fedele's *et al.* results [16–19], the Madelung fluid description of derivative and cubic+quintic NLS equations is given. In this approach, a correspondence between NLS-type equations and a generalized KdV equation is established, at least for traveling wave solutions. Integrating the KdV-type equation, a large class of periodic and stationary wave solutions are found.

The last chapter is dedicated to the results obtained in the study of nonlinear excitations in dusty plasmas. Here again, the KdV equation plays a major rôle. The results refer to the dust acoustic (DA) solitons and dust ion-acoustic (DIA) solitons. The influence of charge variation of the dust grain on the dust acoustic solitons is determined in a *local equilibrium approximation* (LEA). The LEA is based on the fact that the charging time τ_c of the dust grain is several orders smaller than the hydrodynamic time τ_h characteristic for the dust acoustic wave, and consequently the charge accumulated on the dust grain has enough time to accommodate itself to the local value of the electrostatic potential. The condition $\tau_c \ll \tau_h$ is well satisfied in laboratory experiments. Also the effect of the distribution of the grain radius on the dust acoustic solitons is investigated (unpublished results) for a model with two types of spherical grains. The dust ion-acoustic solitons are studied for a dusty plasma with positive and a small amount of lighter, negative ions. A critical concentration of the light negative ions is determined for which the KdV equation is no longer valid and a modified KdV equation is found to describe the phenomenon in this range.

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Chapter II: Modulational Instability

II.1. Modulational Instability in Hydrodynamics and Plasma Physics

The phenomenon of modulational instability is one of the basic subjects in the theory of nonlinear waves. It was predicted theoretically by Benjamin and Feir (1967) [4] for waves in deep water and by Bespalov and Talanov (1966) in their study [6] of electromagnetic waves propagating through nonlinear media with cubic nonlinearity and had proved since then a characteristic phenomenon for many other branches of physics such as nonlinear optics (nonlinear optical fibers, short pulse lasers, optical self-focusing), plasma physics (nonlinear Langmuir waves), condensed matter physics (quasi-one dimensional molecular systems, long Josephson junctions, Bose-Einstein condensates). It is a general phenomenon that manifests itself whenever a quasi-monochromatic wave is propagating through a dispersive and weakly nonlinear medium as the instability of its amplitude against weak modulations with wave numbers lower than some critical value (in the long wavelength region), in general [1]. Long time evolution leads to the growth of sidebands and the periodic exchange of energy between these sidebands, due to the nonlinearity, has a tendency to reduce the dispersion/diffraction effects. When the two phenomena, the dispersion and the nonlinearity, balance each other, coherent structures may appear as the envelope of the original oscillation characterized by variations in time and space which are much more slower than the carrier (original) wave.

To better understand the side-band resonance, let us briefly recall the ideas of Eckhaus and Benjamin-Feir (see [39]). Consider a nonlinear system which admits a small amplitude plane wave (Stoke's wave)

$$\Psi(x,t) = a \exp\left[i(kx - \omega t)\right] \tag{2.1}$$

where a is the amplitude, k the wavenumber and ω its frequency. Due to the nonlinear terms of the evolution equation, higher harmonics of this mode will emerge and let us consider the first harmonic in particular which is proportional to $\varepsilon a^2 \exp\left[2i(kx - \omega t)\right]$. To study the stability of this harmonic, two modal perturbations will be introduced, $\varepsilon a_1 \exp\left[i(k_1 x - \omega_1 t)\right]$ in the upper side band and $\varepsilon a_2 \exp\left[i(k_2 x - \omega_2 t)\right]$ in the lower side band, with amplitudes even smaller than the original wave, $\Psi(x,t)$, indicated by the small parameter $\varepsilon \ll 1$. Therefore the plane wave is written

$$\begin{aligned} \Psi(x,t) &= a e^{i(kx-\omega t)} + \varepsilon a_1 e^{i(k_1 x - \omega_1 t)} + \varepsilon a_2 e^{i(k_2 x - \omega_2 t)} \\ \Psi^*(x,t) &= a^* e^{-i(kx-\omega t)} + \varepsilon a_1^* e^{-i(k_1 x - \omega_1 t)} + \varepsilon a_2^* e^{-i(k_2 x - \omega_2 t)}. \end{aligned} \tag{2.2}$$

One of the simplest nonlinear terms has the form $|\Psi|^2\Psi$ which, for the perturbed wave after some calculus and linearizing, writes

$$\begin{aligned} |\Psi|^2\Psi =& |a|^2 a e^{i(kx-\omega t)} + 2\varepsilon |a|^2 a_1 e^{i(k_1 x - \omega_1 t)} + 2\varepsilon |a|^2 a_2 e^{i(k_2 x - \omega_2 t)} \\ &+ \varepsilon a^2 e^{i(2kx - 2\omega t)} \left(a_1^* e^{-i(k_1 x - \omega_1 t)} + a_2^* e^{-i(k_2 x - \omega_2 t)} \right). \end{aligned} \tag{2.3}$$

The last two terms clearly show the interaction between the first harmonic and the side bands. Suppose that the following situation arises

$$k_1 + k_2 = 2k, \quad \omega_1 + \omega_2 = 2\omega, \tag{2.4}$$

that is the sidebands are symmetric to the carrier wave. Then the two interaction terms become, respectively,

$$\begin{aligned} &\varepsilon a^2 a_1^* \exp\left[i(k_2 x - \omega_2 t)\right] \\ &\varepsilon a^2 a_2^* \exp\left[i(k_1 x - \omega_1 t)\right] \end{aligned} \tag{2.5}$$

showing a mutual reinforcement of the two side bands in association with the first harmonic or in other words, resonance. For the initial Stokes wave this resonance is the main mechanism of instability.

Intrinsically nonlinear systems as the one considered above are very common in everyday life. For instance the amplitude modulated (AM) radio waves are fast oscillating carrier waves with a relatively slowly varying envelope which contains the actual information (the sound heard by the listener). Also the pulse from a pulsed laser with a duration of the order of nanoseconds is actually a train of a few cycles of the carrier wave (the electromagnetic oscillation corresponding to the wavelength of the coherent light) contained in the envelope (in time). In this case of coherent light sources however, the nonlinear medium should not be resonant with the incident radiation in order to eliminate complications due to absorption and re-emission of the radiation.

The modulational instability (Benjamin-Feir instability) can be discussed in two distinct ways, a deterministic approach and a statistical one, which will be presented, in detail, in the following section.

II.2. Deterministic and Statistical Approach of Modulational Instability for Nonlinear Schrödinger Type Equations

II.2.1. D.A.M.I. for Cubic Nonlinear Schrödinger Equation

The first method of treating the Benjamin-Feir instability is the deterministic approach to the modulational instability (D.A.M.I.). It is the most common approach that can be found in any textbook on the nonlinear wave propagation. Initially, a linear analysis around a Stokes wave (a plane wave with a dispersion relation dependent on its amplitude) is performed. Then the instability (due to nonlinear mechanisms like the mutual reinforcement of side-bands) is related to complex values of the frequency obtained from the linear problem and the dependence of the instability domain(s) on the parameters of the studied problem can be carefully investigated.

As an example of the procedure described above, we'll consider the nonlinear Schrödinger equation which describes the evolution of the amplitude of a quasi-monochromatic wave propagating in a weakly (cubic) nonlinear medium

$$i\frac{\partial \Psi}{\partial t} + \alpha \frac{\partial^2 \Psi}{\partial x^2} + \beta |\Psi|^2 \Psi = 0. \tag{2.6}$$

The NLS equation is satisfied by a Stokes wave of the form

$$\Psi(x,t) = a \exp[i(kx - \omega(k;|a|)t)], \quad \omega(k;|a|) = \alpha k^2 - \beta |a|^2. \tag{2.7}$$

Let's consider a small modulation of the Stokes solution of (2.6) as

$$\Psi(x,t) = a\,[1 + \varepsilon A(x,t)] \exp[i(kx - \omega t)], \tag{2.8}$$

where $\omega(k;|a|)$ is given by the dispersion relation in (2.7), $\varepsilon << 1$ is a small parameter (in the theory of nonlinear ocean waves it is usually the steepness of the wave $\varepsilon = k|A|$) and $A(x,t)$ is the time dependent modulation. Then the linear equation (first order of ε) satisfied by $A(x,t)$ is

$$i\frac{\partial A}{\partial t} + 2i\alpha k \frac{\partial A}{\partial x} + \alpha \frac{\partial^2 A}{\partial x^2} + \beta |a|^2 (A + A^*) = 0. \tag{2.9}$$

Looking for plane wave solutions of (2.9)

$$A(x,t) = M \exp[i(Qx - \Omega t)] + N^* \exp[-i(Qx - \Omega^* t)], \tag{2.10}$$

after straightforward calculations, one obtains the following expression for the angular frequency Ω

$$\Omega - 2\alpha k Q = i|\alpha Q| \sqrt{2\frac{\beta}{\alpha}|a|^2 - Q^2}. \tag{2.11}$$

The modulational instability is associated to complex value of Ω having $\mathrm{Im}\,\Omega > 0$. This situation occurs if the NLS equation is in the focusing case and the wave number Q has small values, that is

$$\begin{aligned} &\alpha\beta > 0 \ (\alpha,\ \beta \text{ have the same sign}) \\ &Q^2 < 2\frac{\beta}{\alpha}|a|^2. \end{aligned} \tag{2.12}$$

This result shows that the M.I. manifests itself only in the long wave length region. Further discussions about the influence of the nonlinearity type on the instability domain and a qualitative, graphical representation of the condition (2.12) will be given in the next section.

II.2.2. S.A.M.I. for Cubic Nonlinear Schrödinger Equation

The statistical approach is a less used method to analyze the phenomenon of modulational instability. However, recently, the statistical approach was intensely applied to investigate phenomena in different fields of physics ranging from propagation of deep water waves in hydrodynamics (the theory of surface gravity waves) [3, 7, 21, 23, 33–35] to wave propagation in non-stationary inhomogeneous plasma (plasma physics) [40,41], the study of the longitudinal dynamics of charged particle beams in accelerators [8, 9], the dynamics of Bose-Einstein condensates [10] or the problem of incoherent light propagation in nonlinear media in nonlinear optics [18, 22]. It was even extended to the study of M.I. in discrete systems [12–14, 16] and in coupled nonlinear Schrödinger equations (Manakov's system) in theoretical physics [13, 15, 36].

The aim of this complementary approach is to emphasize the influence of the statistical properties of the medium on the instability [19, 26, 43]. In a seminal paper [3], Alber set the basics of this method in an effort to provide "*a further bridge between the deterministic and random schools, by examining the stability properties of a weakly nonlinear random wave train*". It is assumed that the degree of randomness (the spectral spread around the central carrier wavenumber, k_0) is small (narrow-band process) and the equations governing the evolution of the complex field (wave amplitude) $\Psi(x, t)$ remain valid when Ψ becomes a stochastic variable. Most of the nonlinear partial differential equations are the result of applying a multiple scale analysis on the dynamic laws governing the studied system. Therefore the envelope function must variate over distances much larger than the wavelength of the carrier wave. It means that these variations must be characterized by a small parameter ε which is identified with the wave slope $\varepsilon = (k_0^2 \bar{a_0^2})^{1/2}$, where $\bar{a_0^2}$ is the mean square amplitude of a given unperturbed, spatially homogeneous wave field. Under these assumptions Alber finds an evolution equation for a two-point correlation function written for the complex amplitude of the Davey and Stewardson equation.

Let us define the two-point correlation function as

$$W(1,2) = W(x_1, x_2) = \langle \Psi(x_1)\Psi^*(x_2)\rangle \,, \tag{2.13}$$

where $\langle \ldots \rangle$ represents the average over the statistical ensemble characterizing the medium. Introducing the "center-of-mass" coordinate $X = \frac{1}{2}(x_1 + x_2)$ and the relative coordinate $\xi = x_1 - x_2$, the correlation function writes

$$W(1,2) = \left\langle \Psi\left(X + \frac{\xi}{2}\right)\Psi^*\left(X - \frac{\xi}{2}\right)\right\rangle = W(\xi, X, t). \tag{2.14}$$

In the case of cubic NLS equation (2.6), the evolution equation for $W(1, 2)$ is obtained by adopting the following procedure introduced by Wigner in quantum mechanics [42]. We write the equation (2.6) at the point x_1 and multiply it by $\Psi^*(x_2)$ then add it to the equation for $\Psi^*(x_2)$ (c.c. of (2.6)) multiplied by $\Psi(x_1)$. Taking the ensemble average we get

$$i\frac{\partial}{\partial t}\langle\Psi(x_1)\Psi^*(x_2)\rangle + \alpha\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2}\right)\langle\Psi(x_1)\Psi^*(x_2)\rangle$$
$$+ \beta\left[\langle\Psi(x_1)\Psi^*(x_1)\Psi(x_1)\Psi^*(x_2)\rangle - \langle\Psi(x_2)\Psi^*(x_2)\Psi^*(x_2)\Psi(x_1)\rangle\right] = 0. \quad (2.15)$$

In order to evaluate the forth-order correlation terms in the equation above, one has to assume that the stochastic variable $\Psi(x,t)$ corresponds initially to a Gaussian random process and that it maintains the same Gaussian statistical properties through-out its evolution [5]. Only for Gaussian statistics the fourth-order cumulants decompose exactly in a sum of products of pairs of second-order correlations. In general, such a decomposition is allowed if one considers a Gaussian approximation, therefore in (2.15), we get

$$\langle\Psi(x_1)\Psi^*(x_1)\Psi(x_1)\Psi^*(x_2)\rangle \simeq 2n(x_1,t)W(x_1,x_2,t),$$
$$\langle\Psi(x_2)\Psi^*(x_2)\Psi^*(x_2)\Psi(x_1)\rangle \simeq 2n(x_2,t)W(x_1,x_2,t),$$

where $n(x,t)$ is the ensemble average of the mean square amplitude of the field (the pulse intensity in optics or the fluid density in hydrodynamics)

$$n(x,t) = \langle\Psi(x,t)\Psi^*(x,t)\rangle\,.$$

In the expansions above, the terms involving ensemble averages of the form $\langle\Psi(x,t)\Psi(x,t)\rangle$ vanish because correlations must be invariant to the addition of a random phase. The kinetic equation for the two-point correlation function writes

$$i\frac{\partial W(1,2)}{\partial t} + \alpha\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2}\right)W(1,2)$$
$$+ 2\beta\left(n(x_1,t) - n(x_2,t)\right)W(1,2) = 0. \quad (2.16)$$

Further on, a Wigner-Moyal transform [31, 42] can be performed on this kinetic equation obtaining a transport equation for the wave-envelope power spectral density $\rho(k,X,t)$ (the Wigner's function) defined as the Fourier transform of the two-point correlation function

$$\rho(k,X,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ik\xi}\left\langle\Psi(X+\frac{\xi}{2},t)\Psi^*(X-\frac{\xi}{2},t)\right\rangle\,\mathrm{d}\xi. \quad (2.17)$$

In deriving the evolution equation for $\rho(k,X,t)$ one uses the following properties and formulae

$$n(x_1,t) = n(X+\frac{\xi}{2},t) = \sum_{j=0}^{\infty} \frac{\xi^j}{2^j j!}\left(\frac{\partial^j n(X,t)}{\partial X^j}\right)_{X=0},$$
$$n(x_2,t) = n(X-\frac{\xi}{2},t) = \sum_{j=0}^{\infty} \frac{(-)^j\xi^j}{2^j j!}\left(\frac{\partial^j n(X,t)}{\partial X^j}\right)_{X=0},$$
$$n(x_1,t)-n(x_2,t) = 2\sum_{j=0}^{\infty} \frac{\xi^{2j+1}}{2^{2j+1}(2j+1)!}\left(\frac{\partial^{2j+1} n(X,t)}{\partial X^{2j+1}}\right)_{X=0} \tag{2.18}$$
$$W(x_1,x_2) = \int_{-\infty}^{+\infty} e^{ik'\xi}\rho(k',X,t)\,\mathrm{d}k', \quad \xi^j e^{-ik\xi} = (i)^j \frac{\partial^j}{\partial k^j} e^{-ik\xi}$$
$$\delta(k'-k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(k'-k)\xi}\,\mathrm{d}\xi.$$

Transforming the equation (2.16) one gets

$$\frac{\partial\rho(k,X,t)}{\partial t} + 2\alpha k\frac{\partial\rho(k,X,t)}{\partial X} + 4\beta n(X,t)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho(k,X,t) = 0, \tag{2.19}$$

where the **sin** operator is defined in terms of its Taylor expansion and the arrows give the direction of differentiation. Starting from this equation, a linear stability analysis can be done, assuming

$$\rho(k,X,t) = \rho_0(k) + \varepsilon\rho_1(k,X,t), \quad n(X,t) = n_0 + \varepsilon n_1(X,t), \tag{2.20}$$

where

$$n_0 = \int \rho_0(k)\,\mathrm{d}k, \quad n_1(X,t) = \int \rho_1(k,X,t)\,\mathrm{d}k. \tag{2.21}$$

Then the first order perturbation $\rho_1(k,X,t)$ satisfies the following linear equation

$$\frac{\partial\rho_1}{\partial t} + 2\alpha k\frac{\partial\rho_1}{\partial X} + 4\beta n_1(X)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho_0(k) = 0. \tag{2.22}$$

Looking for plane wave solutions of the form

$$\begin{aligned}\rho_1(k,X,t) &= g(k)\exp[i(QX-\Omega t)],\\ n_1(X,t) &= G\exp[i(QX-\Omega t)], \quad G = \int g(k)\,\mathrm{d}k,\end{aligned} \tag{2.23}$$

straightforward algebraic manipulations lead to the following integral form of the dispersion relation

$$1 + \frac{\beta}{\alpha}\frac{1}{Q}\int_{-\infty}^{+\infty} \frac{\rho_0\left(k+\frac{Q}{2}\right) - \rho_0\left(k-\frac{Q}{2}\right)}{k-\frac{\Omega}{2\alpha Q}}\,\mathrm{d}k = 0. \tag{2.24}$$

As Ω is a complex quantity, the development of the modulational instability is associated with positive values of its imaginary part, $\mathrm{Im}\,\Omega > 0$. To study the effects of the statistical properties of the medium, different equilibrium distributions $\rho_0(k)$ need to be considered, then the equation (2.24) must be solved in order to find the domain of parameter values that satisfy the instability condition.

II.2.2.1. δ-spectrum

If the equilibrium spectral power density is a δ function of the wave number this corresponds to a situation in which all the points of the field are equally correlated to each other, that is they have a white noise distribution in real space.

$$\rho_0(k) = n_0 \delta(k), \qquad W_0(\xi) = n_0 = \text{const.} \tag{2.25}$$

Introducing (2.25) into (2.24) the integration can be easily done and assuming Ω is a purely imaginary quantity $\Omega = i\Omega_i$, one finds

$$\Omega_i = \alpha Q \sqrt{\frac{4\beta n_0}{\alpha} - Q^2}. \tag{2.26}$$

Then, the instability condition is satisfied if α and β have the same sign and $Q^2 < 4\frac{\beta}{\alpha} n_0$ (in the long wavelength region). These results are very similar to the ones obtained in the deterministic approach. Comparing the inequalities that give the instability domain, one may find a correspondence between the square of the modulus of the unperturbed wave amplitude $|a|^2$ (the intensity of the wave) and the ensemble averaged mean square amplitude of the statistical field at equilibrium n_0

$$|a|^2 \leftrightarrow 2n_0.$$

II.2.2.2. Limited white spectrum

In [17], we also considered a "limited" white spectrum distribution as the equilibrium spectral power density

$$\rho_0(k) = \begin{cases} \frac{1}{2\Lambda} n_0, & |k| \le \Lambda \\ 0, & |k| > \Lambda \end{cases}. \tag{2.27}$$

In the real space, the corresponding equilibrium two-point correlation function is

$$W_0(\xi) = n_0 \frac{\sin(\Lambda\xi)}{\Lambda\xi}.$$

Introducing (2.27) into (2.24) and performing the integration, the dispersion relation writes

$$\Omega^2 = (2\alpha Q)^2 \left[\left(\frac{Q^2}{4} + \Lambda^2 \right) - Q\Lambda \coth\left(\frac{\alpha Q \Lambda}{\beta n_0} \right) \right]. \tag{2.28}$$

Assuming a purely imaginary $\Omega = i\Omega_i$, the instability is associated with positive values of Ω_i. Due to the odd parity of coth, this condition is met when $\alpha\beta > 0$ and in the domains where

$$\Omega_i^2 = [2\alpha Q \lambda(Q,\Lambda)]^2 > 0,$$
$$\lambda^2(Q,\Lambda) = Q\Lambda \coth\left(\frac{\alpha Q \Lambda}{\beta n_0}\right) - \left(\frac{Q^2}{4} + \Lambda^2\right).$$

It should be noted that when $\Lambda \to 0$, the instability domain is the same as for the δ spectrum. However, a qualitative representation of the instability domain is given in figure II.1 as a function of Q for a given set of values for $\Lambda \in \{0, 0.5, 1, \sqrt{2}, 1.5\}$. The instability domains are the filled areas under the curves corresponding to increasing values of Λ as their color darkens. It is easily seen that at first the long wavelength region is unstable but around a critical value $\Lambda \simeq \sqrt{2}$ it becomes stable, then the instability domain decreases rapidly as the parameter Λ increases.

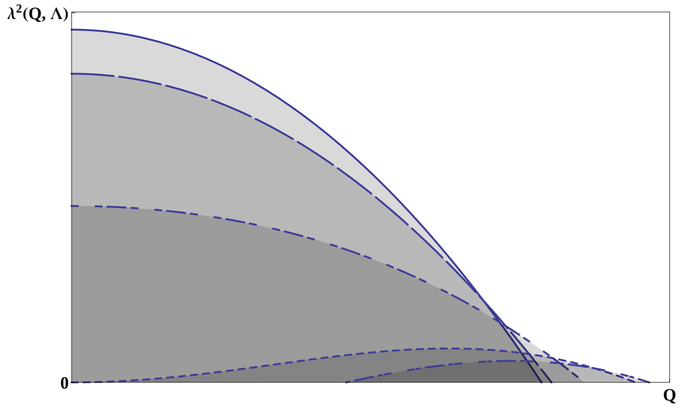

Figure II.1: The M.I. domains for equilibrium spectral power densities represented by a larger and larger "limited" white spectra.

II.2.2.3. Lorentzian spectrum

Let us consider that, at equilibrium, the spectral distribution in the k-space has the form of a Lorentzian distribution

$$\rho_0(k) = \frac{n_0}{\pi} \frac{p}{p^2 + k^2}, \tag{2.29}$$

where p is the scale parameter of the distribution (the half-width at half-maximum). In the real space this corresponds to an exponentially decreasing initial two-point correlation function

$$W_0(\xi) = n_0 e^{-p|\xi|}.$$

Again, assuming $\Omega = i\Omega_i$ the dispersion relation is quickly computed, leading to the following expression for the imaginary part of the angular frequency of the perturbation

$$\Omega_i = 2\alpha Q \left(\sqrt{\frac{\beta}{\alpha} n_0 - \frac{Q^2}{4}} - p \right). \tag{2.30}$$

In this case the instability depends also on the correlation length p^{-1} of the initial distribution. Therefore the M.I. manifests only for initial long range correlations and as the scale parameter p increases it inhibits the development of the instability which is similar to the well-known process of Landau damping in plasma physics [24,38]. This analogy is enforced by the form of (2.24) similar to the stability equation found in the study of the Landau damping phenomenon.

II.2.2.4. Gaussian spectrum

A more realistic equilibrium spectral power density is a Gaussian distribution

$$\rho_0(k) = \frac{n_0}{\sigma\sqrt{2\pi}} \exp\left(-\frac{k^2}{2\sigma^2}\right), \tag{2.31}$$

that obviously corresponds to a Gaussian initial two-point correlation function

$$W_0(\xi) = n_0 \exp\left(-\frac{\sigma^2\xi^2}{2}\right).$$

In this case the Gaussian decoupling of the fourth order cumulants is also exact. Using (2.31) in (2.24), the stability equation writes

$$\begin{aligned} 1 =& \frac{\beta}{\alpha Q}\frac{n_0}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{-\eta^2}\left(\frac{1}{z-\eta} - \frac{1}{-z^*-\eta}\right)\,\mathrm{d}\eta, \\ \eta =& \frac{k}{\sqrt{2}\sigma}, \qquad z = \frac{1}{2\sqrt{2}\sigma}\left(Q + i\frac{\Omega_i}{\alpha Q}\right) \end{aligned} \tag{2.32}$$

and we've assumed that $\Omega = i\Omega_i$ is purely imaginary. Using the integral representation of the complex error (Faddeeva) function [2]

$$w(z) = \frac{i}{\pi}\int_{-\infty}^{+\infty} e^{-\eta^2}\frac{\mathrm{d}\eta}{z-\eta},$$

the equation (2.32) becomes

$$1 = \frac{\beta n_0\sqrt{2\pi}}{\alpha\sigma Q}\mathrm{Im}\,[w(z)]. \tag{2.33}$$

When the initial correlation radius is increased, in the asymptotic region $\sigma \to 0$ ($z \to \infty$) corresponding to a δ spectrum for the equilibrium spectral power density $\rho_0(k)$, the function $w(z)$ may be approximated [2] by

$$w(z) = \frac{i}{\sqrt{\pi}}\frac{1}{z}\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2z^2)^n}.$$

Then in the leading order one gets

$$\operatorname{Im} w(z) = \frac{1}{\sqrt{\pi}} \frac{x}{x^2 + y^2},$$

where

$$z = x + iy, \quad x = \frac{1}{2\sqrt{2}} \frac{Q}{\sigma}, \quad y = \frac{1}{2\sqrt{2}} \frac{\Omega_i}{\alpha \sigma Q}. \tag{2.34}$$

Using this in (2.33), one immediately recovers the result (2.26) for the δ spectrum.

In order to determine the domain of instability, one has to solve the transcendent dispersion equation (2.33) for each given Q imposing that the instability condition $\Omega_i > 0$ is satisfied. One should note that, using the previous notations and introducing the full width half maximum (FWHM) parameter (for a Gaussian distribution FWHM $= 2\sqrt{2 \ln 2}\sigma$), the equation (2.33) writes

$$\frac{\alpha}{\beta} \frac{1}{\sqrt{2 \ln 2}} \frac{\text{FWHM}}{\rho_0(0)} x = \operatorname{Im}[w(x + iy)]. \tag{2.35}$$

In figure II.2 one has the qualitative representation of modulational instability domain in the plane (Q, Ω_i). It should be noted that the instability is possible in the long wavelength region where the frontier of the domain has an almost linear dependence of the wavenumber Q. Besides the M.I. is limited to a specific interval on the Q-axis the width of which depends on the other parameters of the problem $n_0, \sigma, \alpha, \beta$.

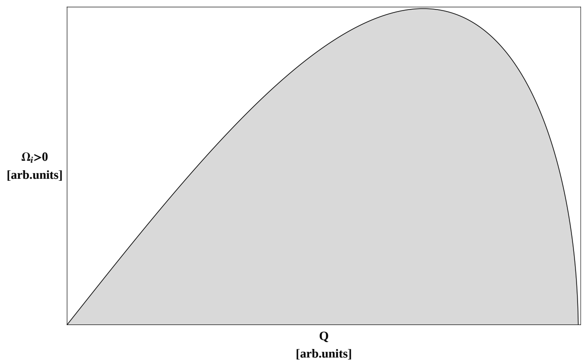

Figure II.2: The M.I. domain for a Gaussian two-point correlation function at equilibrium (the axes are in arbitrary units: $xy = \frac{1}{8\alpha\sigma^2}\Omega_i$ on the ordinate, $x = \frac{1}{2\sigma\sqrt{2}}Q$ on the abscissa).

II.3. Deterministic and Statistical Approach of Modulational Instability for Derivative Nonlinear Schrödinger Type Equations

In order to determine the effect of the type of nonlinearity on the modulational instability phenomenon, we'll investigate, in the following, the derivative nonlinear Schrödinger equation both from a deterministic and statistical point of view. It is another completely integrable equation that appears in various fields of physics like nonlinear optics or plasma physics (for detailed references see §III.3). Recently, the modulational instability phenomenon was analyzed in astrophysics [37] and plasma physics [32] in connection with the dynamics of magnetic field-aligned dust Alfvén waves respectively parallel propagating Alfvén wave turbulence which are both described, in appropriate approximations, by the type-1 derivative nonlinear Schrödinger equation.

II.3.1. D.A.M.I. for Derivative NLS Equations

The first equations in the derivative NLS class are the so-called dNLS-1 and dNLS-2, having the form

$$i\frac{\partial\Psi}{\partial t}+\alpha\frac{\partial^2\Psi}{\partial x^2}+i\gamma_1\frac{\partial\left(|\Psi|^2\Psi\right)}{\partial x}=0, \tag{2.36}$$

respectively

$$i\frac{\partial\Psi}{\partial t}+\alpha\frac{\partial^2\Psi}{\partial x^2}+i\gamma_2|\Psi|^2\frac{\partial\Psi}{\partial x}=0. \tag{2.37}$$

They can be obtained from a general NLS equation (gNLS), that includes the cubic nonlinearity discussed in the previous section,

$$i\frac{\partial\Psi}{\partial t}+\alpha\frac{\partial^2\Psi}{\partial x^2}+\beta|\Psi|^2\Psi+i\gamma_1\frac{\partial|\Psi|^2}{\partial x}\Psi+i\gamma_2|\Psi|^2\frac{\partial\Psi}{\partial x}=0, \tag{2.38}$$

by taking $\beta=0$ and $\gamma_1=\gamma_2$ or $\gamma_1=0$ respectively. For a Stokes wave (2.8), the dispersion relation has the same form for each of the derivative equations:

$$\omega(k)=\alpha k^2+\gamma_j k|a|^2,\quad j=\overline{1,2},$$

while for the general NLS equation (2.38) it writes

$$\omega(k)=\alpha k^2+\gamma_2|a|^2k-\beta|a|^2.$$

Considering the perturbed Stokes wave (2.8) and introducing this solution into (2.36), (2.37), (2.38), in the linear approximation (order ε) one obtains the following evolution equations for the small perturbation $A(x,t)$,

$$\text{dNLS-1: } i\frac{\partial A}{\partial t}+i(2\alpha k+\gamma_1|a|^2)\frac{\partial A}{\partial x}+\alpha\frac{\partial^2 A}{\partial x^2}+ \\ +i\gamma_1|a|^2\left(\frac{\partial A}{\partial x}+\frac{\partial A^*}{\partial x}\right)-\gamma_1k|a|^2\left(A+A^*\right)=0, \tag{2.39}$$

$$\text{dNLS-2: } i\frac{\partial A}{\partial t}+i(2\alpha k+\gamma_2|a|^2)\frac{\partial A}{\partial x}+\alpha\frac{\partial^2 A}{\partial x^2}-\gamma_2k|a|^2\left(A+A^*\right)=0, \tag{2.40}$$

$$\text{gNLS: } i\frac{\partial A}{\partial t}+i(2\alpha k+\gamma_2|a|^2)\frac{\partial A}{\partial x}+\alpha\frac{\partial^2 A}{\partial x^2}+ \\ +i\gamma_1|a|^2\left(\frac{\partial A}{\partial x}+\frac{\partial A^*}{\partial x}\right)+|a|^2(\beta-\gamma_2k)\left(A+A^*\right)=0, \tag{2.41}$$

respectively. Looking for plane wave solutions (2.10), the compatibility condition of the homogeneous system for M and N leads to the following expression for the complex angular frequency Ω in the general case (2.41)

$$\Omega - Q\left[2\alpha k + (\gamma_1 + \gamma_2)|a|^2\right] = iQ\sqrt{\left[2\alpha(\beta - \gamma_2 k) - \gamma_1^2|a|^2\right]|a|^2 - \alpha^2 Q^2}. \quad (2.42)$$

Again, by taking $\beta = 0$ and considering $\gamma_1 = \gamma_2$ and $\gamma_1 = 0$ one obtains the expressions corresponding to the dNLS-1 and dNLS-2 respectively:

$$\Omega_{dNLS-1} - 2Q(\alpha k + \gamma_1|a|^2) = iQ\sqrt{|a|^2|\gamma_1|(2|\alpha|k - |\gamma_1||a|^2) - \alpha^2 Q^2}, \quad (2.43)$$

$$\Omega_{dNLS-2} - Q(2\alpha k + \gamma_2|a|^2) = iQ\sqrt{2|\alpha\gamma_2|k|a|^2 - \alpha^2 Q^2}. \quad (2.44)$$

The instability develops when $\operatorname{Im}\Omega > 0$. This condition is satisfied for the dNLS-2 equation (2.44) when the coefficients α and γ_2 have different signs and

$$Q^2 < 2\left|\frac{\gamma_2}{\alpha}\right||a|^2 k.$$

A qualitative view of the instability domain is given in figure II.3 (c), showing that in the long wavelength region the instability no longer manifests as opposed to the case of the cubic NLSE – figure II.3 (a) . There is also less restriction on the instability domain since it grows linearly with k. For the dNLS-1 type equations the instability domain is also linearly dependent on k but the long wavelength region is strictly stable up to a specific value $k_0 = \frac{1}{2}\left|\frac{\gamma_1}{\alpha}\right||a|^2$ when the coefficients α, γ_1 have opposite signs – figure II.3 (b). For $k \in (k_0, +\infty)$ the modulational instability domain is given by

$$Q^2 < 2\left|\frac{\gamma_1}{\alpha}\right||a|^2 k - \left(\frac{\gamma_1|a|^2}{\alpha}\right)^2.$$

When considering the general NLS equation[1], the interplay of the two types of nonlinearities (cubic and derivative) yields two different situations for the instability to develop. Let us first take notice that by denoting $\beta_1 = 2\alpha\beta - \gamma_1^2|a|^2$, $\operatorname{Im}\Omega$ from (2.42) writes

$$\operatorname{Im}\Omega = Q\sqrt{(\beta_1 - 2\alpha\gamma_2 k)|a|^2 - \alpha^2 Q^2}. \quad (2.45)$$

Analyzing the sign of the quantity under the square root one finds, qualitatively, only two distinct situations, regardless of the sign of β_1 and γ_1, corresponding to the cases when α, β have the same sign and γ_2 has the same or the opposite sign to α. The only difference is on the limits imposed to the coefficient γ_1, namely

$$\beta_1 < 0 \Rightarrow |\gamma_1| > \frac{\sqrt{\alpha\beta}}{|a|}$$
$$\beta_1 > 0 \Rightarrow |\gamma_1| < \frac{\sqrt{\alpha\beta}}{|a|}.$$

[1] Here one should keep in mind that the coefficients γ_1 and γ_2 are not necessarily related to the ones in the purely derivative NLS equations and $\gamma_1 \neq \gamma_2$.

As γ_1 is by definition a real quantity, the relations above imposed that $\alpha\beta > 0$ (they must have the same sign). In the following we'll consider that β_1 is positive. Then, in the first case ($\alpha\beta(-\gamma_2) > 0$) the instability is highly favored in the long wavelength region and its domain grows linearly with k – figure II.3(d). For the second case ($\alpha\beta\gamma_2 > 0$), the M.I. is restricted to an interval $(0, k_0)$ around the long wavelength region – figure II.3 (e), where

$$k_0 = \frac{1}{2}\left|\frac{\beta_1}{\alpha\gamma_2}\right|.$$

These results were published by the author in [12]. Further more, if one takes $\gamma_1 = 0$ the results published in [11] are easily reproduced for the deterministic approach of the modified NLS equation studied therein, with similar instability domains as (2.38) (figure II.3 d,e).

II.3.2. S.A.M.I. for Derivative NLS Equations

The statistical approach to modulational instability in the family of derivative nonlinear Schrödinger equations is quite similar to the procedure described for the cubic NLS equation. With respect to the cubic NLS case, the derivative nonlinear terms in (2.36), (2.37), (2.38) introduce new fourth order cumulants in the kinetic equation for the two-point correlation function $W(x_1, x_2)$, obtained through Wigner's procedure. For the equation (2.38), the Gaussian decoupling of these terms leads to

$$\begin{aligned}
&\left\langle \frac{\partial|\Psi(x_1)|^2}{\partial x_1}\Psi(x_1)\Psi^*(x_2)\right\rangle \simeq \left[q(1) + n(1)\frac{\partial}{\partial x_1} + 2q^*(1)\right]W(1,2)\\
&\left\langle \frac{\partial|\Psi(x_2)|^2}{\partial x_2}\Psi^*(x_2)\Psi(x_1)\right\rangle \simeq \left[q^*(2) + n(2)\frac{\partial}{\partial x_2} + 2q(2)\right]W(1,2),\\
&\left\langle \Psi(x_1)\Psi^*(x_1)\frac{\partial\Psi(x_1)}{\partial x_1}\Psi^*(x_2)\right\rangle \simeq \left[q(1) + n(1)\frac{\partial}{\partial x_1}\right]W(1,2)\\
&\left\langle \Psi(x_2)\Psi^*(x_2)\frac{\partial\Psi^*(x_2)}{\partial x_2}\Psi(x_1)\right\rangle \simeq \left[q^*(2) + n(2)\frac{\partial}{\partial x_2}\right]W(1,2),
\end{aligned} \tag{2.46}$$

for the coefficient γ_1 and γ_2 respectively. Here $n(1) = n(x_1, t)$ and $n(2) = n(x_2, t)$ are the quantities defined in (2.18), while $q(j) = q(x_j, t)$, $q^*(j) = q^*(x_j, t)$ $(j = \overline{1, 2})$ denote

$$\begin{aligned}
q(x,t) &= \left\langle \frac{\partial\Psi(x,t)}{\partial x}\Psi^*(x,t)\right\rangle\\
q^*(x,t) &= \left\langle \Psi(x,t)\frac{\partial\Psi^*(x,t)}{\partial x}\right\rangle.
\end{aligned} \tag{2.47}$$

The quantity $q(x,t)$ is related to $n(x,t)$ by

$$\frac{\partial n(x,t)}{\partial x} = q(x,t) + q^*(x,t). \tag{2.48}$$

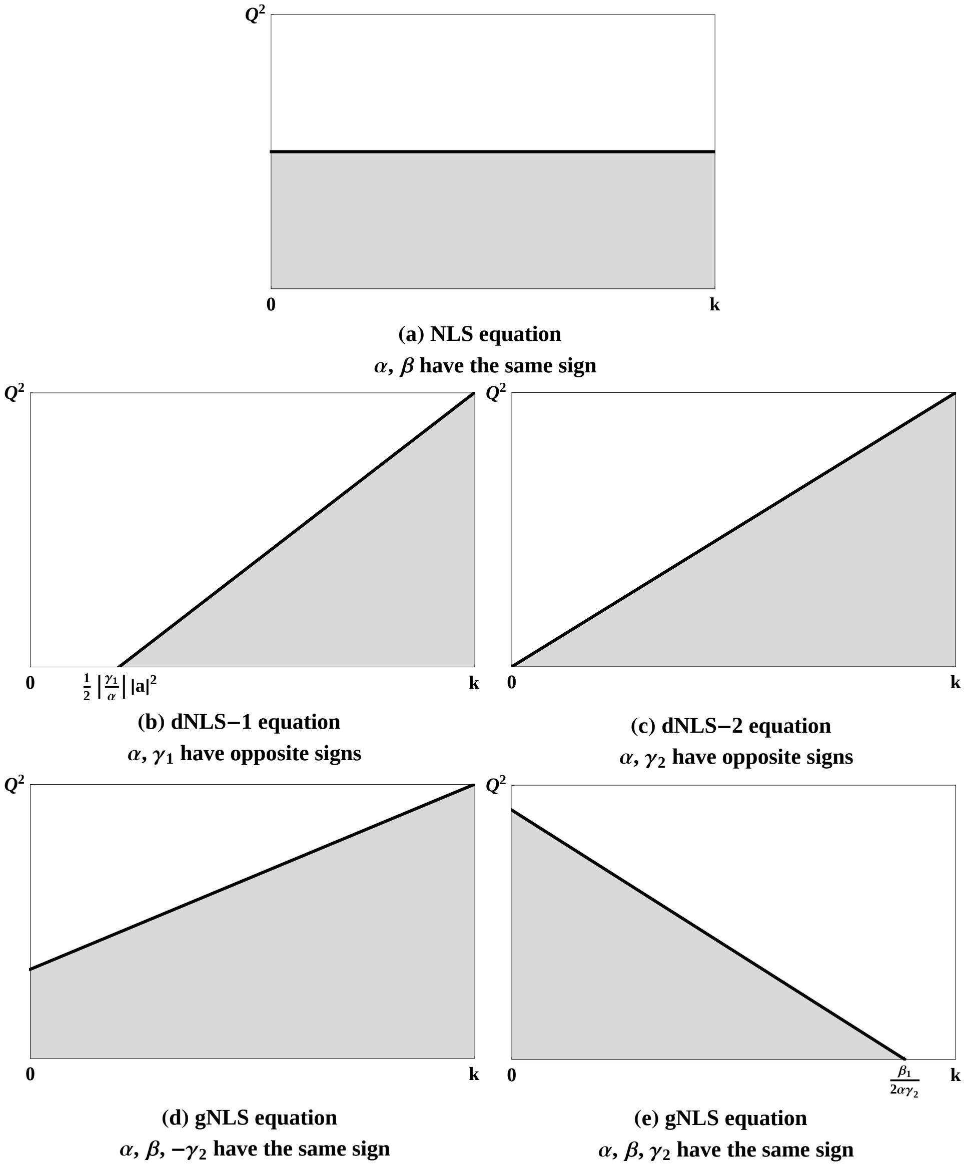

Figure II.3: A qualitative view of the modulational instability domains for (a) NLSE, (b,c) dNLSE-1 and dNLSE-2 and (d,e) gNLSE.

A second relation for $q(x,t)$ and its complex conjugate may be obtained from the conservation law of $n(x,t)$, namely

$$\frac{\partial n(x,t)}{\partial t} + \frac{\partial}{\partial x}\left\{\left(\gamma_1 + \frac{\gamma_2}{2}\right)\left\langle|\Psi|^4\right\rangle - i\alpha\left[q(x,t) - q^*(x,t)\right]\right\} = 0, \qquad (2.49)$$

which has the form of a fluid continuity equation. Here, a further the Gaussian decoupling of the fourth order cumulant yields $\left\langle|\Psi|^4\right\rangle \simeq 2\left(n(x,t)\right)^2$. Thus the equations (2.48) and (2.49) allow us to express the statistical quantity $q(x,t)$ and its complex conjugate in terms of $n(x,t)$ for every point of the field.

Then, the field evolution equation (2.38) leads to the following kinetic equation for the two-point correlation function

$$
\begin{aligned}
& i\frac{\partial W(1,2)}{\partial t} + \alpha \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) W(1,2) \\
& + 2\beta \left[n(1) - n(2) \right] W(1,2) + i\gamma_1 \left[\frac{\partial n(1)}{\partial x_1} + \frac{\partial n(2)}{\partial x_2} \right] W(1,2) \\
& + i(\gamma_1 + \gamma_2) \left[n(1)\frac{\partial}{\partial x_1} + n(2)\frac{\partial}{\partial x_2} \right] W(1,2) \\
& + i\gamma_1 \left[q^*(1) + q(2) \right] W(1,2) + i\gamma_2 \left[q(1) + q^*(2) \right] W(1,2) = 0,
\end{aligned} \tag{2.50}
$$

where the dependence on time is only omitted for brevity. Applying the Wigner-Moyal transform, which will be detailed in appendix II.3.2.3, one gets the evolution equation for the spectral power density $\rho = \rho(k, X, t)$

$$
\begin{aligned}
& \frac{\partial \rho}{\partial t} + 2\alpha k \frac{\partial \rho}{\partial x} + 4\beta n(X) \sin\left(\frac{1}{2} \frac{\overleftarrow{\partial}}{\partial X} \frac{\overrightarrow{\partial}}{\partial k} \right) \rho(k) \\
& + (3\gamma_1 + \gamma_2) \frac{\partial n(X)}{\partial X} \cos\left(\frac{1}{2} \frac{\overleftarrow{\partial}}{\partial X} \frac{\overrightarrow{\partial}}{\partial k} \right) \rho(k) \\
& - 2(\gamma_1 + \gamma_2) n(X) \sin\left(\frac{1}{2} \frac{\overleftarrow{\partial}}{\partial X} \frac{\overrightarrow{\partial}}{\partial k} \right) [k\rho(k)] \\
& + (\gamma_1 + \gamma_2) n(x) \cos\left(\frac{1}{2} \frac{\overleftarrow{\partial}}{\partial X} \frac{\overrightarrow{\partial}}{\partial k} \right) \frac{\partial \rho(k)}{\partial X} \\
& + i(\gamma_2 - \gamma_1) \left[q(X) - q^*(X) \right] \sin\left(\frac{1}{2} \frac{\overleftarrow{\partial}}{\partial X} \frac{\overrightarrow{\partial}}{\partial k} \right) \rho(k) = 0.
\end{aligned} \tag{2.51}
$$

Again, the dependence on some of the coordinates was omitted were possible and the **sin** and **cos** operators are defined in terms of their Taylor series with the over head arrows indicating the terms on which the derivatives act. It should be noted that the equation (2.51) is, however, a real equation for the real Wigner function $\rho(k, X, t)$, since the difference in the last term is a purely complex quantity. The kinetic equation (2.50) and its Fourier transform (2.51) for the field evolution equations (2.36), (2.37) are obtained with the same coefficient transformations that were used in the introduction of the D.A.M.I. For dNLS-1 it is easily seen that the corresponding evolution equation for the power spectral density doesn't include $q(X, t)$, thus this quantity must be defined only when a dNLS-2 type nonlinearity appears.

At equilibrium, we consider the system to be homogeneous meaning that the two-point correlation function $W_0(\xi)$ does not depend on the center-of-mass coordinate X nor time t and the density n_0 is constant. Besides if we assume that the system, at equilibrium, is isotropic then $W_0 = W_0(|\xi|)$ and its Fourier transform $\rho_0(k)$ is an even function of k. In these conditions, let us perform a first order (linear) perturbation analysis by introducing

$$\begin{aligned}
&\rho(k,X,t)=\rho_0(k)+\varepsilon\rho_1(k,X,t), \qquad n(X,t)=n_0+\varepsilon n_1(X,t),\\
&n_0=\int\rho_0(k)\,\mathrm{d}k, \qquad n_1(X,t)=\int\rho_1(k,X,t)\,\mathrm{d}k
\end{aligned} \tag{2.52}$$

into the kinetic equation (2.51). The statistical quantity $q(X,t)$ exists only for the perturbed system and it satisfies (2.48) and a linearized equation (2.49) for the center-of-mass coordinate

$$\frac{\partial n_1(X,t)}{\partial t}+\frac{\partial}{\partial X}\left\{2(2\gamma_1+\gamma_2)n_0n_1(X,t)-i\alpha\left[q_1(X,t)-q_1^*(X,t)\right]\right\}=0. \tag{2.53}$$

Then, the kinetic equation satisfied by the perturbative terms ρ_1, n_1 writes

$$\begin{aligned}
&\frac{\partial\rho_1(k,X,t)}{\partial t}+\left[2\alpha k+(\gamma_1+\gamma_2)n_0\right]\frac{\partial\rho_1(k,X,t)}{\partial X}\\
&+4\beta n_1(X,t)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho_0(k)\\
&+(3\gamma_1+\gamma_2)\frac{\partial n_1(X,t)}{\partial X}\cos\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho_0(k)\\
&-2(\gamma_1+\gamma_2)n_1(X,t)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\left[k\rho_0(k)\right]\\
&+i(\gamma_2-\gamma_1)\left[q_1(X,t)-q_1^*(X,t)\right]\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho_0(k)=0.
\end{aligned} \tag{2.54}$$

Derivating (2.54) with respect to X and using (2.53), we get

$$\begin{aligned}
&\frac{\partial^2\rho_1(k,X,t)}{\partial X\partial t}+\left[2\alpha k+(\gamma_1+\gamma_2)n_0\right]\frac{\partial^2\rho_1(k,X,t)}{\partial X^2}\\
&+4\beta\frac{\partial n_1(X,t)}{\partial X}\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho_0(k)\\
&+(3\gamma_1+\gamma_2)\frac{\partial^2 n_1(X,t)}{\partial X^2}\cos\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho_0(k)\\
&-2(\gamma_1+\gamma_2)\frac{\partial n_1(X,t)}{\partial X}\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)h(k)\\
&+\frac{\gamma_2-\gamma_1}{\alpha}\left(\frac{\partial}{\partial t}+2(2\gamma_1+\gamma_2)n_0\frac{\partial}{\partial X}\right)n_1(X,t)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho_0(k)=0,
\end{aligned} \tag{2.55}$$

where $h(k)=k\rho_0(k)$.

In the following we seek plane wave solutions of (2.55) of the form

$$\begin{aligned} \rho_1(k,X,t) &= g(k)e^{i(QX-\Omega t)} + \text{cc.}\ , \\ n_1(x,t) &= Ge^{i(QX-\Omega t)} + \text{cc.}\ , \\ G &= \int_{-\infty}^{+\infty} g(k)\,\mathrm{d}k. \end{aligned} \tag{2.56}$$

The effect of the spatial and time derivatives on this solutions is to reproduce the plane waves with specific coefficients, $\frac{\partial}{\partial X} \to iQ$ and $\frac{\partial}{\partial t} \to -i\Omega$ respectively. The differential operators defined in terms of Taylor series transform accordingly

$$\begin{aligned} \sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right) &= \sin\left(i\frac{Q}{2}\frac{\overrightarrow{\partial}}{\partial k}\right) = i\sinh\left(\frac{Q}{2}\frac{\partial}{\partial k}\right), \\ \cos\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right) &= \cos\left(i\frac{Q}{2}\frac{\overrightarrow{\partial}}{\partial k}\right) = \cosh\left(\frac{Q}{2}\frac{\partial}{\partial k}\right). \end{aligned}$$

The effect of these hyperbolic differential operators defined in terms of their Taylor expansion on an arbitrary function of k, $f(k)$, is

$$\begin{aligned} 2\sinh\left(\frac{Q}{2}\frac{\partial}{\partial k}\right)f(k) &= f\left(k+\frac{Q}{2}\right) - f\left(k-\frac{Q}{2}\right), \\ 2\cosh\left(\frac{Q}{2}\frac{\partial}{\partial k}\right)f(k) &= f\left(k+\frac{Q}{2}\right) + f\left(k-\frac{Q}{2}\right). \end{aligned}$$

With these transformations, the equation (2.55) for the solutions (2.56) writes

$$\begin{aligned} &\left[\Omega Q - 2\alpha Q^2 k - (\gamma_1+\gamma_2)n_0 Q^2\right]g(k) - 2\beta QG\left[\rho_0\left(k+\frac{Q}{2}\right) - \rho_0\left(k-\frac{Q}{2}\right)\right] \\ &\quad - \frac{3\gamma_1+\gamma_2}{2}Q^2 G\left[\rho_0\left(k+\frac{Q}{2}\right) + \rho_0\left(k-\frac{Q}{2}\right)\right] \\ &\quad + (\gamma_1+\gamma_2)QG\left[h\left(k+\frac{Q}{2}\right) - h\left(k-\frac{Q}{2}\right)\right] \\ &\quad + \frac{\gamma_2-\gamma_1}{2\alpha}\left[\Omega - 2(2\gamma_1+\gamma_2)n_0 Q\right]G\left[\rho_0\left(k+\frac{Q}{2}\right) - \rho_0\left(k-\frac{Q}{2}\right)\right] = 0. \end{aligned} \tag{2.57}$$

Dividing (2.57) by $2\alpha Q^2$, one can denote

$$\omega = \frac{\Omega}{2\alpha Q} - \frac{\gamma_1+\gamma_2}{2\alpha}n_0. \tag{2.58}$$

Then, an implicit dispersion relation for $\omega(Q)$ is obtained dividing by $(\omega - k)$ and integrating over all values of k

$$1 + \left[\frac{\gamma_2-\gamma_1}{2\alpha}\left(\omega - \frac{3\gamma_1+\gamma_2}{2\alpha}n_0\right) - \frac{\beta}{\alpha}\right]\mathcal{I} + \frac{\gamma_1+\gamma_2}{2\alpha}\mathcal{J} - \frac{3\gamma_1+\gamma_2}{2\alpha}\mathcal{K} = 0, \tag{2.59}$$

where we denoted the integrals

$$\begin{aligned}
\mathcal{I} &= \frac{1}{Q}\int_{-\infty}^{+\infty} \frac{\rho_0\left(k+\frac{Q}{2}\right)-\rho_0\left(k-\frac{Q}{2}\right)}{\omega-k}\,\mathrm{d}k,\\
\mathcal{J} &= \frac{1}{Q}\int_{-\infty}^{+\infty} \frac{h\left(k+\frac{Q}{2}\right)-h\left(k-\frac{Q}{2}\right)}{\omega-k}\,\mathrm{d}k, \qquad (2.60)\\
\mathcal{K} &= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\rho_0\left(k+\frac{Q}{2}\right)+\rho_0\left(k-\frac{Q}{2}\right)}{\omega-k}\,\mathrm{d}k.
\end{aligned}$$

The dispersion relations for dNLS-1 and dNLS-2 are obtained by taking $\beta = 0$ and $\gamma_1 = \gamma_2$, respectively $\gamma_1 = 0$ in the equation (2.59). Further more, one should notice that the dispersion relation (2.24) for the cubic NLS is easily recovered using $\gamma_1 = \gamma_2 = 0$.

In the following we'll consider only the δ and Lorentzian spectrum as power spectral distributions at equilibrium and solve the equation (2.59) in order to determine the instability conditions.

II.3.2.1. δ-spectrum

When $\rho_0(k) = n_0\delta(k)$ the integrals $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ are easily performed

$$\mathcal{I} = -\frac{n_0}{\omega^2-\frac{Q^2}{4}};\quad \mathcal{J}=0;\quad \mathcal{K} = \frac{n_0\omega}{\omega^2-\frac{Q^2}{4}}. \qquad (2.61)$$

The equation (2.59) becomes a second order equation for the complex angular frequency $\omega = \omega_r + i\omega_i$

$$\omega^2 - n_0\frac{\gamma_1+\gamma_2}{\alpha}\omega + \frac{\beta}{\alpha}n_0 + \frac{(3\gamma_1+\gamma_2)(\gamma_2-\gamma_1)}{4\alpha^2}n_0^2 - \frac{Q^2}{4} = 0. \qquad (2.62)$$

Separating the complex and real parts in the equation above one obtains

$$\begin{aligned}
\omega_r &= \frac{\gamma_1+\gamma_2}{2\alpha}n_0,\\
\omega_i &= \frac{\Omega_i}{2\alpha Q} = \sqrt{\frac{\beta}{\alpha}n_0 - \left(\frac{n_0\gamma_1}{\alpha}\right)^2 - \frac{Q^2}{4}}.
\end{aligned} \qquad (2.63)$$

These results reproduce the ones obtained in the deterministic approach (for the δ spectrum) at the point $k = 0$. It is easily seen that for both types of derivative NLS equation, there is no instability in the long wave-length region. A modified NLS equation, obtained from (2.38) when $\beta \neq 0$, $\gamma_2 \neq 0$ and $\gamma_1 = 0$, will however exhibit the modulational instability phenomenon as discussed in [11].

II.3.2.2. Lorentzian spectrum

Using a Lorentzian spectrum for the equilibrium spectral power density (2.29), the integrals in (2.59) can be done in the complex plane where the integrated functions have three singularities above the abscissa and only one below. The integrals are thus performed on a contour in the lower half-plane using the well-known Cauchy's residue theorem

$$\mathcal{I} = \frac{-n_0}{(\omega+ip)^2 - \frac{Q^2}{4}}; \quad \mathcal{J} = i\frac{n_0 p}{(\omega+ip)^2 - \frac{Q^2}{4}}; \quad \mathcal{K} = \frac{n_0(\omega+ip)}{(\omega+ip)^2 - \frac{Q^2}{4}}. \quad (2.64)$$

Using these expressions in (2.59) and denoting $\tilde{\omega} = \omega + ip = \tilde{\omega_r} + i\tilde{\omega_i}$, one finds the following bi-quadratic equation for the coefficient of the imaginary part $\tilde{\omega_i}$

$$\tilde{\omega_i}^4 - A_2\tilde{\omega_i}^2 - A_0 = 0, \quad (2.65)$$

where

$$A_2 = \frac{\beta}{\alpha}n_0 - \frac{Q^2}{4} - \left(\frac{n_0\gamma_1}{\alpha}\right)^2, \quad A_0 = \left(\frac{n_0\gamma_2 p}{2\alpha}\right)^2.$$

With $\Delta = A_2^2 + (2A_0)^2$, the only acceptable (positive) solution is

$$\tilde{\omega_i}^2 = \frac{1}{2}(A_2 + \sqrt{\Delta}). \quad (2.66)$$

Then the instability develops for positive values of

$$\omega_i = \frac{\Omega_i}{2\alpha Q} = \frac{1}{\sqrt{2}}\sqrt{A_2 + \sqrt{\Delta}} - p. \quad (2.67)$$

This result is very similar to the one obtained for the cubic NLS equation (also for Lorentzian equilibrium spectral power density). The modulational instability manifests for $\omega_i \sim \Omega_i > 0$ which determines the following domain in the wave-number space

$$\frac{Q^2}{4} \leq \frac{\beta}{\alpha}n_0 - \left(\frac{n_0}{\alpha}\right)^2\left[\gamma_1^2 - \left(\frac{\gamma_2}{2}\right)^2\right] - p^2. \quad (2.68)$$

It is easily seen that for a fixed scale parameter p, there is an upper limit of the wave-number Q for which the instability no longer develops. Also, in order to keep Q a real quantity, the coefficients α and β must have the same sign, $\alpha\beta > 0$. Qualitatively, these results remain the same for the dNLS-1 and dNLS-2 type equations [12] as well as the modified NLS equation discussed in [11], with the corresponding redefinition of the coefficients β, γ_1 and/or γ_2.

II.3.2.3. Formulae used in Wigner-Moyal transform

In this small appendix we'll present some details of the computations involved in performing the Wigner-Moyal transform, especially to nonlinear equations with derivative terms. To this purpose we'll use the formulas (2.18) as a starting point. The decomposition in Taylor series with respect to the relative coordinate $\xi = x_1 - x_2$ and the center-of-mass coordinate

$X = (x_1 + x_2)/2$ in (2.18) is valid for any smooth function $f(x, t)$. Further more, one has

$$f(x_1, t) + f(x_2, t) = 2 \sum_{j=0}^{\infty} \frac{\xi^{2j}}{2^{2j}(2j)!} \left(\frac{\partial^{2j} f(X, t)}{\partial X^{2j}} \right). \tag{2.69}$$

The procedure used in computing the Fourier transform of the various terms encountered in the kinetic equations derived for the two-point correlation functions is more or less the same. It involves expanding all functions $f(x_i) = f(X \pm \xi/2)$, $i = 1, 2$ in Taylor series around the center-of-mass coordinate, the performing the integrations using the formulas (2.18). For instance, denoting the Fourier transforms by $\mathscr{F}$, the term corresponding to the cubic nonlinearity for the NLS equation is computed this way

$$\begin{aligned}
&\mathscr{F}\left\{[n(x_1) - n(x_2)]W(x_1, x_2)\right\} = \\
&= \frac{1}{2\pi} \int \mathrm{d}\xi e^{-ik\xi} 2 \sum_{j=0}^{\infty} \frac{\xi^{2j+1}}{2^{2j+1}(2j+1)!} \frac{\partial^{2j+1} n(X)}{\partial X^{2j+1}} \int \mathrm{d}k' e^{ik'\xi} \rho(k', X) \\
&= 2i \int \frac{\mathrm{d}k'}{2\pi} \int \mathrm{d}\xi \sum_{j=0}^{\infty} \frac{(-)^j}{2^{2j+1}(2j+1)!} \frac{\partial^{2j+1} n(X)}{\partial X^{2j+1}} \frac{\partial^{2j+1}}{\partial k^{2j+1}} e^{i(k'-k)\xi} \rho(k', X) \\
&= 2i \sum_{j=0}^{\infty} \frac{(-)^j}{2^{2j+1}(2j+1)!} \frac{\partial^{2j+1} n(X)}{\partial X^{2j+1}} \frac{\partial^{2j+1}}{\partial k^{2j+1}} \int \mathrm{d}k' \delta(k' - k) \rho(k', X) \\
&= 2i \sum_{j=0}^{\infty} \frac{(-)^j}{2^{2j+1}(2j+1)!} \frac{\partial^{2j+1} n(X)}{\partial X^{2j+1}} \frac{\partial^{2j+1}}{\partial k^{2j+1}} \rho(k, X) \\
&= 2in(X, t) \sin\left(\frac{1}{2} \frac{\overleftarrow{\partial}}{\partial X} \frac{\overrightarrow{\partial}}{\partial k} \right) \rho(k, X, t).
\end{aligned}$$

A similar method of integration is used for terms resulting from derivative nonlinearities

$$\begin{aligned}
&\mathscr{F}\left\{[q(x_1) + q^*(x_2)]W(x_1, x_2)\right\} = \\
&\mathscr{F}\Big\{ W(1, 2) \sum_{j=0}^{\infty} \frac{\xi^{2j}}{2^{2j}(2j)!} \frac{\partial^{2j}}{\partial X^{2j}} [q(X) + q^*(X)] + \\
&+ \frac{\xi^{2j+1}}{2^{2j+1}(2j+1)!} \frac{\partial^{2j+1}}{\partial X^{2j+1}} [q(X) - q^*(X)] \Big\} = \\
&= \frac{\partial n(X)}{\partial X} \cos\left(\frac{1}{2} \frac{\overleftarrow{\partial}}{\partial X} \frac{\overrightarrow{\partial}}{\partial k} \right) \rho(k, X) + i[q(X) - q^*(X)] \sin\left(\frac{1}{2} \frac{\overleftarrow{\partial}}{\partial X} \frac{\overrightarrow{\partial}}{\partial k} \right) \rho(k, X)
\end{aligned}$$

and the complex conjugate

$$\mathscr{F}\{[q^*(x_1)+q(x_2)]W(x_1,x_2)\} = \frac{\partial n(X)}{\partial X}\cos\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho(k,X)$$

$$+\, i[q^*(X)-q(X)]\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho(k,X).$$

For computing further terms due to derivative nonlinearities in (2.38) one has to use

$$\frac{\partial}{\partial x_1} = \frac{1}{2}\frac{\partial}{\partial X}+\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x_2} = \frac{1}{2}\frac{\partial}{\partial X}-\frac{\partial}{\partial \xi}.$$

The Fourier transforms of the terms involving the densities $n(x_1)$, $n(x_2)$ and their derivatives write

$$\mathscr{F}\left\{\left[n(x_1)\frac{\partial}{\partial x_1}+n(x_2)\frac{\partial}{\partial x_2}\right]W(x_1,x_2)\right\} =$$

$$\mathscr{F}\left\{\frac{1}{2}[n(x_1)+n(x_2)]\frac{\partial}{\partial X}W(\xi,X)+[n(x_1)-n(x_2)]\frac{\partial}{\partial \xi}W(\xi,X)\right\} =$$

$$n(X)\cos\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\frac{\partial\rho(k,X)}{\partial X} - 2n(X)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)[k\rho(k,X)]$$

and

$$\mathscr{F}\left\{\left[\frac{\partial n(x_1)}{\partial x_1}+\frac{\partial n(x_2)}{\partial x_2}\right]W(x_1,x_2)\right\} = \mathscr{F}\left\{\frac{\partial}{\partial X}\sum_{j=0}^{\infty}\frac{\xi^{2j}}{2^{2j}(2j)!}\frac{\partial^{2j}n(X)}{\partial X^{2j}}+\right.$$

$$\left.+\,2\frac{\partial}{\partial \xi}\sum_{j=0}^{\infty}\frac{\xi^{2j+1}}{2^{2j+1}(2j+1)!}\frac{\partial^{2j+1}n(X)}{\partial X^{2j+1}}W(1,2)\right\} =$$

$$= \mathscr{F}\left\{2\frac{\partial}{\partial X}\sum_{j=0}^{\infty}\frac{\xi^{2j}}{2^{2j}(2j)!}\frac{\partial^{2j}n(X)}{\partial X^{2j}}W(1,2)\right\} = 2\frac{\partial n(X)}{\partial X}\cos\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho(k,X),$$

where in the first relation $\partial W(\xi,X)/\partial\xi$ yields a factor ik in the final result.

The operators used through-out this chapter are defined by their Taylor expansions, with the overhead arrows indicating the terms on which the respective derivatives act. We give here an example of such expansions acting on two arbitrary function $g(k)$ and $f(X)$ (the functions could depend on other variables as well)

$$f(X)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)g(k) = \sum_{j=0}^{\infty}\frac{(-)^j}{2^{2j+1}(2j+1)!}\frac{\partial^{2j+1}f(X)}{\partial X^{2j+1}}\frac{\partial^{2j+1}g(k)}{\partial k^{2j+1}},$$

$$f(X)\cos\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)g(k) = \sum_{j=0}^{\infty}\frac{1}{2^{2j}(2j)!}\frac{\partial^{2j}f(X)}{\partial X^{2j}}\frac{\partial^{2j}g(k)}{\partial k^{2j}}.$$

The formulae presented above were used to derive the kinetic equations for the power spectral densities corresponding to each type of nonlinear Schrödinger equation discussed in the statistical approach to modulational instability with a Gaussian approximation employed in decoupling the nonlinear fourth order cumulants.

II.4. Deterministic and Statistical Approach of Modulational Instability for Spherical and Cylindrical NLS Equations

The one-dimensional approach to the study of the nonlinear Schrödinger type equations can be further enriched by considering their formulation in restricted geometries like the cylindrical and spherical ones. In these geometries the oscillations of the field propagate along the radial axis being described by NLS equations of the form

$$i\frac{\partial \Phi}{\partial t} + \alpha \frac{\partial^2 \Phi}{\partial r^2} + \beta |\Phi|^2 \Phi + i\frac{m}{2t}\Phi = 0, \tag{2.70}$$

where $m = 1$ for the cylindrical equation and $m = 2$ for the spherical one. These equations were found to model various processes in laboratory, plasma and astrophysical environments, especially when dust contaminated plasmas appear [25,44], or in fluid dynamics [20]. They are obtained from a multiple scale analysis of the given physical problem, so the variables r and t in (2.70) are the "*slow variables*" introduced by this procedure which usually relate to the real radial and temporal coordinates ρ and τ through expression of the form $r = \varepsilon(\rho - v\tau)$ and $t = \varepsilon^2\tau$ (here $\varepsilon \ll 1$ is a small parameter and v is an arbitrary constant).

In the following the deterministic and statistical analysis method will be applied to the equations (2.70).

II.4.1. D.A.M.I. for Cylindrical and Spherical NLS Equations

The deterministic approach presented below, follows closely the analysis in [44]. Instead of working with the field variable Φ one may use a simple transformation

$$\Phi = \frac{1}{t^{\frac{m}{2}}}\Psi \tag{2.71}$$

so that the equation satisfied by the new variable Ψ writes

$$i\frac{\partial \Psi}{\partial t} + \alpha \frac{\partial^2 \Psi}{\partial r^2} + \frac{\beta}{t^m} |\Psi|^2 \Psi = 0. \tag{2.72}$$

This has the form of a cubic NLS equation with the coefficient of the nonlinear term β replaced by β/t^m. Therefore the results of the linear stability analysis of a slowly modulated wave packet,

$$\Psi(r,t) = a\left[1 + \varepsilon A(r,t)\right] \exp\left(i\int_{t_0}^{t} \Delta(t')\, \mathrm{d}t' - \frac{m}{2}\ln t\right), \quad \Delta(t) = \frac{\beta}{t^m}|a|^2$$

($t_0 \neq 0$ arbitrary), are similar to the case of the one-dimensional cubic NLS equation with a time dependent coefficient of the nonlinearity. The plane wave solutions for the perturbation have now the form

$$A(r,t) = M \exp\left[i\left(Qr - \int_{t_0}^{t} \Omega(t')\,\mathrm{d}t'\right)\right] + N^* \exp\left[-i\left(Qr - \int_{t_0}^{t} \Omega^*(t')\,\mathrm{d}t'\right)\right]$$

and

$$\mathrm{Im}\,\Omega(t) = Q\sqrt{2\alpha\beta\frac{|a|^2}{t^m} - \alpha^2 Q^2}. \tag{2.73}$$

The instability manifests when the coefficient of the imaginary part of $\Omega(t)$ is positive, thus when α and β have the same sign (the focusing case of the NLS equation) and in a domain of long wavelength

$$Q^2 < Q_C^2(t) = 2\frac{\beta}{\alpha}\frac{|a|^2}{t^m}. \tag{2.74}$$

Unlike the one-dimensional cubic NLS equation, the instability growth for the cylindrical and spherical equations will cease when

$$t > t_{max} = \left(2\frac{\alpha}{\beta}\frac{|a|^2}{Q^2}\right)^{1/m} \tag{2.75}$$

for a given initial wavelength of the perturbation (fixed Q). This is a special property of the modulational instability phenomenon manifesting for the cylindrical and spherical NLS equations which shows the influence of the symmetry of the processes on the instability domain, *i. e.* the M. I. due to an initial perturbation develops for only a limited interval of time.

One can define a total growth of the modulation denoted by $\exp(G)$, where

$$G = \int_{t_0}^{t_{max}} \mathrm{Im}\,\Omega(t')dt' = |\alpha| Q^2 t_0 \int_1^{R^{1/m}} \mathrm{d}\lambda \sqrt{\frac{R}{\lambda^m} - 1} = \frac{\beta |a|^2}{t_0^{m-1}} f_m(R). \tag{2.76}$$

Here we used the notations

$$\lambda = \frac{t}{t_0}, \quad R = 2\frac{\beta}{\alpha}\frac{|a|^2}{Q^2}\frac{1}{t_0^m} \geq 1,$$

and the functions $f_m(R)$, obtained through straightforward integration, write

$$\begin{aligned} f_1(R) &= \arctan\sqrt{R-1} - \frac{\sqrt{R-1}}{R} \quad \text{(cylindrical NLS)} \\ f_2(R) &= \frac{1}{R}\left(\sqrt{R}\ln\frac{\sqrt{R}+\sqrt{R-1}}{\sqrt{R}-\sqrt{R-1}} - 2\sqrt{R-1}\right) \quad \text{(spherical NLS).} \end{aligned} \tag{2.77}$$

Thus the results in [44] are reproduced. Analyzing the two functions $f_m(R)$, one sees in figure II.4 that, as a general conclusion, the spherical waves are more structurally stable to perturbations than the cylindrical ones.

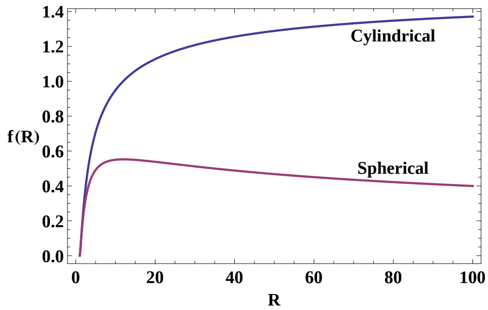

Figure II.4: Comparative dependence of the exponent of growth G on the parameter R for cylindrical (*up*) and spherical (*down*) nonlinear Schrödinger equations.

II.4.2. S.A.M.I. for Cylindrical and Spherical NLS Equations

The statistical approach to modulational instability for the cylindrical and spherical NLS equations having the form (2.72) is the same as the procedure employed for the cubic NLS equation only with a time dependent coefficient of the nonlinearity. Consequently, the kinetic equation for the two-point correlation function $W(x_1, x_2, t)$ writes

$$i\frac{\partial W(1,2)}{\partial t} + \alpha\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2}\right)W(1,2) + \frac{2\beta}{t^m}\left[n(x_1) - n(x_2)\right]W(1,2) = 0. \quad (2.78)$$

Applying the Wigner-Moyal transform, (2.78) becomes

$$\frac{\partial \rho(k,X,t)}{\partial t} + 2\alpha k\frac{\partial \rho(k,X,t)}{\partial X} + \frac{4\beta}{t^m}n(X,t)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho(k,X,t) = 0. \quad (2.79)$$

describing the evolution of the spectral power distribution $\rho(k, X, t)$.

Performing a linear stability analysis of (2.79), we'll consider the same form (2.52) for the perturbation of the equilibrium power spectral density. The linearized equation, satisfied by $\rho_1(k, X, t)$, is

$$\frac{\partial \rho_1}{\partial t} + 2\alpha k\frac{\partial \rho_1}{\partial X} + \frac{4\beta}{t^m}n_1(X)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho_0(k) = 0. \quad (2.80)$$

Looking for plane wave solutions of the form

$$\begin{aligned} \rho_1(k,X,t) &= g(k)\, e^{i\left[Qx - \int_{t_0}^{t}\Omega(t')\,\mathrm{d}t'\right]} + \mathrm{cc.}\,, \\ n_1(X,t) &= G\, e^{i\left[Qx - \int_{t_0}^{t}\Omega(t')\,\mathrm{d}t'\right]} + \mathrm{cc.}\,, \quad G = \int_{-\infty}^{+\infty} g(k)\,\mathrm{d}k, \end{aligned} \quad (2.81)$$

the same algebraic manipulations used in the statistical approach of the cubic NLS equation lead us to the following integral dispersion relation

$$1 + \frac{1}{t^m}\frac{\beta}{\alpha}\frac{1}{Q}\int_{-\infty}^{+\infty}\frac{\rho_0(k+\frac{Q}{2}) - \rho_0(k-\frac{Q}{2})}{k - \frac{\Omega}{2\alpha Q}}\,\mathrm{d}k = 0. \quad (2.82)$$

The modulational instability develops only for positive coefficients of imaginary part of Ω, $\operatorname{Im}\Omega = \Omega_i > 0$. To study these instability domains, we shall consider in the following, different equilibrium power spectral densities $\rho_0(k)$.

II.4.2.1. δ-spectrum

Using

$$\rho_(k) = n_0 \delta(k)$$

in the equation (2.82) and assuming, for simplicity, that $\Omega = i\Omega_i$ is purely imaginary, one finds

$$\Omega_i = |\alpha| Q \sqrt{4 \frac{\beta}{\alpha} \frac{n_0}{t^m} - Q^2}. \tag{2.83}$$

This is exactly the result (2.73) obtained in the deterministic approach if one uses the correspondence $|a|^2 \leftrightarrow 2n_0$ between the deterministic squared amplitude (wave intensity) and the mean value of the squared stochastic field amplitude. Therefore, a total growth of the instability may be defined as $\exp(\mathscr{G})$, where $\mathscr{G}$ is a new notation for G in (2.76) and the functions $f_m(R)$ are defined in (2.77) for the cylindrical and spherical equations.

II.4.2.2. Lorentzian spectrum

Considering a Lorentzian distribution for the power density at equilibrium (2.29), a straightforward integration in (2.82) will yield a similar result as the one for the cubic NLS equation but with a time dependent coefficient β

$$\Omega_i = 2|\alpha| Q \left[\sqrt{\frac{\beta}{\alpha} \frac{n_0}{t^m} - \frac{Q^2}{4}} - p \right]. \tag{2.84}$$

The instability exists if $\alpha\beta > 0$ (have the same sign) and for Q is in the long wave length limit, namely

$$\frac{Q^2}{4} < \frac{n_0}{t^m} \frac{\beta}{\alpha} - p^2. \tag{2.85}$$

For fixed Q and p, there is a cut-off time, t_{max}

$$t_{max} = \left(4 \frac{\beta}{\alpha} \frac{n_0}{Q^2} \frac{1}{1 + 4p^2/Q^2} \right)^{1/m} \tag{2.86}$$

and the instability is also limited in time, $t < t_{max}$. The exponent of the total growth of the instability , defined as

$$\mathscr{G} = \int_{t_0}^{t_{max}} \Omega_i(t')\, \mathrm{d}t',$$

can be easily calculated with the following result

$$\mathscr{G}_m = 4|\beta| \frac{n_0}{t_0^{m-1}} \left\{ f_m(R) - \Delta f_m(\zeta) \right\} \tag{2.87}$$

where $f_m(R)$ are defined in (2.77) and $\zeta = 2p/Q$. As for the Lorentzian corrections $\Delta f_m(\zeta)$, one obtains

$$\Delta f_1(\zeta) = \arctan\zeta - \frac{\zeta}{R} \tag{2.88}$$

for the cylindrical case and

$$\Delta f_2(\zeta) = \frac{1}{\sqrt{R}}\left\{\ln\left(\zeta + \sqrt{1+\zeta^2}\right) - \frac{\zeta}{\sqrt{R}}\right\} \tag{2.89}$$

for the spherical one. When the scale parameter $p = 0$, $\Delta f_m(\zeta) = 0$ and one gets the results obtained previously for the case of δ-spectrum.

One should notice that for a given equilibrium distribution (n_0, p fixed), $\zeta = \sigma\sqrt{R}$ with

$$\sigma = \left(2\pi\frac{\beta}{\alpha}\frac{1}{t_0^{m-1}}\right)^{-1/2}\sqrt{\frac{\mathrm{FWHM}}{\rho_0(0)}},$$

where FWHM $= 2p$ is the full width half maximum of a Lorentzian function centered in origin. Thus, σ^2 is inversely proportional to the steepness of the equilibrium distribution. In figure II.5, the functions $f_m(R)$ are plotted with their corresponding Lorentzian corrections for three increasingly wider equilibrium distributions. The blue curves correspond to the cylindrical function

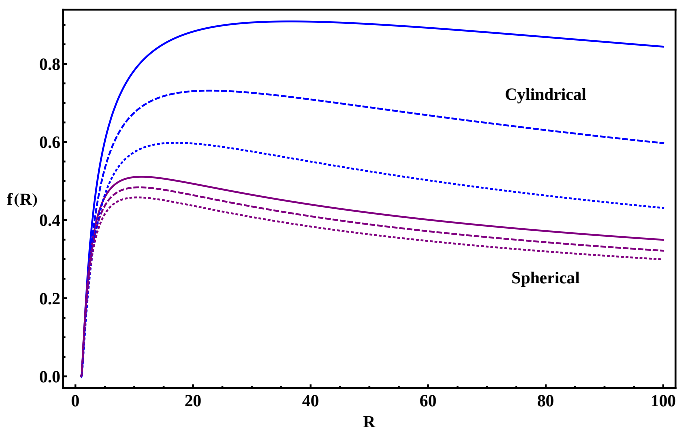

Figure II.5: Comparative plots of the exponent of growth $\mathscr{G}(R)$ for cylindrical (*blue*) and spherical (*purple*) nonlinear Schrödinger equations. The continuous, dashed and dotted curves correspond to increasingly wider Lorentzian power spectral densities at equilibrium.

$f_1(R) - \Delta f_1(R)$ and the purple curves to the spherical one, $f_2(R) - \Delta f_2(R)$. As the scale parameter p is increased, keeping the maximum $\rho_0(0)$ constant, the corresponding curves are continuous, dashed and dotted, respectively. It is easily seen that the effect of widening the equilibrium distribution is an increased stability to perturbations for the both geometries considered, but greater for the cylindrical NLS equation.

II.4.2.3. Gaussian Spectrum

If one considers a Gaussian equilibrium distribution (2.31), the results are more realistic as the decoupling of the fourth order ensemble averages is exact. Again, for the cylindrical and spherical NLS equations the computations are similar to the ones already performed for the cubic NLS equation but with the β coefficient substituted with β/t^m. If one takes $\Omega = i\Omega_i$ purely imaginary, using the integral representation of the complex error function (see [2], p. 297, formula 7.1.4), the equation (2.82) becomes

$$1 = \frac{1}{t^m}\frac{\beta}{\alpha}\frac{\sqrt{2\pi}n_0}{Q\sigma}\operatorname{Im} w(z), \tag{2.90}$$

where z is defined in (2.34). This is an implicit, integral equation that can only be solved numerically. To this purpose we shall use the relation between the real and imaginary part of the complex variable z and the important quantities of the physical problem Q and Ω_i

$$Q = 2\sqrt{2}\sigma x, \qquad \Omega_i = 8\sigma^2\alpha x y.$$

Also the parameters of the Gaussian distribution (like the full width half maximum) will be considered in order to put (2.90) in a very similar form to the equation (2.35)

$$\frac{\alpha}{\beta}\frac{t^m}{\sqrt{2\ln 2}}\frac{\text{FWHM}}{\rho_0(0)}x = \operatorname{Im}[w(x+iy)]. \tag{2.91}$$

Then, at each moment t this equation is solved numerically for positive x values, imposing the instability condition $y \sim \Omega_i > 0$ so that the instability domain can be determined as the area under the curve $\Omega_i(Q)$ which is hashed in the figure II.6. The darker shades under the curves correspond to

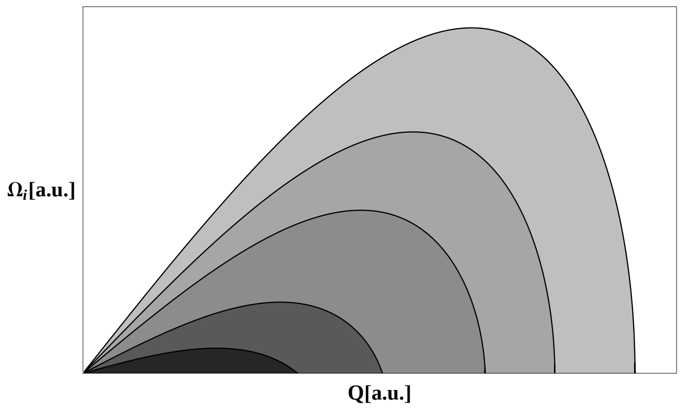

Figure II.6: Qualitative view of the modulational instability domain corresponding to cylindrical and spherical NLS equations for a Gaussian power spectral density at equilibrium. The gradually darker shades of gray indicate the instability domain as it shrinks with the passage of time.

increasing time values so as time passes the instability domain shrinks until the modulational instability ceases at a cut-off time which can be numerically estimated from the equation (2.91) for a given set of parameters of the physical problem.

II.5. Conclusions

The modulational instability is a common and well-known phenomenon in nonlinear physics. Usually a plane wave solution of a nonlinear partial differential equation is unstable to slow modulations of its amplitude. Finding the conditions for this phenomenon to take place is a basic problem of the nonlinear physics with important applications in various physical situations. There are two different ways of studying the M.I., reflecting distinct practical realities.

The first method consider the field variable $\Psi(x,t)$ as a plane wave with slowly varying amplitude

$$\Psi(x,t) = a\,[1+\varepsilon A(x,t)]\exp[i(kx-\omega t)]. \tag{2.92}$$

Plane wave solutions

$$A(x,t) = M\exp[i(Qx-\Omega t)] + \text{cc.}\ ,$$

of the linearized equation satisfied by the perturbation $A(x,t)$ are considered and the instability domain is easily determined by the condition $\operatorname{Im}\Omega > 0$. Usually, this domain contains the long wavelength limit (small values of the wavenumber Q). For instance, the M.I. develops for a cubic NLS equation if the coefficients α and β have the same sign and $Q^2 < 2\frac{\beta}{\alpha}|a|^2$. For derivative NLS equations the instability domain depends on the wave number k of the carrier wave and it occurs only when $k > 0$. Indeed, for the generalized derivative NLS equation

$$i\frac{\partial \Psi}{\partial t} + \alpha\frac{\partial^2\Psi}{\partial x^2} + \beta|\Psi|^2\Psi + i\gamma_1\frac{\partial|\Psi|^2}{\partial x}\Psi + i\gamma_2|\Psi|^2\frac{\partial\Psi}{\partial x} = 0, \tag{2.93}$$

the linear equation satisfied by $A(x,t)$ is

$$i\frac{\partial A}{\partial t} + i(2\alpha k+\gamma_2|a|^2)\frac{\partial A}{\partial x} + \alpha\frac{\partial^2 A}{\partial x^2} + \\ + i\gamma_1|a|^2\left(\frac{\partial A}{\partial x}+\frac{\partial A^*}{\partial x}\right) + |a|^2(\beta-\gamma_2 k)\,(A+A^*) = 0 \tag{2.94}$$

and the modulational instability domain is determined by

$$\left[2\left(\frac{\beta}{\alpha}-\frac{\gamma_2}{\alpha}k\right) - \left(\frac{\gamma_1}{\alpha}\right)^2|a|^2\right]|a|^2 - Q^2 \geq 0,$$

showing a strong dependence of the M.I. phenomenon on the wave number k of the carrier wave (details are presented in section §II.3.1). Such a deterministic approach of the modulational instability (D.A.M.I.) is appropriate for discussing the propagation of coherent pulses in nonlinear media.

The opposite approach is well-suited for the study of partially incoherent light beams propagating in nonlinear media. In this case the discussion makes use of the mutual coherence function [27]

$$\Gamma(\vec{r}_1, \vec{r}_2, z; t_1, t_2) = \langle \Psi(\vec{r}_1, x; t_1) \Psi^*(\vec{r}_2, z; t_2) \rangle \,. \tag{2.95}$$

Moreover, when temporal coherence effects are of no particular interest, the time dependence may be dropped and one can consider the spatial coherent function

$$J(1,2) = \langle \Psi(\vec{r}_1, x) \Psi^*(\vec{r}_2, z) \rangle \,, \tag{2.96}$$

which is the appropriate mathematical object to study the spatial properties of the incoherent beams. Experimentally, such a beam can be obtained by sending a laser beam through a rotating diffuser that changes the beam phase in a random manner as noted by Mitchel *et al.* [29, 30]. The self-trapping phenomenon of such partially coherent beams in nonlinear media leads to the concept of "*incoherent solitons*" (see [22], ch. 13 for a historical overview and theoretical methods). The concepts and theoretical methods used for dealing with incoherent solitons have many in common with the statistical approach to modulational instability (S.A.M.I.). In this approach, following Alber (1978) [3], a kinetic equation for the two-point correlation function

$$W(1,2) = W(x_1, x_2) = \langle \Psi(x_1) \Psi^*(x_2) \rangle \tag{2.97}$$

is written. Here $\langle \ldots \rangle$ means an average over the statistical ensemble. The field variables $\Psi(x,t)$ are considered stochastic quantities. The studied mathematical object is, however, the Fourier transform of $W(1,2)$

$$\rho(k, X, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik\xi} \left\langle \Psi(X + \frac{\xi}{2}, t) \Psi^*(X - \frac{\xi}{2}, t) \right\rangle \, \mathrm{d}\xi, \tag{2.98}$$

where $X = \frac{1}{2}(x_1 + x_2)$, $\xi = x_1 - x_2$. This is called the Wigner's function, introduced in statistical quantum mechanics in 1932. It is a real function and thus the number of equations to be solved is reduced by a factor of 2. The procedure to obtain the kinetic equation describing the time and space evolution of the two-point correlation function and its Fourier transform (Wigner's function) is well-known. For the cubic nonlinear Schrödinger equation (carefully analyzed in §II.2) the equation satisfied by $\rho(k, X, t)$ writes

$$\frac{\partial \rho(k,X,t)}{\partial t} + 2\alpha k \frac{\partial \rho(k,X,t)}{\partial X} + 4\beta n(X,t) \sin\left(\frac{1}{2} \overleftarrow{\frac{\partial}{\partial X}} \overrightarrow{\frac{\partial}{\partial k}} \right) \rho(k,X,t) = 0. \tag{2.99}$$

Here $n(X,t) = \left\langle |\Psi(X,t)|^2 \right\rangle$ is the pulse intensity in optics or the fluid density in hydrodynamics and it is related to the power spectral distribution $\rho(k,X,t)$ by

$$n(X,t) = \int_{-\infty}^{+\infty} \mathrm{d}k\rho(k,X,t).$$

The sin operator appearing in this equation is defined in terms of its Taylor expansion and the arrows over the differential operators indicate the direction in which these derivatives act (on the surrounding functions). One should remark that by changing $t \to z$, the equation (2.99) has the same form as the one obtained using the "Wigner Transform Method" in the study of incoherent solitons, see [22], ch. 13.

To analyze the M.I. one considers small perturbations around the equilibrium state. Due to the homogeneity and isotropy of the equilibrium state, one can write

$$\begin{aligned} \rho(k,X,t) = \rho_0(k) + \varepsilon\rho_1(k,X,t), \quad n(X,t) = n_0 + \varepsilon n_1(X,t), \\ n_0 = \int \rho_0(k)\,\mathrm{d}k, \quad n_1(X,t) = \int \rho_1(k,X,t)\,\mathrm{d}k, \end{aligned} \tag{2.100}$$

where $\rho_0(k)$ is an even function of k corresponding to a two-point correlation function at equilibrium $W(1,2) = W(|\xi|)$. Looking for plane wave solutions of the linearized equation satisfied by $\rho_1(k,X,t)$, the following implicit dispersion relation is found $\omega = \Omega/2\alpha Q$

$$1 + \frac{\beta}{\alpha}\frac{1}{Q}\int_{-\infty}^{+\infty} \frac{\rho_0\left(k+\frac{Q}{2}\right) - \rho_0\left(k-\frac{Q}{2}\right)}{k - \frac{\Omega}{2\alpha Q}}\,\mathrm{d}k = 0. \tag{2.101}$$

The instability is associated to positive values of the imaginary part of the angular frequency Ω and detailed analyses for different equilibrium distributions $\rho_0(k)$,

$$\begin{aligned} \delta - \text{spectrum} \quad & \rho_0(k) = n_0\delta(k), \\ \text{limited white spectrum} \quad & \rho_0(k) = \begin{cases} \frac{1}{2\Lambda}n_0, & |k| \le \Lambda \\ 0, & |k| > \Lambda \end{cases} \\ \text{Lorentzian spectrum} \quad & \rho_0(k) = \frac{n_0}{\pi}\frac{p}{p^2+k^2}, \\ \text{Gaussian spectrum} \quad & \rho_0(k) = \frac{n_0}{\sigma\sqrt{2\pi}}\exp\left(-\frac{k^2}{2\sigma^2}\right) \end{aligned}$$

is done in §II.2.

For the derivative NLS equations (2.36), (2.37) and (2.38), the S.A.M.I. is more complex as beside the mean value $n(X,t)$ one needs to introduce other stochastic quantities (mean values), namely

$$\begin{aligned} q(x,t) &= \left\langle \frac{\partial\Psi(x,t)}{\partial x}\Psi^*(x,t) \right\rangle \\ q^*(x,t) &= \left\langle \Psi(x,t)\frac{\partial\Psi^*(x,t)}{\partial x} \right\rangle. \end{aligned} \tag{2.102}$$

They are related to $n(x,t)$ by

$$q(x,t) + q^*(x,t) = \frac{\partial n(x,t)}{\partial x} \tag{2.103}$$

and by the relation (corresponding to the generalized NLS equation (2.38))

$$\frac{\partial n(x,t)}{\partial t} + \frac{\partial}{\partial x}\left\{\left(\gamma_1 + \frac{\gamma_2}{2}\right)\langle|\Psi|^4\rangle - i\alpha\left[q(x,t) - q^*(x,t)\right]\right\} = 0. \tag{2.104}$$

This second equation is nothing else than the conservation law of $n(x,t)$ for the generalized NLS equation. With these notations the kinetic equation satisfied by $\rho(k,X,t)$ is given by

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + 2\alpha k \frac{\partial \rho}{\partial x} + 4\beta n(X)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho(k) \\
&+ (3\gamma_1 + \gamma_2)\frac{\partial n(X)}{\partial X}\cos\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho(k) \\
&- 2(\gamma_1 + \gamma_2)n(X)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)[k\rho(k)] \\
&+ (\gamma_1 + \gamma_2)n(x)\cos\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\frac{\partial \rho(k)}{\partial X} \\
&+ i(\gamma_2 - \gamma_1)\left[q(X) - q^*(X)\right]\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\rho(k) = 0,
\end{aligned} \tag{2.105}$$

which still contains the quantities q and q^* defined above. The equation is however real because the difference $(q - q^*)$ is purely imaginary. It must be emphasized that a S.A.M.I. for generalized NLS equations was studied previously in [12,28] but, in both papers an incomplete decoupling of higher order correlation functions was used. The present results, unpublished yet, are the correct way to apply the Gaussian decoupling to higher order cumulants containing derivative terms - see (2.46) - with the introduction of the mean values $q(X,t)$ and $q^*(X,t)$.

In the first order of a perturbative problem, the difference $(q - q^*)$ can be eliminated and one get the equation (2.54) satisfied by $\rho_1(k,X,t)$. Looking for plane wave solutions the implicit dispersion relation for the same generalized NLS equation (2.38) writes

$$1 + \left[\frac{\gamma_2 - \gamma_1}{2\alpha}\left(\omega - \frac{3\gamma_1 + \gamma_2}{2\alpha}n_0\right) - \frac{\beta}{\alpha}\right]\mathcal{I} + \frac{\gamma_1 + \gamma_2}{2\alpha}\mathcal{J} - \frac{3\gamma_1 + \gamma_2}{2\alpha}\mathcal{K} = 0, \tag{2.106}$$

where the integrals $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ are defined in (2.60). They are easily calculated for an equilibrium spectral power density of type δ-spectrum as well as Lorentzian distribution. The instability domains are then determined resolving (2.106) and imposing that the imaginary part of ω be positive. For the δ-spectrum one gets

$$\omega_i \sim \sqrt{4\frac{\beta}{\alpha}n_0 - \left(\frac{2n_0\gamma_1}{\alpha}\right)^2 - Q^2}$$

which is exactly the result of the deterministic approach at $k = 0$ if $2n_0 \leftrightarrow |a|^2$.

The last paragraph is dedicated to the study of M.I. for cylindrical and spherical NLS equations

$$i\frac{\partial \Phi}{\partial t} + \alpha \frac{\partial^2 \Phi}{\partial r^2} + \beta|\Phi|^2\Phi + i\frac{m}{2t}\Phi = 0, \tag{2.107}$$

where $m = 1$ for the cylindrical equation and $m = 2$ for the spherical one. With the change of variable

$$\Phi(x,t) = \frac{1}{t^{\frac{m}{2}}}\Psi(x,t) \tag{2.108}$$

the equation (2.108) becomes

$$i\frac{\partial \Psi}{\partial t} + \alpha \frac{\partial^2 \Psi}{\partial r^2} + \frac{\beta}{t^m}|\Psi|^2\Psi = 0, \tag{2.109}$$

which is a cubic NLS equation having the coefficient β of the nonlinear term replaced by a time dependent one β/t^m. Therefore all the results known for the NLS equation can be applied to the cylindrical and spherical ones by changing β into β/t^m. The main qualitative difference from the NLS case is the time dependence of the instability domain, namely the instability growth will cease when

$$t > t_{max} = \left(2\frac{\alpha}{\beta}\frac{|a|^2}{Q^2}\right)^{1/m}. \tag{2.110}$$

Thus, one can define a total growth of the modulational instability by $\exp G$

$$G = \int_{t_0}^{t_{max}} \mathrm{Im}\,\Omega(t')dt' = \frac{\beta|a|^2}{t_0^{m-1}}f_m(R), \tag{2.111}$$

where $R = 2\frac{\beta}{\alpha}\frac{|a|^2}{Q^2}\frac{1}{t_0^m} \geq 1$ and the functions $f_m(R)$ are defined in (2.77).

The original results, in this last section, refer to the S.A.M.I. applied to the two equations and are presented in §II.4.2. As expected the results known for the NLS case are transposed for these equation by changing β into β/t^m. For instance, the implicit dispersion relation writes

$$1 + \frac{1}{t^m}\frac{\beta}{\alpha}\frac{1}{Q}\int_{-\infty}^{+\infty} \frac{\rho_0(k+\frac{Q}{2}) - \rho_0(k-\frac{Q}{2})}{k - \frac{\Omega}{2\alpha Q}}\,\mathrm{d}k = 0 \tag{2.112}$$

and the time-dependent instability domains can be calculated considering different power spectral densities (δ spectrum, Lorentzian, Gaussian). The total growth increment of the M.I., $\mathscr{G}$, can also be determined and for a Lorentzian spectrum it writes

$$\mathscr{G}_m = 4|\beta| \frac{n_0}{t_0^{m-1}} \{f_m(R) - \Delta f_m(\zeta)\} \tag{2.113}$$

where $f_m(R)$ are defined in (2.77) and $\Delta f_m(R)$ are given by (2.88) and (2.89) for the cylindrical and the spherical equation respectively ($\zeta = 2p/Q$).

Summarizing, the main results of this chapter are:

- A detailed analysis of the modulational instability for the class of nonlinear Schrödinger equations (cubic NLS, derivative NLS, cylindrical and spherical NLS) was performed from a deterministic point of view. For the most part, these results are not new but, they are presented here for comparison with the results of the statistical approach.

- A detail analysis of the M.I. for the NLS equation from a statistical point of view. (S.A.M.I.). Although some of these outcomes are well known, new results are obtained using a limited white spectrum and a Gaussian distribution for the equilibrium power spectral density $\rho_0(k)$.

- Completely original results are presented in §II.3.2 and §II.4.2 concerning the S.A.M.I. for derivative and cylindrical/spherical NLS equations. In §II.3.2 a complete and correct procedure is applied to the class of derivative NLS equations (in previous works an incomplete Gaussian decoupling method was used). The main improvement is the introduction of the mean value q(X,t)=$\left\langle \frac{\partial \Psi}{\partial X} \Psi^* \right\rangle$ besides $n(X,t) = \left\langle |\Psi|^2 \right\rangle$. Thus, a complete kinetic equation for the Wigner's function $\rho(k, X, t)$ may be written (2.51). In a linear perturbation analysis the stochastic quantity $q(X,t)$ and its complex conjugate can be eliminated, and finally a compact implicit dispersion relation is found (2.54), which is explicitly solved for a δ spectrum type and Lorentzian power spectral density $\rho_0(k)$ at equilibrium. All these results are new and not yet published. It should be noted that the equation (2.51) has many things in common with the Wigner method applied to the study of incoherent solitons, an aspect which will be analyzed in future works.

- In section §II.4.2 the statistical approach of M.I. for the cylindrical and spherical NLS equations was studied. The main observation is that the problem can be reduced to the problem of a cubic NLS equation by changing the coefficient β into β/t^m. The characteristic of the cylindrical and spherical problems is the time dependence of the instability domain - the fact that the M.I. ceases after a certain interval of time. This allows one to use the total growth of the modulation to describe the phenomenon. Explicit expressions of the total growth increment are calculated for a Lorentzian equilibrium spectrum.

These original results were published in:

- ISI indexed Romanian and international journals:

    1. "*Deterministic and Statistical Approach of Modulational Instability in the Class of Non-Linear Schrödinger Equations*", **A. T. Grecu**, D. Grecu, A. Vişinescu, Rom. Rep. Phys. **61**(3), 467–477 (2009).
    2. "*Modulational Instability of Cylindrical and Spherical NLS Equations. Statistical Approach*", **A. T. Grecu**, S. de Nicola, R. Fedele, D. Grecu, A. Visinescu, in *AIP Proc. 7th International Conference of the Balkan Physical Union*, vol. CP1203, pp. 1239–1244 (2009).
    3. "*Statistical Approach to Modulational Instability in the Class of Derivative Nonlinear Schrödinger Equations*", **A. T. Grecu**, D. Grecu, A. Visinescu, Int. J. Theor. Phys. **46**(5), 1190–1204 (2007).

- Romanian annals, journals of the Romanian Academy and books:

    1. "*Deterministic and Statistical Approach of Modulational Instability in the Class of Non-linear Schrödinger Equations*", **A. T. Grecu**, D. Grecu, A. Visinescu, Rom. Journ. Phys. **50**(1-2), 127–135 (2005).
    2. "*Modulational Instability of Derivative Non-Linear Schrödinger Equation*", **A. T. Grecu**, Physics AUC **15**(I), 177–183 (2005).
    3. "*Modulational Instability and Soliton Generation in Nonlinear Evolution Equations*", D. Grecu, A. Visinescu, A. S. Carstea, **A. T. Grecu**, Physics AUC **15**(I), 111–122 (2005).
    4. "*Modulational Instability Phenomenon of Quasi-Monochromatic Waves in Dispersive and Weakly Nonlinear Media*", D. Grecu, A. Visinescu, **A. T. Grecu**, in *Interdisciplinary Applications of Fractal and Chaos Theory* (R. Dobrescu, C. Vasilescu, eds.), pp. 406–413 (Rom. Acad. Publ., Bucharest, 2004).

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Chapter III: Madelung Fluid Description of Generalized Nonlinear Schrödinger Equations

III.1. Madelung Fluid Description of Quantum Mechanics

In 1924, in his doctoral thesis, Louis de Broglie introduced his revolutionary theory of electron waves. It had drawn little attention from the scientific world but a copy of his "*Recherches sur la théorie des quanta*" reached Albert Einstein who was very enthusiastic about this new idea of "matter waves". In 1926, adopting Louis de Broglie's proposal (for which he was awarded the Nobel Prize in Physics in 1929), Erwin Schrödinger derived his famous wave equation (Nobel Prize in Physics in 1933). The same year, starting from Schrödinger's equation, the German professor Erwin Madelung proposed the first hydrodynamical model of quantum/wave mechanics [1, 35].

In the description proposed by Madelung, the one-dimensional Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U(x)\Psi \tag{3.1}$$

is considered and one seeks solutions of the form

$$\Psi(x,t) = \sqrt{\rho(x,t)}\exp\left(\frac{i}{\hbar}\theta(x,t)\right). \tag{3.2}$$

Introducing (3.2) into (3.1) and separating the real and imaginary part, one obtains the following system of coupled partial differential equations

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \tag{3.3}$$

$$m\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)v = \frac{\partial}{\partial x}\left[\frac{\hbar^2}{2m}\left(\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}\right) - U\right] \tag{3.4}$$

where

$$v(x,t) = \frac{1}{m}\frac{\partial\theta(x,t)}{\partial x}. \tag{3.5}$$

The first equation (3.3) is a continuity equation for the *fluid density* $\rho = |\Psi|^2$ where $v(x,t)$ is the *fluid velocity.* The second equation is a Navier-Stokes-like equation of motion for the fluid velocity in which besides the usual force term (the gradient of the potential) we encounter another derivative term, known in the literature as *Bohm's potential*, which gives the quantum interaction between the fluid particles. The interpretation of $v(x,t)$ as the velocity of the fluid can also be seen from the expression of the quantum current density

$$j = \frac{\hbar}{2im}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right) = \rho v.$$

This continuum description of the Schrödinger equation suffers though of many flaws being, for instance, unable to give a proper solution to the hydrogen atom problem or a complete and satisfactory explanation of the quantum emission and absorption processes as mentioned by Madelung himself [35]. It turned however quite fruitful in a number of application like stochastic mechanics [7], quantum cosmology [42, 51] or more recently in the description of classical, quantum-like systems [36] or solving nonlinear partial differential equations [1, 20, 23].

In the followings we'll restrict ourselves to investigate the usage of this method for solving two classes of nonlinear partial differential equations the nonlinear Schrödinger equations and their variants with derivative nonlinearity.

III.2. Madelung Fluid Description of NLS Equations with Cubic and Quintic Nonlinearity

III.2.1. Madelung Fluid Description of Generalized NLS Equations

The (cubic) nonlinear Schrödinger equation is a member of the family of completely integrable nonlinear evolution equations. It appears in various fields of physics whenever a quasi-monochromatic wave is propagating in a dispersive and weakly nonlinear medium.

Let us apply the Madelung fluid description to the general form of the NLS equation

$$i\frac{\partial \Psi}{\partial t} + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2} + U(|\Psi|^2)\Psi = 0, \tag{3.6}$$

where $U(|\Psi|^2)$ is a polynomial of $|\Psi|^2$ with real, constant coefficients. Considering the special form of the solutions for the field variable

$$\Psi(x,t) = \sqrt{\rho(x,t)}e^{i\theta(x,t)} \tag{3.7}$$

and introducing it in (3.6) then separating the real and imaginary part, one obtains

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \tag{3.8}$$

$$\frac{\partial \theta}{\partial t} + \frac{1}{2}v^2 = \frac{1}{2}\frac{1}{\sqrt{\rho}}\frac{\partial^2 \sqrt{\rho}}{\partial x^2} + U. \tag{3.9}$$

Here $\rho = |\Psi|^2$ is the fluid density while $v(x,t) = \frac{\partial \theta(x,t)}{\partial x}$ is its velocity. Derivating (3.9) with respect to x one obtains

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)v = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho}}\frac{\partial^2 \sqrt{\rho}}{\partial x^2}\right) + \frac{\partial U(\rho)}{\partial x}. \tag{3.10}$$

Thus the general NLS equation has become a fully equivalent system of coupled partial differential equation, the first of which is a continuity equation

(3.8) for the fluid density ρ while the second (3.10) is an evolution equation for the fluid velocity. From this system, an evolution equation for the fluid density can be deducted, using a series of transformations (presented in detail in the Appendix III.4) [23], [19]

$$-\rho\frac{\partial v}{\partial t}+v\frac{\partial \rho}{\partial t}+2\left[C_0(t)-\int\frac{\partial v}{\partial t}dx\right]\frac{\partial \rho}{\partial x}+ \\ +\frac{1}{4}\frac{\partial^3\rho}{\partial x^3}+\left(\rho\frac{dU}{d\rho}+2U\right)\frac{\partial \rho}{\partial x}=0. \quad (3.11)$$

Here $C_0(t)$ is an arbitrary function of t (integration quantity). Under suitable assumptions for the fluid velocity $v(x,t)$, the evolution equation (3.11) can be put in a form of a generalized KdV equation [23]. There are two conditions that facilitate the solving of (3.11) corresponding to specific physical situations.

1. For $v = v_0 =$ arbitrary const. one has motion with constant velocity. From the continuity equation (3.8) it follows that $\rho(x,t) = \rho(\xi)$, $\xi = x - v_0 t$. It is also obvious from the equation (3.11) that the integration constant C_0 no longer depends on time. Then, (3.11) becomes

$$\frac{1}{4}\frac{d^3\rho}{d\xi^3}+E\frac{d\rho}{d\xi}+\left(\rho\frac{dU}{d\rho}+2U\right)\frac{d\rho}{d\xi}=0, \quad (3.12)$$

where $E = 2C_0 - v_0^2$ is an arbitrary constant. Introducing a new function $G(\rho)$ defined by

$$\rho\frac{dU}{d\rho}+2U=\frac{dG(\rho)}{d\rho}, \quad (3.13)$$

one can integrate (twice) the equation (3.12) obtaining

$$\frac{1}{4}\left(\frac{d\rho}{d\xi}\right)^2=-\mathcal{G}(\rho)-E\rho^2+A\rho+B, \quad \mathcal{G}(\rho)=2\int G(\rho)d\rho. \quad (3.14)$$

Here A and B are two arbitrary integration constants.

On the other hand for constant velocity the equation (3.10) becomes

$$\frac{1}{2}\frac{\partial^2\sqrt{\rho}}{\partial x^2}+U(\rho)\sqrt{\rho}=\lambda\sqrt{\rho} \quad (3.15)$$

which is a nonlinear eigenvalue equation for $\sqrt{\rho}$ (λ constant). After straightforward calculations, see [25], it can be written in the form

$$\left(\frac{E}{2}+\lambda\right)\rho+\frac{B}{2\rho}-\left(\frac{\mathcal{G}}{2\rho}+\rho U-G\right)=0,$$

which must be satisfied for any value of ρ. Therefore the following restrictions must apply

$$\begin{aligned} B = 0, \quad \lambda = -\frac{E}{2} \\ \frac{\mathcal{G}}{2\rho} + \rho U - G = 0. \end{aligned} \tag{3.16}$$

It is easily seen that the last condition is satisfied when $U(\rho)$ is a power function of ρ, $U(\rho) = \beta\rho^p$, with $\beta \in \mathbb{R}$ and $p \in \mathbb{R}\backslash\{-1, -2\}$ constant. For $p = -2$ (the anticubic case), $dG/d\rho = 0$ thus the equation (3.12) becomes linear (trivial). For $p = -1$ one obtains $G(\rho) = \beta \ln \rho$ and the last restriction (3.16) cannot be satisfied anymore as the equation is no longer polynomial.

For the power function form of $U(\rho)$, $\rho(\xi)$ satisfies a stationary modified KdV-type equation

$$E\frac{d\rho}{d\xi} + \beta(p+2)\rho^p\frac{d\rho}{d\xi} + \frac{1}{4}\frac{d^3\rho}{d\xi^3} = 0. \tag{3.17}$$

The phase $\theta(x,t)$ for motion with constant velocity is immediately computed starting from (3.9) using the definition of $v(x,t)$ and relations derived during the previous calculus [25]

$$\theta(x,t) = v_0\xi - \frac{1}{2}\left(v_0^2 + E\right)t. \tag{3.18}$$

2. The second simplifying situation corresponds to the motion with stationary profile when both $\rho(x,t)$ and $v(x,t)$ depend only on $\xi = x - u_0 t$ with u_0 an arbitrary constant. The equation (3.11) transforms into the same equation (3.12) with a different expression of E, $E = 2C_0 + u_0^2$. In this case, no supplementary restrictions exist, therefore it will yield a larger class of solutions. If $U(\rho)$ is a power function of ρ, as in the previous case, the same restrictions apply on the exponent p values. In the anticubic case the problem becomes linear as before. When $p = -1$, however, the right-hand side of the differential equation (3.14) is no longer a polynomial of ρ therefore its solution cannot be included in the special class of periodic solutions discussed below.

   Doing the integration in the ordinary differential equation (3.8), it gives

$$v = u_0 + \frac{A_0}{\rho}, \tag{3.19}$$

   where A_0 is an integration constant. One has to take this constant equal to zero for single solitary wave solutions which vanish at infinity (white solitons).

The phase $\theta(x,t) = \theta(\xi)$ will have a more complex expression (depending on $\rho(\xi)$)

$$\theta(\xi) = u_0 \xi + A_0 \int_0^{\xi} \frac{d\xi'}{\rho(\xi')}. \tag{3.20}$$

In both cases presented above, an equation similar to (3.17) must be satisfied by $\rho(x,t)$. This result constitutes the basis of recent studies, within the framework of Madelung fluid description, of the correspondence between a wide family of generalized nonlinear Schrödinger equations and a wide family of generalized Korteweg-de Vries equations. A review of their results can be found in [21] where it is shown that the surprising correspondence between the NLSE and KdVE families (which describe very different categories of phenomena in physics), proven for the case of power law nonlinearities in NLSE, is maintained for derivative-type NLSE and even cylindrical NLSE.

III.2.2. Periodic and Solitary Solutions of NLSE with Cubic Nonlinearity

The first equation from the nonlinear Schrödinger hierarchy has a cubic nonlinearity for which $U(\rho)$ has the form $U = \beta\rho$, where $\beta = \pm 1$ is a constant. We assume for simplicity that the magnitude of β is included in the definition of $\rho = |\Psi|^2$. Also it is worth mentioning that the cubic NLS equation is a completely integrable equation both for $\beta = +1$ (the focusing case) and for $\beta = -1$ (the defocussing case). Then the equation (3.12) becomes

$$\frac{1}{4}\frac{d^3\rho}{d\xi^3} + E\frac{d\rho}{d\xi} + 3\beta\rho\frac{d\rho}{d\xi} = 0, \tag{3.21}$$

a stationary KdV equation which integrated twice gives

$$\frac{1}{4}\left(\frac{d\rho}{d\xi}\right)^2 = P_3(\rho), \quad P_3(\rho) = -\beta\rho^3 - E\rho^2 + A\rho + B. \tag{3.22}$$

Here A and B are integration constants. Also, when integrating (3.22), one needs to restrict to solutions with physical significance that correspond to real, positive and finite ρ and therefore to the domains where the polynomial $P_3(\rho) > 0$. Let us denote by ρ_1, ρ_2, ρ_3 the three roots of the polynomial $P_3(\rho)$. Obviously, when any of these roots is complex, its complex conjugate is also a root and restricted domains of finite, positive solutions can not exist. Thus only when all the three roots are real one may get physically acceptable solutions and let's assume $\rho_1 > \rho_2 > \rho_3$.

For $\beta = +1$, $P_3(\rho)$ satisfies the requirements if at least two of the roots are positive $\rho_1 > \rho_2 > 0$ in the domain $\rho \in [\rho_2, \rho_1]$. Then the solution of

(3.22) is written in terms of the Jacobi elliptic functions (see [9], p. 79)

$$\begin{aligned}
&\int_{\rho}^{\rho_1} \frac{dt}{\sqrt{(\rho_1 - t)(t-\rho_2)(t-\rho_3)}} = gu = 2\xi \\
\operatorname{sn}^2 u =&\frac{\rho_1 - \rho}{\rho_1 - \rho_2}, \quad \rho = \rho_1 - (\rho_1 - \rho_2)\operatorname{sn}^2 u, \quad u = \frac{2}{g}\xi \\
k^2 =& \frac{\rho_1 - \rho_2}{\rho_1 - \rho_3}, \quad g = \frac{1}{\sqrt{\rho_1 - \rho_3}}.
\end{aligned} \tag{3.23}$$

In the limit case $\rho_3 = \rho_2$, $k^2 = 1$, the solution becomes a solitary wave

$$\rho = \rho_1 - (\rho_1 - \rho_2)\tanh^2 u, \tag{3.24}$$

namely a shifted bright soliton (a bright type soliton with a nonvanishing value at infinity). In the case of constant velocity ($v = v_0$) the supplementary condition $B = 0$ has to be imposed. This can be respected if any ρ_2 or ρ_3 is zero. In the degenerate case $\rho_2 = \rho_3$ the solution (3.24) transforms into the bright soliton solution

$$\rho = \rho_1 \frac{1}{\cosh^2 u}, \quad u = \sqrt{\rho_1}\xi \tag{3.25}$$

It is interesting to note that the equation (3.22) can be solved also in an apparently different way ([9], p. 77)

$$\begin{aligned}
&\int_{\rho_2}^{\rho} \frac{dt}{\sqrt{(\rho_1 - t)(t-\rho_2)(t-\rho_3)}} = gu = 2\xi \\
&k^2 \operatorname{sn}^2 u = \frac{\rho - \rho_2}{\rho - \rho_3}, \quad \rho = \frac{\rho_2 - \rho_3 k^2 \operatorname{sn}^2 u}{1 - k^2 \operatorname{sn}^2 u},
\end{aligned} \tag{3.26}$$

with the same definitions for k^2 and g. Actually the two solutions are not independent. Indeed adding the two integrals (3.23) and (3.26), when $k^2 \neq 1$, we get

$$u_1 + u_2 = \mathrm{K}(k),$$

where $\mathrm{K}(k)$ is the complete elliptic integral of first kind and by u_1 and u_2 we denoted the values of the integral (3.23) and (3.26) respectively. Using the addition formula

$$\operatorname{sn}(u - z) = \frac{\operatorname{sn} u \operatorname{cn} z \operatorname{dn} z - \operatorname{sn} z \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 z}$$

with $u = u_1$ and $z = \mathrm{K}(k)$ we get

$$\operatorname{sn}^2 u_2 = \frac{1 - \operatorname{sn}^2 u_1}{1 - k^2 \operatorname{sn}^2 u_1}.$$

This, using the expressions (3.23) for $\operatorname{sn}^2 u_1$ and (3.26) for $\operatorname{sn}^2 u_2$, becomes an identity. Although of different forms, the two solutions (3.23) and (3.26) represent the same one, the second being the first translated in u-space by $\mathrm{K}(k)$.

If $\beta = -1$ the previous requirements are satisfied only when all the three roots of $P_3(\rho)$ are positive and $\rho \in [\rho_3, \rho_2]$. Then the solution is ([9], p. 72)

$$\int_{\rho_2}^{\rho} \frac{dt}{\sqrt{(\rho_1 - t)(\rho_2 - t)(t - \rho_3)}} = gu = 2\xi$$
$$\mathrm{sn}^2 u = \frac{\rho - \rho_3}{\rho_2 - \rho_3}, \quad \rho = \rho_3 + (\rho_2 - \rho_3)\,\mathrm{sn}^2 u, \quad u = \frac{2}{g}\xi \tag{3.27}$$
$$k^2 = \frac{\rho_2 - \rho_3}{\rho_1 - \rho_3}, \quad g = \frac{2}{\sqrt{\rho_1 - \rho_3}}.$$

In the limit case $\rho_1 = \rho_2$, $k^2 = 1$ we have

$$\rho = \rho_3 + (\rho_1 - \rho_3)\tanh^2 u \tag{3.28}$$

representing a gray soliton ($\rho(0) = \rho_3$, $\rho(\infty) = \rho_2$). For constant velocity, the condition $B = 0$ implies $\rho_3 = 0$ and (3.28) becomes the dark soliton solution

$$\rho = \rho_1 \tanh^2 u, \quad u = \sqrt{\rho_1}\xi. \tag{3.29}$$

For motion with constant profile, the phase $\theta(x,t) = \theta(\xi)$, given by (3.20), writes

$$\theta(\xi) = u_0\xi + \tilde{A}\int_0^u \frac{dt}{1 - \alpha^2\,\mathrm{sn}^2 t}, \tag{3.30}$$

where

$$\tilde{A} = A_0\frac{g}{2\rho_1}, \quad \alpha^2 = \frac{\rho_1 - \rho_2}{\rho_1} \text{ for } \beta = +1$$
$$\tilde{A} = A_0\frac{g}{2\rho_3}, \quad \alpha^2 = -\frac{\rho_2 - \rho_3}{\rho_3} \text{ for } \beta = -1.$$

The integral from the right-hand side of (3.30) is nothing else that the incomplete elliptic integral of third kind ([9], p. 223)

$$\Pi(\varphi, \alpha^2, k) = \int_0^u \frac{dt}{1 - \alpha^2\,\mathrm{sn}^2 t},$$

where $\sin\varphi = \mathrm{sn}\,u$, $u = 2\xi/g$ and the modulus k of the elliptic integral is given in (3.23) for $\beta = +1$ and in (3.27) for $\beta = -1$. Since α^2 and k^2 are both real the incomplete elliptic integral of third kind is in the so-called *circular* or *hyperbolic* case if the sign of $\alpha^2(\alpha^2 - k^2)(\alpha^2 - 1)$ is negative or positive, respectively [2]. Thus for $\beta = +1$ one has

$$\mathrm{sgn}\ \underbrace{\alpha^2}_{>0}(\alpha^2 - k^2)\underbrace{(\alpha^2 - 1)}_{<0} = \begin{cases} -1 \to \text{circular case for } \rho_3 < 0 \\ +1 \to \text{hyperbolic case for } \rho_3 > 0 \end{cases}.$$

When $\beta = -1$ we are in the circular case as

$$\text{sgn} \underbrace{\alpha^2}_{<0} \underbrace{(\alpha^2 - k^2)}_{<0} \underbrace{(\alpha^2 - 1)}_{<0} = -1$$

Therefore, a more compact expression of the phase for constant profile motion is

$$\theta(\xi) = u_0\xi + \tilde{A}\Pi(\varphi, \alpha^2, k). \tag{3.31}$$

Similar results were obtained using a direct method for solving the equation (3.21) – see [25] §3.1.

III.2.3. Periodic and Solitary Solutions of NLSE with Cubic+Quintic Nonlinearity

Let us consider the non-integrable NLS equation with the cubic and quintic nonlinearity of the form

$$U(\rho) = \beta\rho + \frac{3}{2}\gamma\rho^2. \tag{3.32}$$

Then the equation (3.12) writes

$$\frac{1}{4}\frac{d^3\rho}{d\xi^3} + E\frac{d\rho}{d\xi} + \frac{d}{d\xi}\left(\frac{3}{2}\beta\rho^2 + 2\gamma\rho^3\right) = 0 \tag{3.33}$$

which integrated twice becomes

$$\frac{1}{4}\left(\frac{d\rho}{d\xi}\right)^2 = P_4(\rho),\ P_4(\rho) = -\gamma\rho^4 - \beta\rho^3 - E\rho^2 + A\rho + B \tag{3.34}$$

Here E, A, B are arbitrary, real constants. As in the previous section we consider the cases $\beta = \pm 1$. In integrating (3.34) one has to choose positive solutions ρ and those domains of ρ where $P_4(\rho) > 0$ because only a real, finite and positively defined fluid density has meaning in physics. These restrictions imply that $P_4(\rho)$ should have at least two real, positive, distinct solutions (if the polynomial has positive values only on part of the interval determined by two of its roots, the density $\rho(\xi)$ is positive only on restricted intervals of ξ). The are no restrictions on the other two but when they are complex, the one is the conjugate of the other because the polynomial may take only real values.

Further on we shall discuss the influence of the sign of γ.

A. *Case* $\gamma > 0$

Let us consider that the polynomial $P_4(\rho)$ has four distinct, real roots $\rho_1 > \rho_2 > \rho_3 > \rho_4$ and two of them are positive $\rho_1 > \rho_2 > 0$. The physical conditions are met if $\rho \in [\rho_2, \rho_1]$. Then the solution of (3.34) is ([9], p. 124)

$$\begin{aligned}
&\int_\rho^{\rho_1} \frac{dt}{\sqrt{(\rho_1 - t)(t-\rho_2)(t-\rho_3)(t-\rho_4)}} = gu = 2\sqrt{\gamma}\xi \\
&\alpha^2 \operatorname{sn}^2 u = \frac{\rho_1 - \rho}{\rho - \rho_4}, \quad \rho = \frac{\rho_1 + \rho_4 \alpha^2 \operatorname{sn}^2 u}{1 + \alpha^2 \operatorname{sn}^2 u}, \quad u = \frac{2\sqrt{\gamma}}{g}\xi \\
&k^2 = \alpha^2 \frac{\rho_3 - \rho_4}{\rho_1 - \rho_3}, \quad \alpha^2 = \frac{\rho_1 - \rho_2}{\rho_2 - \rho_4} > 0, \quad g = \frac{2}{\sqrt{(\rho_1 - \rho_3)(\rho_2 - \rho_4)}}.
\end{aligned} \tag{3.35}$$

In the limit case $\rho_2 = \rho_3$, $k^2 = 1$, the solution becomes

$$\rho = \frac{\rho_1 + \rho_4 \alpha^2 \tanh^2 u}{1 + \alpha^2 \tanh^2 u} \tag{3.36}$$

describing a shifted bright solitary wave ($\rho(0) = \rho_1$, $\rho(\infty) = \rho_2$).

For constant velocity, the additional condition $B = 0$ has to be imposed and this can be satisfied if one of the roots ρ_2 or ρ_3 is equal to zero (the case with $\rho_4 = 0$ is discussed when all roots are positive). If either $\rho_2 = 0$ or $\rho_3 = 0$, the periodic solution (3.35) maintains its form but with different parameters (g, α^2, k^2). In the limit case $\rho_2 = \rho_3 = 0$ one has $\rho_4 < 0 < \rho_1$. Then $|\rho_4|\alpha^2 = \rho_1$, $g = 2/\sqrt{\rho_1|\rho_4|}$ and the solution (3.36) is the bright solitary wave

$$\rho = \frac{1}{1 + \alpha^2 \tanh^2 u} \frac{\rho_1}{\cosh^2 u}, \quad u = \sqrt{\gamma \rho_1 |\rho_4|}\xi. \tag{3.37}$$

This solution exists regardless of the sign of $\beta = \pm 1$. However, for $\beta = +1$ it can be compared to the bright soliton (3.25) of the cubic NLS equation. Let us assume that ρ_1, the positive root, is the same for both polynomials $P_3(\rho)$ and $P_4(\rho)$. Taking into account the relation between the roots and the coefficients of a polynomial, one finds for $P_4(\rho)$ that $|\rho_4| = \rho_1 + 1/\gamma$. Then the domain of the variable u_c in (3.25) is boosted for u_{c+q} in (3.35), $u_{c+q} = \sqrt{\gamma|\rho_4|}u_c$ and consequently the bright solitary solution (3.37) is much steeper than the bright soliton (3.25). In the figure III.1 the two solutions are represented for $\rho_1 = 1$, $\gamma = 1/2$ ($|\rho_4| = 3$, $\alpha^2 = 1/3$). As the phase is concerned, it is given in both cases by the expression (3.18), but with different values of the constant E, namely $E_c = -\rho_1$ for the cubic NLS equation and $E_{c+q} = -\gamma\rho_1|\rho_4|$ for the cubic + quintic nonlinearity. As a general rule the phase for the solution of the cubic + quintic NLS equation has a more rapid variation than for the cubic case.

If all the roots are positive (possible only if $\beta = -1$ when $\gamma > 0$), a second acceptable situation exists if $\rho \in [\rho_4, \rho_3]$ and the solution (see [9], p.

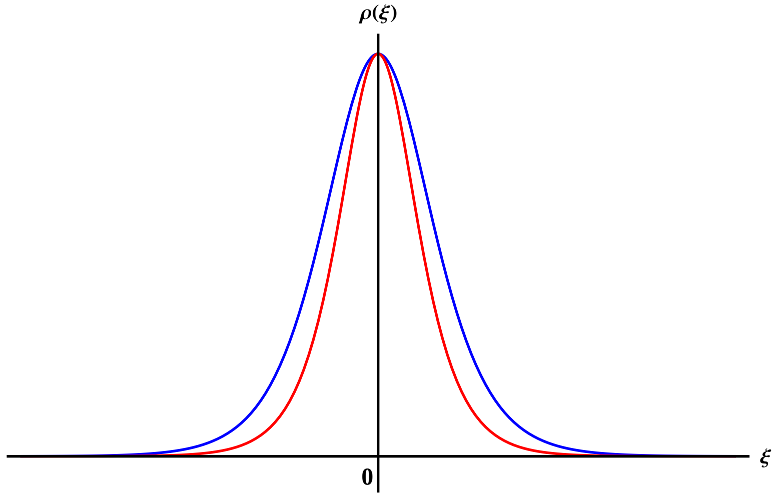

Figure III.1: Bright soliton solution for cubic NLS (blue) and cubic + quintic NLS (red).

103) is given by

$$
\begin{aligned}
&\int_{\rho_4}^{\rho} \frac{dt}{\sqrt{(\rho_1 - t)(\rho_2 - t)(\rho_3 - t)(t - \rho_4)}} = gu = 2\sqrt{\gamma}\xi \\
&\alpha^2 \operatorname{sn}^2 u = \frac{\rho - \rho_4}{\rho_1 - \rho}, \quad \rho = \frac{\rho_4 + \rho_1 \alpha^2 \operatorname{sn}^2 u}{1 + \alpha^2 \operatorname{sn}^2 u}, \quad u = \frac{2\sqrt{\gamma}}{g}\xi \\
&k^2 = \alpha^2 \frac{\rho_1 - \rho_2}{\rho_2 - \rho_4}, \quad \alpha^2 = \frac{\rho_3 - \rho_4}{\rho_1 - \rho_3} > 0, \quad g = \frac{2}{\sqrt{(\rho_1 - \rho_3)(\rho_2 - \rho_4)}}.
\end{aligned}
\tag{3.38}
$$

The limit case $k^2 = 1$ is obtained when $\rho_2 = \rho_3$, and the solution transforms into

$$\rho = \frac{\rho_4 + \rho_1 \alpha^2 \tanh^2 u}{1 + \alpha^2 \tanh^2 u}, \tag{3.39}$$

describing a gray solitary wave ($\rho(0) = \rho_4$, $\rho(\infty) = \rho_3$). In the case of constant velocity, the condition $B = 0$ can be realized only if $\rho_4 = 0$. Then the solution (3.39) becomes

$$\rho = \frac{\rho_1 \alpha^2 \tanh^2 u}{1 + \alpha^2 \tanh^2 u}, \quad \alpha^2 = \frac{\rho_2}{\rho_1 - \rho_2}, \quad u = \sqrt{\gamma \rho_2 (\rho_1 - \rho_2)}\xi, \tag{3.40}$$

describing a dark solitary wave.

The case when $P_4(\rho)$ has two positive roots $\rho_1 > \rho_2 > 0$ and the other two complex conjugated ($\rho_3 = b + ia$, $\rho_4 = \rho_3^* = b - ia$) represents another acceptable situation (if $b \leq \rho_2$ so that ρ is real). Then the solution of (3.34) is ([9], p. 133)

$$\int_{\rho_2}^{\rho} \frac{dt}{\sqrt{(\rho_1 - t)(t - \rho_2)\,[(t-b)^2 + a^2]}} = gu = 2\sqrt{\gamma}\xi$$

$$\operatorname{cn} u = \frac{(\rho_1 - \rho)B - (\rho - \rho_2)A}{(\rho_1 - \rho)B + (\rho - \rho_2)A} \Rightarrow \rho = \frac{(\rho_2 A + \rho_1 B) + (\rho_2 A - \rho_1 B)\operatorname{cn} u}{(A+B) + (A-B)\operatorname{cn} u}$$

$$k^2 = \frac{(\rho_1 - \rho_2)^2 - (A-B)^2}{4AB}, \quad g = \frac{1}{\sqrt{AB}}, \quad u = \frac{2\sqrt{\gamma}}{g}\xi \tag{3.41}$$

$$A^2 = (\rho_1 - b)^2 + a^2, \quad B^2 = (\rho_2 - b)^2 + a^2.$$

The limit case $k^2 = 1$ is attained when $\rho_1 - \rho_2 = A + B$ and $\rho_1 + \rho_2 - 2b = A - B$. Then $A = \rho_1 - b$, $B = -(\rho_2 - b)$, $\rho_2 A + \rho_1 B = b(\rho_1 - \rho_2)$ and $\rho_2 A - \rho_1 B = 2\rho_1\rho_2 - b(\rho_1 + \rho_2)$. As $\lim\limits_{k^2 \to 1} \operatorname{cn} u = \operatorname{sech} u$, the solution (3.41) writes

$$\rho = \frac{b(\rho_1 - \rho_2) + [2\rho_1\rho_2 - b(\rho_1 + \rho_2)]\operatorname{sech} u}{(\rho_1 - \rho_2) + (\rho_1 + \rho_2 - 2b)\operatorname{sech} u}. \tag{3.42}$$

One has $\rho(0) = \rho_2$ and $\rho(\infty)b < \rho_2$ so that (3.42) describes a shifted bright solitary wave. For $b = 0$ it becomes a bright solitary wave

$$\rho = \frac{2\rho_1\rho_2}{(\rho_1 + \rho_2) + (\rho_1 - \rho_2)\cosh u}. \tag{3.43}$$

When periodic solutions with constant profile are sought, the phase $\theta(x,t) = \theta(\xi)$ is calculated using equation (3.20) and the expression of $\rho(\xi)$ from (3.35) (let us restrict only to this case $\rho_1 > \rho_2 > 0$, $\rho_4 < 0$). The result is given in terms of an incomplete elliptic integral of the third kind

$$\theta(\xi) = \left(u_0 + \frac{A_0}{\rho_4}\right)\xi - A_0 \frac{g}{2\sqrt{\gamma}} \frac{\rho_1 - \rho_4}{\rho_1\rho_4} \Pi\left(\varphi, -\frac{\rho_4}{\rho_1}\alpha^2, k\right), \tag{3.44}$$

with $\sin\varphi = \operatorname{sn} u$ and $u = \frac{2\sqrt{\gamma}}{g}\xi$. In order to establish whether the incomplete elliptic integral of the third kind in (3.44) is in the circular or hyperbolic case, one needs to evaluate the sign of the expression $\kappa^2(\kappa^2 - k^2)(\kappa^2 - 1)$, where $\kappa^2 = -\frac{\rho_4}{\rho_1}\alpha^2$. As $0 < \kappa^2 < 1$ is always satisfied in the current conditions, the sign of the expression is given by the term

$$\operatorname{sgn}\,(\kappa^2 - k^2) = \begin{cases} +1 \text{ for } \rho_4 < \rho_3 < 0 \Rightarrow \text{circular case}(-) \\ -1 \text{ for } 0 < \rho_3 < \rho_2 \Rightarrow \text{hyperbolic case}(+) \end{cases}.$$

In the limit case $k^2 = 1$ the expression of the phase $\theta(\xi)$ becomes

$$\theta(\xi) = \left(u_0 - \frac{A_0}{|\rho_4|}\right)\xi + A_0 \frac{g}{2\sqrt{\gamma}} \frac{\rho_1 + |\rho_4|}{\rho_1|\rho_4|} \int_0^u \frac{dt}{1 - \eta^2\tanh^2 t}, \tag{3.45}$$

where we denoted $\eta^2 = \frac{|\rho_4|}{\rho_1}\alpha^2 \leq 1$ ($\eta^2 = 1 \to$ constant velocity case). The integral is easily calculated with the change of variable $x = \tanh t$

$$\begin{aligned}
&\int_0^u \frac{dt}{1-\eta^2\tanh^2 t} = \int_0^X \frac{dx}{1-x^2}\frac{1}{1-\eta^2 x^2}, \qquad X = \tanh u \leq 1 \\
&= \frac{1}{1-\eta^2}\left\{\frac{1}{2}\ln\frac{1+X}{1-X} - \eta\frac{1}{2}\ln\frac{1+\eta X}{1-\eta X}\right\} = \\
&= \frac{1}{1-\eta^2}\left[u - \eta\,\mathrm{artanh}(\eta\tanh u)\right].
\end{aligned} \tag{3.46}$$

For $u \to \pm\infty$, the second term is finite, namely $\pm\,\mathrm{artanh}\,\eta$.

B. *Case* $\gamma < 0$

In this situation the asymptotic behavior of the polynomial $P_4(\rho)$ is $P_4(-\infty) = \infty$, $P_4(\infty) = \infty$ and then the two conditions mentioned before ($\rho > 0$, $P_4(\rho) > 0$) can be satisfied only if $P_4(\rho)$ has at least three positive, distinct roots $\rho_1 > \rho_2 > \rho_3 > 0$ and $\rho \in [\rho_3, \rho_2]$. The solution ([9], p. 116) is

$$\begin{aligned}
&\int_\rho^{\rho_3} \frac{dt}{\sqrt{(\rho_1-t)(\rho_2-t)(t-\rho_3)(t-\rho_4)}} = gu = 2\sqrt{|\gamma|}\xi \\
&\alpha^2\,\mathrm{sn}^2\,u = \frac{\rho_2-\rho}{\rho_1-\rho}, \quad \rho = \frac{\rho_2-\rho_1\alpha^2\,\mathrm{sn}^2\,u}{1-\alpha^2\,\mathrm{sn}^2\,u}, \quad u = \frac{2\sqrt{|\gamma|}}{g}\xi \\
&k^2 = \alpha^2\frac{\rho_1-\rho_4}{\rho_2-\rho_4}, \quad \alpha^2 = \frac{\rho_2-\rho_3}{\rho_1-\rho_3}, \quad g = \frac{2}{\sqrt{(\rho_1-\rho_3)(\rho_2-\rho_4)}}.
\end{aligned} \tag{3.47}$$

The degenerate case ($k^2 = 1$) is obtained when $\rho_3 = \rho_4$ and the solution (3.47) becomes

$$\rho = \frac{\rho_2 - \rho_1\alpha^2\tanh^2 u}{1-\alpha^2\tanh^2 u}, \tag{3.48}$$

describing a shifted bright solitary wave ($\rho(0) = \rho_2$, $\rho(\infty) = \rho_3$).

For motion with constant velocity, the condition $B = 0$ implies that one of the two roots ρ_3 and ρ_4 is zero. Taking either $\rho_3 = 0$ or $\rho_4 = 0$ modifies the expressions of the parameters of the periodic solution (3.47). An interesting fact is that, expressing the polynomial coefficients in terms of its roots, one gets $\beta = |\gamma|(\rho_1 + \rho_2 + \rho_3)$ for $\rho_4 = 0$, therefore β can be only $+1$. On the other hand, for $\rho_3 = 0$, one has $\beta = |\gamma|(\rho_1 + \rho_2 + \rho_4)$ and the sign of β depends on the magnitude of $\rho_4 < 0$ so both ± 1 values are allowed. In the limit case $\rho_3 = \rho_4 = 0$ ($k^2 = 1$), $\beta = |\gamma|(\rho_1 + \rho_2) = +1$ and the solitary wave (3.48) transforms into a bright soliton solution

$$\rho = \frac{1}{1-\alpha^2\tanh^2 u}\frac{\rho_2}{\cosh^2 u}, \quad \alpha^2 = \frac{\rho_2}{\rho_1}, \quad u = \sqrt{|\gamma|\rho_2(\rho_1-\rho_2)}\xi. \tag{3.49}$$

The phase $\theta(x,t)$ is given by the formula (3.18) where, in the previous limit case ($\rho_3 = \rho_4 = 0$), $E = \gamma\rho_1\rho_2 < 0$.

For constant profile solutions, the phase $\theta(\xi)$ is computed introducing the expression (3.47) for $\rho(\xi)$ into (3.20) and integrating. One gets

$$\theta(\xi) = \left(u_0 + \frac{A_0}{\rho_1}\right)\xi + A_0\frac{g}{2\sqrt{|\gamma|}}\frac{\rho_1 - \rho_2}{\rho_1\rho_2}\Pi\left(\varphi, \frac{\rho_1}{\rho_2}\alpha^2, k\right), \qquad (3.50)$$

where $\sin\varphi = \operatorname{sn} u$ and $u = \frac{2\sqrt{|\gamma|}}{g}\xi$. Denoting $\sigma^2 = \frac{\rho_1}{\rho_2}\alpha^2$, the incomplete elliptic integral of third order in (3.50) will be in the circular or hyperbolic case depending on the sign of the expression (one always has $1 > \sigma^2 > 0$ in the present conditions)

$$\operatorname{sgn}\underbrace{\sigma^2}_{>0}(\sigma^2 - k^2)\underbrace{(\sigma^2 - 1)}_{<0\ (\rho_3>0)} = \begin{cases} -1 \text{ for } \rho_4 < 0 \Rightarrow \text{circular case} \\ +1 \text{ for } 0 < \rho_4 < \rho_3 \Rightarrow \text{hyperbolic case} \end{cases}.$$

In the limit case $k^2 = 1$ the expression (3.50) for the phase of the shifted bright soliton solution writes

$$\theta(\xi) = \left(u_0 + \frac{A_0}{\rho_1}\right)\xi + A_0\frac{g}{2\sqrt{|\gamma|}}\frac{\rho_1 - \rho_2}{\rho_1\rho_2}\int_0^u \frac{dt}{1 - \sigma^2\tanh^2 t}. \qquad (3.51)$$

The integral in (3.51) is the same as the one in (3.46) leading to the same result and asymptotic behavior with respect to the variable u.

III.3. Madelung Fluid Description of Derivative NLS Equations

The Madelung fluid description, used in the previous section, may also be employed in the study of another class of nonlinear partial differential equations, namely the class of derivative nonlinear Schrödinger type equations. The most general form of a derivative NLS equation (in 1+1 dimensions) is

$$i\alpha\frac{\partial\Psi}{\partial t} + \beta\frac{\partial^2\Psi}{\partial x^2} = f\left(\Psi, \Psi^*, \frac{\partial\Psi}{\partial x}, \frac{\partial\Psi^*}{\partial x}\right),$$

where f is an analytic function of Ψ, its spatial derivative(s) and their complex conjugates which must contain at least a term of the form $\Psi|\Psi|^2$ (the second derivative of f vanishes when one takes $u = 0$).

In the present paper we'll consider two forms of the equations from this class, the generalized derivative NLS equations of the first kind (denoted by gdNLS-1)

$$i\alpha\frac{\partial\Psi}{\partial t} + \frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} + i\beta\frac{\partial}{\partial x}\left(U(|\Psi|^2)\Psi\right) = 0 \qquad (3.52)$$

and the generalized derivative NLS equations of the second kind (denoted gdNLS-2)

$$i\alpha\frac{\partial\Psi}{\partial t} + \frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} + i\beta U(|\Psi|^2)\frac{\partial\Psi}{\partial x} = 0. \qquad (3.53)$$

When $U = |\Psi|^2$ they become the completely integrable cubic derivative NLS equations (denoted by dNLS-1 and dNLS-2 respectively)

$$i\alpha \frac{\partial \Psi}{\partial t} + \frac{\alpha^2}{2} \frac{\partial^2 \Psi}{\partial x^2} + i\beta \frac{\partial}{\partial x} \left(|\Psi|^2 \Psi \right) = 0 \tag{3.54}$$

and

$$i\alpha \frac{\partial \Psi}{\partial t} + \frac{\alpha^2}{2} \frac{\partial^2 \Psi}{\partial x^2} + i\beta |\Psi|^2 \frac{\partial \Psi}{\partial x} = 0. \tag{3.55}$$

The equation (3.54), especially, is encountered in several problems in physics. In plasma physics it describes the evolution of small amplitude Alfvén waves propagating quasi-parallel to a constant magnetic field in a low-β plasma [28, 37–40] (the parameter β of a plasma denotes the ratio between kinetic and magnetic pressure) and also the behavior of large amplitude magneto-hydrodynamic waves propagating in an arbitrary direction in a high β-plasma [46, 47]. Nonlinear Alfvén waves are important for particle acceleration in solar corona and plasma heating in tokamaks and other laboratory plasma [8, 11–13, 18, 43].

In nonlinear optics, as the duration of the quasi-monochromatic light pulses decreases to pico-seconds range, the Kerr nonlinearity must be supplemented with higher order or derivative terms in order to correctly describe their propagation through the weakly nonlinear wave guides [5,15,50]. Thus, dNLS-1 arises as good approximation of the usual nonlinear Schrödinger equation. Yet, as the pulse duration decreases even lower into the femto-second domain, it was recently found [3, 4, 14, 48] that the integrable Short Pulse Equation becomes the correct evolution equation.

The dNLS-1 equation (3.54) is a completely integrable system, solved using the Inverse Scattering Transform method by Kaup and Newell [33] for vanishing boundary conditions and by Kawata and Inoue [34] for non-vanishing conditions [10]. Many other methods were used to find N-soliton solutions of the derivative NLS equations of which we mention Hirota's bilinear formalism [41], the Darboux transformation technique [27, 52] or the Bäcklund transformations approach [49].

In the following, let us return to the Madelung fluid description of the equations (3.52) and (3.53) that will be used to derive periodic solutions for the gdNLS-type equations which, in the integrable case (dNLS-1), will be compared to the known solutions [32].

According to the Madelung fluid approach, introducing

$$\Psi(x,t) = \sqrt{\rho(x,t)} \exp \left[\frac{i}{\alpha} \theta(x,t) \right]$$

into (3.52) and (3.53) and separating the real and imaginary part, one obtains the continuity equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left(\rho v + \frac{\beta}{\alpha} G(\rho) \right) = 0 \tag{3.56}$$

for the fluid density ρ from the imaginary part and the equation of motion for the fluid velocity ($v(x,t) = \frac{\partial\theta(x,t)}{\partial x}$)

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) v = \frac{\alpha^2}{2}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}\right) - \frac{\beta}{\alpha}\frac{\partial}{\partial x}\left[vU(\rho)\right] \tag{3.57}$$

from the real part. The latter has the same form for both types of derivative nonlinearity. In the equations (3.56), $G(\rho)$ is defined by

$$\frac{\mathrm{d}G}{\mathrm{d}\rho} = U + 2\rho\frac{\mathrm{d}U}{\mathrm{d}\rho} \tag{3.58}$$

when the gdNLS-1 (3.52) is considered and by

$$\frac{\mathrm{d}G}{\mathrm{d}\rho} = U \tag{3.59}$$

for the gdNLS-2 (3.53) equation.

For the derivative nonlinearities, the method described in §III.4 applied to the system (3.56), (3.57) yields the following integro-differential equation

$$\begin{aligned} -\rho\frac{\partial v}{\partial t} + v\frac{\partial \rho}{\partial t} + 2\left[C_0(t) - \int\left(\frac{\partial v}{\partial t}\right)\,\mathrm{d}x\right]\frac{\partial\rho}{\partial x} - \\ -\frac{\beta}{\alpha}\rho U\frac{\partial v}{\partial x} + \frac{\beta}{\alpha}v\frac{\partial\rho}{\partial x}\left(-U \pm \rho\frac{\mathrm{d}U}{\mathrm{d}\rho}\right) + \frac{\alpha^2}{4}\frac{\partial^3\rho}{\partial x^3} = 0, \end{aligned} \tag{3.60}$$

where the sign (+) is for gdNLS-1 and (-) for gdNLS-2, respectively.

When considering the constant velocity case ($v = v_0$) the first notable difference from the results for the generalized nonlinear Schrödinger equation appears. Namely, while the gNLS equations have solutions (periodic, solitary – bright, dark solitons) in this case, for the gdNLS class the continuity equation (3.56) is dispersionless

$$\frac{\partial\rho}{\partial t} + \left(v_0 + \frac{\beta}{\alpha}\frac{\mathrm{d}G}{\mathrm{d}\rho}\right)\frac{\partial\rho}{\partial x} = 0, \tag{3.61}$$

having the implicit solution

$$\rho(x,t) = f\left[x - \left(v_0 + \frac{\beta}{\alpha}\frac{\mathrm{d}G}{\mathrm{d}\rho}\right)t\right], \tag{3.62}$$

where $f(x) = \rho(x, t = 0)$ is just the initial condition. This is incompatible with the nonlinear, dispersive equation (3.60), therefore the dNLS-type equations do not posses solutions for the motion with constant velocity.

In the second simplifying case, the stationary profile current velocity, we have both $\rho(x,t)$ and $v(x,t)$ depending only on $\xi = x - u_0 t$. Straight-forward integration of the continuity equation (3.56) gives

$$v = u_0 + \frac{A_0}{\rho} - \frac{\beta}{\alpha}\frac{G(\rho)}{\rho}, \tag{3.63}$$

where A_0 is an integration constant. Considering $U(\rho)$ a power function of ρ as in the previous section of this chapter, $U(\rho) = \rho^p$, the expression of $G(\rho)$ obtained from (3.58) and (3.59) writes

$$G(\rho) = \frac{2p+1}{p+1}\rho^{p+1} \text{ for gdNLS-1,} \tag{3.64}$$

$$G(\rho) = \frac{1}{p+1}\rho^{p+1} \text{ for gdNLS-2,} \tag{3.65}$$

respectively. Denoting by $G_1(\rho)$ the function $G(\rho)$ for gdNLS-1 given in (3.64) and by $G_2(\rho)$ the function defined in (3.65) for gdNLS-2, it is easily seen that one may write $G_{1,2}(\rho) = \frac{\gamma}{p+1}\rho^{p+1}$, where $\gamma = 2p+1$ for gdNLS-1 and $\gamma = 1$ for gdNLS-2.

III.3.1. Solitary Solutions of dNLS Equations

Under the previous assumptions about the form of $U(\rho)$, we are looking for localized solutions (solitary solutions) which have to satisfy the boundary conditions $\lim_{\xi\to\pm\infty} \rho(\xi) = 0$ in order to be valid for physics problems. Therefore it is required that in (3.63) one takes $A_0 = 0$ and $p > 0$. Then the equation (3.60) transforms into an ordinary differential equation

$$\frac{\alpha^2}{4}\frac{\mathrm{d}^3\rho}{\mathrm{d}\xi^3} + \left(2C_0 + u_0^2\right)\frac{\mathrm{d}\rho}{\mathrm{d}\xi} - u_0\frac{\beta}{\alpha}(p+2)\rho^p\frac{\mathrm{d}\rho}{\mathrm{d}\xi} + \left(\frac{\beta}{\alpha}\right)^2\frac{2p+1}{p+1}\rho^{2p}\frac{\mathrm{d}\rho}{\mathrm{d}\xi} = 0, \tag{3.66}$$

the same regardless the kind of derivative nonlinear Schrödinger equations that was initially considered (gdNLS-1 or gdNLS-2). Integrating twice with respect to ξ and taking into account that ρ and its first and second derivatives vanish when $|\xi| \to \infty$, one gets

$$\frac{\alpha^2}{4}\left(\frac{\mathrm{d}\rho}{\mathrm{d}\xi}\right)^2 = -\rho^2\left[\left(\frac{\beta}{\alpha}\right)^2\frac{\rho^{2p}}{(p+1)^2} - 2u_0\frac{\beta}{\alpha}\frac{\rho^p}{p+1} + (u_0^2 + 2C_0)\right]. \tag{3.67}$$

With the change of variable $z = \rho^{-p}$, the equation (3.67) transforms into

$$\frac{\alpha^2}{4p^2}\left(\frac{\mathrm{d}z}{\mathrm{d}\xi}\right)^2 = -(u_0^2 + 2C_0)z^2 + 2u_0\frac{\beta}{\alpha}\frac{1}{p+1}z - \left(\frac{\beta}{\alpha}\right)^2\frac{1}{(p+1)^2}. \tag{3.68}$$

The discriminant of the second order polynomial in the right hand side of (3.68) is

$$\Delta = -\frac{8C_0}{(p+1)^2}\left(\frac{\beta}{\alpha}\right)^2, \tag{3.69}$$

thus it will have real roots when $C_0 < 0$ and complex conjugated roots for positive values of C_0. For the case with complex roots ($C_0 > 0$) the polynomial is always negative because $u_0^2 + 2C_0 > 0$. The situation does not present interest since valid solutions for physics problem correspond to positive domains on the z-axis ($\rho > 0$) containing the point at infinity where the second order polynomial ($P_2(z)$) is also positive.

For $C_0 < 0$, the expression of the real roots (z_1, z_2, $z_2 > z_1$) is given by

$$z_{1,2} = \left|\frac{\beta}{\alpha}\right| \frac{1}{p+1} \frac{u_0 \operatorname{sgn}(\alpha\beta) \mp \sqrt{2|C_0|}}{u_0^2 - 2|C_0|}. \tag{3.70}$$

If $u_0^2 - 2|C_0| > 0$, the second order polynomial is asymptotically negative. Both its roots are either negative or positive as the coefficients α and β have opposite ($\alpha\beta < 0$) or the same sign ($\alpha\beta > 0$), respectively. In the latter case, the previous conditions are met if $z \in [z_1, z_2]$, but it is easily seen that this situation is not physically interesting because the domain does not contain the point $z \to \infty$ (unless $2|C_0| = u_0^2$).

When $u_0^2 - 2|C_0| = -b^2 < 0$ ($C_0 < -\frac{1}{2}u_0^2$), regardless of the sign of $\alpha\beta$, the polynomial has one negative (z_1) and one positive root (z_2), therefore the region of interest will be the domain (z_2, ∞). Denoting $\eta = \frac{2p}{|\alpha|}b$, the equation (3.68) writes

$$\frac{\mathrm{d}z}{\mathrm{d}\xi} = \eta\sqrt{(z - z_1)(z - z_2)}. \tag{3.71}$$

Integrating (3.71), one obtains

$$z(\xi) = z_m + z_M \cosh \eta\xi, \tag{3.72}$$

where

$$z_m = \frac{z_1 + z_2}{2} = \left|\frac{\beta}{\alpha}\right| \frac{1}{p+1} \frac{u_0 \operatorname{sgn}(\alpha\beta)}{u_0^2 - 2|C_0|},\ z_M = \frac{z_2 - z_1}{2} = \left|\frac{\beta}{\alpha}\right| \frac{1}{p+1} \frac{\sqrt{2|C_0|}}{u_0^2 - 2|C_0|}.$$

Then

$$\rho(\xi) = (z_m + z_M \cosh \eta\xi)^{-\frac{1}{p}}. \tag{3.73}$$

When $p = 1$, the gdNLS-1 becomes the completely integrable equation dNLS-1 and (3.66) transforms into the stationary Gardner's equation

$$\frac{\alpha^2}{4}\frac{\mathrm{d}^3\rho}{\mathrm{d}\xi} + \left(2C_0 + u_0^2\right)\frac{\mathrm{d}\rho}{\mathrm{d}\xi} - 3u_0\frac{\beta}{\alpha}\rho\frac{\mathrm{d}\rho}{\mathrm{d}\xi} + \frac{3}{2}\left(\frac{\beta}{\alpha}\right)^2\rho^2\frac{\mathrm{d}\rho}{\mathrm{d}\xi} = 0, \tag{3.74}$$

which has the solution

$$\rho(\xi) = \frac{1}{z_m + z_M \cosh \eta\xi}. \tag{3.75}$$

Here η, z_m and z_M are the values of the previously defined parameters corresponding to $p = 1$.

The phase $\theta(\xi)$ is calculated by introducing the expression (3.73) of $\rho(\xi)$ into the equation (3.63) (for solitary solutions one takes $A_0 = 0$, $p > 0$) and integrating. Then the equation (3.63) writes (for $p = 1$)

$$v = \frac{\mathrm{d}\theta}{\mathrm{d}\xi} = u_0 - \frac{\beta}{\alpha}\frac{\gamma}{p+1}\frac{1}{z_m + z_M \cosh \eta\xi}, \tag{3.76}$$

and the phase is

$$\theta(x,t) = u_0\xi - \frac{\beta}{\alpha}\frac{\gamma}{p+1}\frac{2}{\eta z_M\sqrt{1-a^2}}\arctan\left[\sqrt{\frac{1-a}{1+a}}\tanh\frac{\eta}{2}\xi\right] - 2C_0 t - \theta_0, \tag{3.77}$$

where a denotes

$$a = \frac{z_m}{z_M} = \frac{u_0 \operatorname{sgn}\alpha\beta}{\sqrt{2|C_0|}}, \quad |a| < 1$$

and θ_0 is the initial phase (an integration constant).

These results for the dNLS-1 equation have the same form as the ones obtained by Kaup and Newell utilizing the Inverse Scattering Transform (IST) method – see [26]. Expressions with similar form as (3.75) were also found by Mjølhus [40] as the one parameter soliton solutions that model the change of form of MHD waves propagating at small angle to the ambient magnetic field in a plasma. More recently, they were confirmed by Nakamura and Chen [41] using the bilinear transform method, Huang and Chen [27] and Xiao [52] using Darboux transformations or Steudel [49] utilizing Bäcklund transformations to derive multi-soliton solutions of the derivative nonlinear Schrödinger equation (3.54).

III.3.2. Periodic Solutions of dNLS Equations

Besides the solitary solutions, the completely integrable equations have another interesting class of periodic solutions. The problem of finding periodic solutions is well-known and implies using completely different methods (for a review see [16]) such as the "finite-band" integration method. The latter was adopted by Kamchatnov who used it, under assumptions that reduce the complexity of the results, to obtain periodic solution for a number of important equations, dNLS-1 included [29–32]. In the following, periodic solutions of a similar form will be obtained for dNLS-1 using the method of Madelung fluid description.

Starting from the equation (3.60) with $U(\rho)$ a power function of ρ as before but considering the less restrictive case $A_0 \neq 0$, one gets the ordinary differential equation

$$\frac{\alpha^2}{4}\frac{\mathrm{d}^3\rho}{\mathrm{d}\xi^3} + \left(2C_0 + u_0^2\right)\frac{\mathrm{d}\rho}{\mathrm{d}\xi} \pm A_0 p\frac{\beta}{\alpha}\rho^{p-1}\frac{\mathrm{d}\rho}{\mathrm{d}\xi} - u_0\frac{\beta}{\alpha}(\gamma+1\mp p)\rho^p\frac{\mathrm{d}\rho}{\mathrm{d}\xi} + \left(\frac{\beta}{\alpha}\right)^2\frac{\gamma}{p+1}\rho^{2p}(p+1\mp p)\frac{\mathrm{d}\rho}{\mathrm{d}\xi} = 0 \tag{3.78}$$

in the general case, where the upper sign corresponds to the gdNLS-1 class and the lower to the gdNLS-2 one. In the particular case when $p = 1$ that corresponds to dNLS-1 and dNLS-2, this equation becomes the Gardner's equation

$$\frac{\alpha^2}{4}\frac{\mathrm{d}^3\rho}{\mathrm{d}\xi^3} + \left(2C_0 + u_0^2 \pm \frac{\beta}{\alpha}A_0\right)\frac{\mathrm{d}\rho}{\mathrm{d}\xi} - 3u_0\frac{\beta}{\alpha}\rho\frac{\mathrm{d}\rho}{\mathrm{d}\xi} + \frac{3}{2}\left(\frac{\beta}{\alpha}\right)^2\rho^2\frac{\mathrm{d}\rho}{\mathrm{d}\xi} = 0. \tag{3.79}$$

Integrating the equation (3.79) twice with respect to ξ, one obtains

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}\xi}\right)^2 = -\left(\frac{\beta}{\alpha^2}\right)^2 P_4(\rho), \tag{3.80}$$

where $P_4(\rho)$ is a fourth order polynomial of ρ

$$P_4(\rho) = \rho^4 - 4u_0\frac{\alpha}{\beta}\rho^3 + 4\left(\frac{\alpha}{\beta}\right)^2\left(2C_0 + u_0^2 \pm \frac{\beta}{\alpha}A_0\right)\rho^2 + B\rho + C, \tag{3.81}$$

with B and C arbitrary integration constants. The solutions that are valid for physics problems correspond to domains where $\rho(\xi)$ is finite and positively defined and also the right hand side of (3.80) is positive, that is the polynomial $P_4(\rho) < 0$. Denoting the four roots of $P_4(\rho)$ by $\rho_{1,\ldots,4}$, the previous conditions translate that the polynomial must have at least two real, positive roots and its values must be negative in the interval determined by these roots. In the following, when all the roots are real, one assumes that they are ordered so that $\rho_1 > \rho_2 > \rho_3 > \rho_4$. Since, asymptotically, the polynomial $P_4(\rho)$ is positive only the cases listed below will yield valid solutions.

When all four roots are real, the conditions are met if at least two of them are positive, $\rho_1 > \rho_2 > 0$, and $\rho \in [\rho_2, \rho_1)$. The solution of (3.80) is given by

$$\int_{\rho_2}^{\rho} \frac{\mathrm{d}t}{\sqrt{(t-\rho_4)(t-\rho_3)(t-\rho_2)(\rho_1-t)}} = \frac{|\beta|}{\alpha^2}\xi. \tag{3.82}$$

The expression of $\rho(\xi)$ is ([9], formula 256.00)

$$\begin{aligned}
&\operatorname{sn}^2 u = \frac{(\rho_1-\rho_3)(\rho-\rho_2)}{(\rho_1-\rho_2)(\rho-\rho_3)}, \quad \rho(\xi) = \frac{\rho_2 - \rho_3\mu^2\operatorname{sn}^2 u}{1-\mu^2\operatorname{sn}^2 u},\\
&\mu^2 = \frac{\rho_1-\rho_2}{\rho_1-\rho_3}, \quad k^2 = \frac{(\rho_1-\rho_2)(\rho_3-\rho_4)}{(\rho_1-\rho_3)(\rho_2-\rho_4)}, \quad k^2 < \mu^2 < 1,\\
&g = \frac{2}{\sqrt{(\rho_1-\rho_3)(\rho_2-\rho_4)}}, \quad u = \frac{|\beta|}{g\alpha^2}\xi
\end{aligned} \tag{3.83}$$

in terms of Jacobi elliptical functions. It is easily seen that $\rho(\xi)$ is a periodic function with the period $2\mathrm{K}(k)$, where $\mathrm{K}(k)$ is the complete elliptic integral of the first kind ($\rho(u=0) = \rho_2$, $\rho(u = \mathrm{K}(k)) = \rho_1$, $\rho(u = 2\mathrm{K}(k)) = \rho_2$).

If all four roots are real and positive, besides the previous solution, another interesting situation appears for $\rho \in [\rho_4, \rho_3)$ ($\rho_3 > \rho_4 > 0$). The integration of (3.80) leads to

$$\int_{\rho_4}^{\rho} \frac{\mathrm{d}t}{\sqrt{(\rho_1-t)(\rho_2-t)(\rho_3-t)(t-\rho_4)}} = \frac{|\beta|}{\alpha^2}\xi. \tag{3.84}$$

Then the solution is ([9], formula 252.00)

$$\operatorname{sn}^2 u = \frac{(\rho_1-\rho_3)(\rho-\rho_4)}{(\rho_3-\rho_4)(\rho_1-\rho)}, \quad \rho(\xi) = \frac{\rho_4-\rho_1\mu^2\operatorname{sn}^2 u}{1-\mu^2\operatorname{sn}^2 u},$$
$$\mu^2 = \frac{\rho_4-\rho_3}{\rho_1-\rho_3} < 0, \quad k^2 = \frac{(\rho_1-\rho_2)(\rho_3-\rho_4)}{(\rho_1-\rho_3)(\rho_2-\rho_4)} \tag{3.85}$$
$$g = \frac{2}{\sqrt{(\rho_1-\rho_3)(\rho_2-\rho_4)}}, \quad u = \frac{|\beta|}{g\alpha^2}\xi,$$

also a periodic function of ξ.

When two of the real roots coincide, $\rho_3=\rho_4$ and the other two are positive, the equation (3.80) has the form

$$\int_{\rho_2}^{\rho} \frac{\mathrm{d}t}{(t-\rho_3)\sqrt{(t-\rho_2)(\rho_1-t)}} = \frac{|\beta|}{\alpha^2}\xi. \tag{3.86}$$

With a change of variable, the integration in the left hand side is performed directly, see [2] (p.13, formula 3.3.36), and the solution writes

$$\rho(\xi) = \rho_3 + \frac{2ab}{(a+b)+(a-b)\cos(\eta\xi)}, \tag{3.87}$$

where $a=\rho_1-\rho_3$, $b=\rho_2-\rho_3$ and $\eta = \frac{|\beta|}{\alpha^2}\sqrt{(\rho_1-\rho_3)(\rho_2-\rho_3)}$. The period of the solution is $2\pi/\eta$.

The last situation, that yields valid physical solutions, is with two real, positive roots $\rho_1>\rho_2>0$ and the other two complex conjugated $\rho_3=c$, $\rho_4=c^*$, $c\in\mathbb{C}$. Then, from (3.80), one gets

$$\int_{\rho_2}^{\rho} \frac{\mathrm{d}t}{\sqrt{P_2(t)(t-\rho_2)(\rho_1-t)}} = \frac{|\beta|}{\alpha^2}\xi, \tag{3.88}$$

where $P_2(t)$ is a second order polynomial of t with the complex roots c and c^*. The solution $\rho(\xi)$ writes ([9], formula 259.00)

$$\operatorname{cn} u = \frac{(\rho_1-\rho)Q-(\rho-\rho_2)P}{(\rho_1-\rho)Q+(\rho-\rho_2)P}, \quad \rho(\xi) = \frac{P\rho_2+Q\rho_1+(P\rho_2-Q\rho_1)\operatorname{cn} u}{(P+Q)+(P-Q)\operatorname{cn} u}$$
$$P^2 = (\rho_1-b_1)^2+a_1^2,\ Q^2=(\rho_2-b_1)^2+a_1^2,\ a_1^2 = -\frac{1}{4}(c-c*)^2,\ b_1 = \frac{c+c^*}{2}$$
$$k^2 = \frac{(\rho_1-\rho_2)^2-(P-Q)^2}{4PQ}, \quad g = \frac{1}{\sqrt{PQ}}, \quad u = \sqrt{PQ}\frac{|\beta|}{\alpha^2}\xi. \tag{3.89}$$

It is a periodic function with the period $4\mathrm{K}(k)$ (in the variable $u(\xi)$).

The phase $\theta(x,t)=\theta(\xi)$ is found starting from the equation (3.63) for the dNLS family of equations

$$v(x,t) = \frac{\mathrm{d}\theta}{\mathrm{d}\xi} = u_0 + \frac{A_0}{\rho(\xi)} - \frac{\gamma}{2}\frac{\beta}{\alpha}\rho(\xi), \tag{3.90}$$

where $\gamma = 3$ for dNLS-1 and $\gamma = 1$ for dNLS-2. Introducing any of the expressions above for $\rho(\xi)$ into (3.90) and integrating with respect to ξ, one obtains the expression of the phase. For instance, considering ρ given by (3.83) and using formula 340.01 from [9], the phase has the following expression

$$\theta(\xi) = \left(\frac{|\beta|}{g\alpha^2}u_0 + \frac{A_0}{\rho_3} - \frac{\gamma}{2}\frac{\beta}{\alpha}\rho_3\right)\xi - \theta_0 - \\ - \frac{g\alpha^2}{\rho_3|\beta|}\frac{\rho_2-\rho_3}{\rho_2}\left[A_0\Pi(u, \frac{\rho_3}{\rho_2}\mu^2, k) + \frac{\beta\gamma\rho_2}{2\alpha}\Pi(u,\mu^2,k)\right]. \quad (3.91)$$

III.4. Appendix

In this section we retrace the steps taken in [23] (§3) for deducing equation (3.11) starting from the system of equations (3.8),(3.10), using the notations and quantities defined in this chapter.

Multiplying the equation (3.8) by v, we get the following expression

$$\rho\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)v = -v\frac{\partial\rho}{\partial t} - v^2\frac{\partial\rho}{\partial x} + \rho\frac{\partial v}{\partial t}. \quad (3.92)$$

On the other hand, multiplying the equation (3.10) by ρ and utilizing the identity

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho^{1/2}}\frac{\partial^2\rho^{1/2}}{\partial x^2}\right) = \frac{1}{\rho}\left(\frac{1}{2}\frac{\partial^3\rho}{\partial x^3} - 4\frac{\partial\rho^{1/2}}{\partial x}\frac{\partial^2\rho^{1/2}}{\partial x^2}\right), \quad (3.93)$$

one obtains

$$\rho\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)v = \rho\frac{\partial U}{\partial x} + \frac{1}{4}\frac{\partial^3\rho}{\partial x^3} - 2\frac{\partial\rho^{1/2}}{\partial x}\frac{\partial^2\rho^{1/2}}{\partial x^2}. \quad (3.94)$$

The equation (3.94), combined with (3.92), leads to

$$-v\frac{\partial\rho}{\partial t} - v^2\frac{\partial\rho}{\partial x} + \rho\frac{\partial v}{\partial t} = \rho\frac{\partial U}{\partial x} + \frac{1}{4}\frac{\partial^3\rho}{\partial x^3} - 2\frac{\partial\rho^{1/2}}{\partial x}\frac{\partial^2\rho^{1/2}}{\partial x^2}. \quad (3.95)$$

Next, integrating the equation (3.10) with respect to x and multiplying the result by $\rho^{1/2}\left(\partial\rho^{1/2}/\partial x\right) = 1/2\partial\rho/\partial x$, one has

$$2\frac{\partial\rho^{1/2}}{\partial x}\frac{\partial^2\rho^{1/2}}{\partial x^2} = 2\frac{\partial\rho}{\partial x}\int\left(\frac{\partial v}{\partial t}\right)dx + v^2\frac{\partial\rho}{\partial x} + 2U\frac{\partial\rho}{\partial x} - 2C_0(t)\frac{\partial\rho}{\partial x} \quad (3.96)$$

where $C_0(t)$ is an arbitrary function of t (sort of an integration "constant"). Then combining (3.95) and (3.96) we get the evolution equation for the density $\rho(x,t)$ and velocity $v(x,t)$

$$-\rho\left(\frac{\partial v}{\partial t}\right) + v\frac{\partial\rho}{\partial t} + 2\left[C_0(t) - \int\left(\frac{\partial v}{\partial t}\right)dx\right]\frac{\partial\rho}{\partial x} - \left(\frac{\partial U}{\partial x}\rho + 2U\frac{\partial\rho}{\partial x}\right) + \frac{1}{4}\frac{\partial^3\rho}{\partial x^3} = 0. \quad (3.97)$$

III.5. Conclusions

In this chapter the hydrodynamic approach of Madelung was applied to generalized nonlinear Schrödinger equations containing higher order (quintic) and derivative nonlinearities. Particular cases of the following generalized NLS equation are considered [1]

$$i\frac{\partial\Psi}{\partial t}+\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2}+U(|\Psi|^2)\Psi+i\gamma_1\frac{\partial}{\partial x}\left(|\Psi|^2\Psi\right)+i\gamma_2|\Psi|^2\frac{\partial\Psi}{\partial x}=0. \quad (3.98)$$

When $\gamma_1=\gamma_2=0$, if $U(|\Psi|^2)=\beta|\Psi|^2$ the equation becomes the well-known, completely integrable, cubic NLS equation, while if $U(|\Psi|^2)=\beta_1|\Psi|^2+\beta_2|\Psi|^4$ the NLS equation with cubic and quintic nonlinear terms is obtained. Taking $U=0$, one recovers the completely integrable derivative NLS equations dNLS-1 and dNLS-2 if $\gamma_1\neq 0$, $\gamma_2=0$, respectively $\gamma_1=0$, $\gamma_2\neq 0$. More complex derivative nonlinearities were also discussed, having the form $i\gamma_1\frac{\partial}{\partial x}\left(U_1(|\Psi|^2)\Psi\right)$ and $i\gamma_2U_2(|\Psi|^2)\frac{\partial\Psi}{\partial x}$ with $U_i(|\Psi|^2)=|\Psi|^{2p}$ where p is an arbitrary positive number. These equations are considered by several authors to describe pulse propagation in weakly nonlinear media and therefore finding periodic and solitary wave solutions are of special interest.

In the Madelung's fluid description the field variable is written as

$$\Psi(x,t)=\sqrt{\rho(x,t)}e^{\frac{i}{\alpha}\theta(x,t)}. \quad (3.99)$$

Separating the real and the imaginary part, the equation (3.98) is equivalent with a coupled system of two equations, the first being a continuity equation for the fluid density $\rho=|\Psi|^2$ and the fluid velocity $v=\partial\theta(x,t)/\partial x$

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}\left(\rho v+\frac{\gamma_i}{\alpha}G(\rho)\right)=0, \quad (3.100)$$

where

$$\frac{\mathrm{d}G}{\mathrm{d}\rho}=\begin{cases}U_1+2\rho\frac{\mathrm{d}U_1}{\mathrm{d}\rho}, & \text{dNLS-1}\\ U_2, & \text{dNLS-2}\\ 0, & \text{NLS (c/c+q)}\end{cases} \quad (3.101)$$

and the second, an equation of motion for the fluid velocity $v(x,t)$. The latter equation, after a series of transformations – called by us "Fedele's transformations" – becomes

$$-\rho\frac{\partial v}{\partial t}+v\frac{\partial\rho}{\partial t}+2\left[C_0(t)-\int\frac{\partial v}{\partial t}dx\right]\frac{\partial\rho}{\partial x}+ \\ +\frac{\alpha^2}{4}\frac{\partial^3\rho}{\partial x^3}+\left(\rho\frac{dU}{d\rho}+2U\right)\frac{\partial\rho}{\partial x}=0, \quad (3.102)$$

[1] With respect to the rest of the chapter and in order to consider the different situations in the same scheme, slightly different notations for the coefficients are used.

when only cubic and cubic+quintic nonlinear terms are considered and

$$-\rho\frac{\partial v}{\partial t}+v\frac{\partial \rho}{\partial t}+2\left[C_0(t)-\int\left(\frac{\partial v}{\partial t}\right)\,\mathrm{d}x\right]\frac{\partial \rho}{\partial x}+ \\ +\frac{\alpha^2}{4}\frac{\partial^3\rho}{\partial x^3}-\frac{\gamma_i}{\alpha}\rho U_i\frac{\partial v}{\partial x}+\frac{\gamma_i}{\alpha}v\left(-U_i\pm\rho\frac{\mathrm{d}U_i}{\mathrm{d}\rho}\right)\frac{\partial \rho}{\partial x}=0, \quad (3.103)$$

when the nonlinearity is given only by the derivative terms. Although the equations (3.102), (3.103) have a quite complicate form, they present the advantage of being defined in real space and allow simple results to be obtained in at least two situations: (a) the case of constant velocity $v = v_0 = \text{const.}$ and (b) the case of motion with stationary profile current velocity, when both $\rho(x,t)$ and $v(x,t)$ are depending only on the variable $\xi = x - u_0 t$ (u_0 an arbitrary constant). In these two cases the equations reduce to generalized stationary Korteweg-de Vries equations.

As a first result, it was proved that NLS equations containing only derivative nonlinear terms have no solutions in the case of constant velocity. Indeed, if $v = v_0 = \text{const.}$, the integration of the continuity equation gives

$$\rho(x,t) = f\left[x-\left(v_0+\frac{\gamma_i}{\alpha}\frac{\mathrm{d}G_i}{\mathrm{d}\rho}\right)t\right], \quad (3.104)$$

where $f(x)$ is the initial condition. This shock wave solution is incompatible with the dispersive equation (3.103).

Starting from the equations (3.102) and (3.103) periodic solutions for the cubic, respectively the derivative NLSE classes were derived in terms of Jacobi elliptic functions. For instance, the equation (3.102) for the cubic + quintic NLSE, described by $U(\rho) = \beta_1\rho + \frac{3}{2}\beta_2\rho^2$, reduces to

$$\frac{1}{4\alpha^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}\xi}\right)^2 = P_4(\rho), \quad (3.105)$$

where $P_4(\rho)$ is a fourth order polynomial in ρ. In order to present interest for physics problems the solutions of this equation must be positive and more over the condition $P_4(\rho) > 0$ must also be satisfied in the domain they determine. The periodic solutions are listed in terms of the roots of the polynomial $P_4(\rho)$ (at least two of them are positive) and they are expressed through Jacobi elliptic functions. An example of such solution is given below for the case when the polynomial has four real roots $\rho_1 > \rho_2 > \rho_3 > \rho_4$ and the first two are positive $\rho_1 > \rho_2 > 0$ while $\beta_2 > 0$.

$$\rho = \frac{\rho_1+\rho_4\alpha^2\,\mathrm{sn}^2\,u}{1+\alpha^2\,\mathrm{sn}^2\,u}, \quad u = \frac{2\sqrt{\beta_2}}{g}\xi \\ k^2 = \alpha^2\frac{\rho_3-\rho_4}{\rho_1-\rho_3}, \quad \alpha^2 = \frac{\rho_1-\rho_2}{\rho_2-\rho_4} > 0, \quad g = \frac{2}{\sqrt{(\rho_1-\rho_3)(\rho_2-\rho_4)}}, \quad (3.106)$$

where $\mathrm{sn}^2(u,k)$ is the elliptic sn Jacobi function of variable u and modulus k. When $k^2 = 1$ ($\rho_2 = \rho_3$), sn $\to$ tanh and the previous solution becomes

$$\rho_{c+q} = \frac{\rho_1 + \rho_4 \alpha^2 \tanh^2 u}{1 + \alpha^2 \tanh^2 u} \tag{3.107}$$

describing a shifted bright solitary wave ($\rho_{c+q}(0) = \rho_1$, $\rho_{c+q}(\infty) = \rho_2$). For $\rho_2 = \rho_3 = 0$ it becomes a true bright solitary wave ($\rho_{c+q}(\infty) = 0$). This latter solitary solution was compared to the bright soliton $\rho_c(\xi)$ of the cubic NLS equation (3.25). $\rho_{c+q}(\xi)$ is a much steeper function than $\rho_c(\xi)$, this fact showing that the NLS equation with cubic and quintic nonlinearity describes in a better way the propagation of short light pulses in a nonlinear medium.

Unfortunately, explicit expressions for periodic solutions of NLS equations with higher order nonlinearities are not known though the solutions of (3.105) (for general expression of $U(\rho)$) are expected to be found in the class of hyperelliptic functions. Yet, a qualitative discussion can be done by writing the equivalent of equation (3.105) in the form

$$(\rho_\xi)^2 - P_{n+2}(\rho) = 0 \tag{3.108}$$

and considering ξ as a "time variable" and ρ as the "position variable". Thus the derivative of ρ with respect to ξ plays the role of a generalized momentum coordinate and then, in (3.108), $(\rho_\xi)^2$ is a kinetic energy while $P_{n+2}(\rho)$ is a potential energy. The trajectory in phase space (ρ, ξ) of a system described by these energies is a zero-energy surface. This "*potential representation*", introduced by Rosenau [44,45], can be used for a qualitative analysis of many nonlinear evolution equations [17]. Here closed trajectories in bounded regions of the phase space correspond to periodic solutions while solitary wave solutions are the separatrix trajectories in the phase space. Mathematical methods of classical mechanics [6] can thus be employed to treat such systems. This is a direction to be followed in the future for further investigating qualitatively the behavior of generalized NLS equations.

For the derivative NLS equations, a general solitary wave solution (asymptotically vanishing at $|\xi| \to \infty$) was obtained for an arbitrary $U(\rho) = \rho^p$ with p positive but not necessarily and integer, namely

$$\rho(\xi) = \frac{1}{(z_m + z_M \cosh A\xi)^{1/p}}, \tag{3.109}$$

where z_m, z_M are constants depending on p, γ_i/α, u_0 and $b^2 = -(u_0^2 + 2C_0) > 0$ (see (3.73)). For $p = 1$ it becomes the solution obtained by Kaup and Newell many years ago (1978) using the inverse scattering transform method.

Periodic solutions were computed only for the case when $p = 1$ (dNLS-1 and dNLS-2 equations) by integrating an equation of the form (3.105). They were expressed through Jacobi elliptic functions and presented depending on the roots of a fourth order polynomial $P_4(\rho)$ in a similar way as for the cubic

+ quintic NLS equation. For higher order nonlinearities $p > 1$, periodic solutions exist only when p is an integer and a qualitative analysis can be developed using the "potential representation" (3.108) mentioned before.

Another benefit of the Madelung's fluid description approach is that it leads to an interesting correspondence between generalized NLS and generalized KdV equations. This was discussed recently and it was proven to be true not only for multiplicative nonlinear terms $U(|\Psi|^2)$ but also for derivative ones and even for the cylindrical variant of NLS and KdV equations [19,21–24]. This unique correspondence allows one to determine a large class of periodic and solitary wave solutions of a generalized NLS equation starting from its corresponding generalized KdV equation.

The original results presented in the current chapter are published in:

1. "*Periodic and Solitary Wave Solutions of Generalized Nonlinear Schrödinger Equation Using a Madelung Fluid Description*", D. Grecu, R. Fedele, S. de Nicola, **A. T. Grecu**, A. Visinescu, *Rom. Journ. Phys.* **55**(9-10), 980–994 (2010).

2. "*Solitary Waves in a Madelung Fluid Description of Derivative NLS Equations*", D. Grecu, **A. T. Grecu**, A. Visinescu, R. Fedele, S. de Nicola, *J. Nonlinear Math. Phys.* **15**(Suppl. 3), 209–219 (2008).

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Chapter IV: Nonlinear Oscillation Modes in Dusty Plasma

The Universe around us is a complex system where numerous physical and chemical processes take place, their result being the fragmentation of macroscopic celestial bodies sometimes up to microscopical particles which we collectively call *dust*. At the same time, the dust is the main form of solid matter present in protoplanetary discs which form planetesimals (macroscopic bodies) through gravitational collapse. Therefore the dust is almost as ubiquitous in the cosmic space as the fourth state of matter, nowadays called *plasma*, which is believed to make up as much as 99% of the matter in the whole Universe. The dust grains immersed in interplanetar or interstellar plasma become charged depending on the characteristics of the plasma and its environment, thus creating a link between dust and plasma dynamics [1, 43].

The existence of plasma was guessed since ancient times as being the element of fire (other than earth, water and air). However, the fire flames are not strictly what we call nowadays *plasma* as they are composed of hot and incandescent nanometric particles of unburnt carbon (soot) and only the thermionic emission of electrons from these particulates (at around 1000°C) elevates the degree of ionization several orders of magnitude above the values given by Saha equations for air at this temperatures. The academic study of plasma began around 1929 when Tonks and Langmuir introduced the term to describe the core of a glowing ionized gas produced by electrical discharge in a tube. Yet, only recently the interplay between plasmas and charged dust grains has opened up the new and fascinating research area of dust-laden and dusty plasmas which grew exponentially from 1981 to 2004 [28]. The major boost to research in the field of dust-plasma interactions was initiated by the theoretical prediction of the *dust acoustic waves* by Rao *et al.* [40] (1990) which were observed by a large number of laboratory experiments since 1995 [6] (see also [45] and references therein). In 1992 the *dust ion-acoustic waves* were predicted by Shukla and Silin which manifest at larger frequencies than dust acoustic waves (tens of kHz versus tens of Hz or below). They were detected in laboratory experiments by Barkan *et al.* (1996) and Nakamura *et al.* (1999) and represent the second type of acoustic modes in uniform, unmagnetized, collisionless dusty plasmas with a weak Coulomb coupling between the charged dust grains. Even before these theoretical and experimental successes, in 1986 Ikezi [16] predicted theoretically the Coulomb crystallization of dust grains interacting via a repulsive Yukawa force in plasma when Coulomb energy density of the dust particles exceeds the thermal energy density by at least two levels of magnitude. These predictions were verified experimentally in 1994 by many scientists who reported observing *dusty plasma crystals* composed of ordered charged dust grains which form various crystalline structures depending on the experimental conditions. The phenomena of phase transition in dusty plasma crystals were also observed (Thomas and Morfill, 1996). Furthermore, in strongly coupled dusty plasmas, dust lattice waves appear analogous to those in solid-state physics. In this case the restoring force derives from the Debye-Hückel interaction between neighboring grains while the dust mass provides the inertia (Melandsø, 1996; Farokhi *et al.*, 1999; Wang *et al.*, 2001). Apart from the linear ones, the presence of the massive dust grains in plasma produces also nonlinear new collective phenomena on the specific space and time scales such as dust acoustic or dust ion-acoustic shock waves, dust acoustic Mach cones, dust microbubbles in dusty plasma liquids or dust vortical motions [42, 43].

IV.1. From Dust-Laden Plasma to Dusty Plasma

IV.1.1. Dusty Plasma in Nature and Laboratory

The presence of charged dust in nature manifests itself mostly in microgravity conditions through phenomena taking place at high altitudes in the Earth's atmosphere and in interplanetary and interstellar space. In Earth's atmosphere, clouds of charged dust are observed during the polar summer in the mesopause region at altitudes between 80 and 90 km and latitudes between 50° and 70° north and south of the equator. These clouds are called *noctilucent clouds* (clouds that glow at night, see figure IV.1) since they reflect solar rays after sunset (see figure IV.2). They were reported for the first time in 1885 after the eruption of the Krakatoa volcano. Though close to the plasma filled thermosphere, the noctilucent clouds produce specific electromagnetic phenomena such as polar mesospheric summer echoes (backscattering of radar at frequencies between 50 MHz and 1.3 GHz) which prove that the particulates in their composition are charged. The sources of dust which form these clouds are both natural, micrometeroids and soot from powerful volcanic eruptions, and man-made, rocket exhausts and industrial contamination [1, 43].

Figure IV.1: Some real-life pictures of noctilucent clouds (*left* – (c) 2007 Edwin van Dijk; *right* – Martin Koitmäe, July 2009)

Even closer to the Earth surface, dusty plasmas appear in ball lightnings

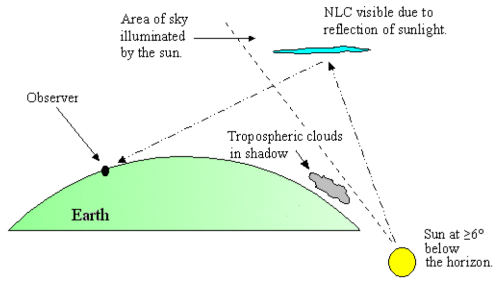

Figure IV.2: Reflection of solar rays by noctilucent clouds after sunset

which appear after the lightning strikes the soil and contains networks of nano particles [42].

As advocated by the Nobel Laureate Hannes Alfvén (1954) the Sun was initially surrounded by a nebula of dust which in time led to the appearance of the planets, comets, asteroids. Nowadays the interplanetary dust is responsible for the phenomenon called *zodiacal light*, a faint, roughly triangular, whitish glow seen in the night sky which appears before sunrise and after sunset and extends upwards from the vicinity of the sun along the ecliptic. The zodiacal light (see figure IV.3) was first investigated by the astronomer Giovanni Domenico Cassini in 1683 and its first explanation was given by Nicolas Fatio de Duillier a year later [1, 3, 43].

The interplanetary dust consists mainly from fragments of debris in comet tails and dust resulted from mutual collisions of asteroids in the asteroid belt and it is collected by planets and the Sun through accretion. Comets are small, irregularly shaped and fragile bodies composed of a mixture of frozen gases and nonvolatile particulates. They have highly elliptical orbits around the Sun and while they approach the star they develop a surrounding cloud of gas and grains carried away by the sublimation of the frozen gases, called *coma*, which grows in size and brightness. The remaining, boiling solid body (nucleus) and the coma make up the head of the comet. The material that is accelerated away from the comet by the solar radiation constitutes the bright tail of the comet. It becomes fluorescent as it absorbs ultraviolet radiation. The Sun's radiation is also responsible for starting processes on the nucleus of the comet that release hydrogen creating an envelope around the head of the comet. The heavy dust particles and the ionized gases in the comet tail are accelerated differently so that the dust tail becomes curved while the ion tail is much less massive and appears as a straight line pointing away from the Sun as in figure IV.4.

Micron- and submicron-sized dust grains are the main constituents of the ring systems around the outer giant planets in the Solar System (Jupiter, Saturn, Uranus, Neptune). The Jovian ring system is the third one discovered in the Solar system by Voyager 1 space probe in 1979. It is very faint and consists almost exclusively of dust, part of it originating from collisions between the four satellites with orbits within the rings, and various unob-

Figure IV.3: Zodiacal light with the Cancer constellation and Venus in background (*left*, Dominic Cantin, August, 2000); Zodiacal light seen from Very Large Telescope facility on Mount. Paranal, Chile (*right*, ESO, November 2009)

served macroscopic bodies. There are four regions: the halo near the planet which contains very fine dust, a very thin, bright main ring and two wide, thick and faint outer rings all of which contain dust grains with a radius about 15 μm. The ring system around Uranus was discovered in 1977 during observations of a stellar occultation by the planet. Though initially only 5 rings were seen, further Earth-based observations indicated there were actually 9. The Voyager 2 spacecraft photographed another two new rings in 1986, while the Hubble spatial telescope found two new outer rings so that as of 2008 the Uranian ring system contains a total of 13 rings with a radii range from 38,000 km to 98,000 km. Of these only two rings (λ and 1986U2Rζ) contain mainly micron-sized dust particles while various, temporary dust bands were observed between the rings of the system. Careful studies of the Voyager 2 images showed variations in the brightness of the λ rings which seem to be periodic and thus resembling the pattern of a standing wave. Though evidences of rings around Neptune were long known, it was believed that only incomplete arcs or debris orbited the planet until the discovery of the five rings in 1989 by the Voyager 2 mission. Images showed that the rings were complete but with bright clumps and one of them had even a complex twisted structure. Analysis by the Voyager 2 instruments

Figure IV.4: The Hale-Bopp comet passing by the North American Nebula; the curved dust tail and the ion tail (blue) are clearly visible. (c) NASA, 1997

proved the rings mainly consisted of micrometer-sized dust grains and intense broadband bursts of radio noise registered at each ring plane is thought to be produced by charged dust grains under the influence of the oddly oriented magnetic fields of Neptune. Last but the most extensive planetary ring system in the Solar system is that formed by the rings of Saturn. Discovered in 1610 by Galileo Galilei and described as a suite of rings surrounding the planet for the first time in 1655 by Cristiaan Huygens, the Saturnian ring system is made up of three major rings, known as C, B, A from outward direction, a faint, inner ring D and three more outer, narrow rings F, G, E. The rings are primarily composed of ice particle that range in size from micrometer to meters and form a series of very close rings with small gaps between them, the largest of which bearing the name of Giovanni Domenico Cassini (the first astronomer to issue this theory in 1675) and stretching between the ring A and B. The scientists' century long fascination with Saturn and its rings grew further as the two Voyager spacecrafts sent back, in 1980-1981, images of what look like frequent and rapidly changing, wedge-shaped spokes in ring B (figure IV.5, left). Along with other spectacular celestial mechanics phenomena the spokes are among the objects of interest for the Cassini-Huygens mission. The Cassini spacecraft reached Saturn's orbit in 2004 after being launched in 1997 and it is predicted to continue studying the Saturn's satellite and ring system till 2017. The spokes are confined to the dense ring B and are the effect of dust-plasma interactions in the planetary magnetosphere. The current model [12, 15, 31],though criticized [11], is based on the assumption that they are formed by electrostatically levitated charged dust-grains and their elongation is due to the rapid radial motion of dense plasma clouds created by meteor impacts on the ring [32] (possibly because of interaction between the later and Saturn's magnetosphere [26]).

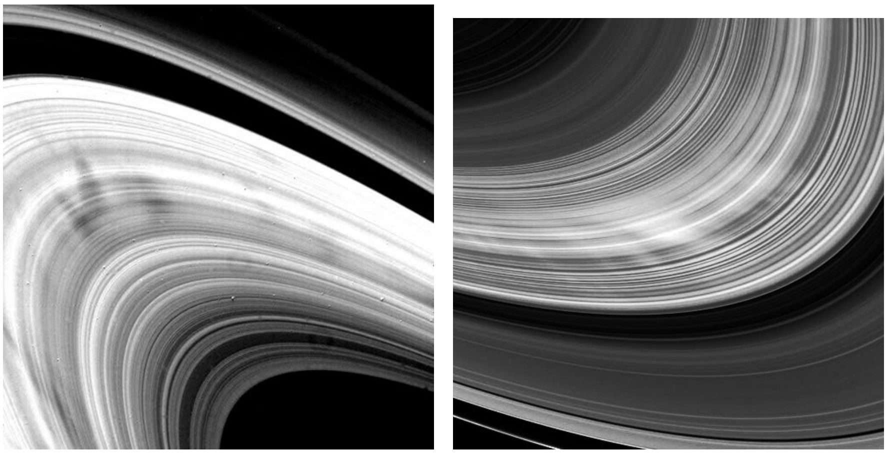

Figure IV.5: Images of nearly radial spokes in Saturn's ring B; dark spokes as viewed by Voyager 2 in 1981 (*left*) and bright one registered by Cassini in 2009 (*right*) (courtesy of NASA/JPL)

The interstellar space if full of gas and dust that form the interstellar medium. The ratio of gas to dust is 99:1 and the gas content is constantly diminishing as giant molecular clouds collapse gravitationally forming new stars. The densities are very low by Earth standards and both the gas and the dust particulates (usually submicronic) get charged because of various radiation (cosmic rays) from surrounding stars and other cosmic sources (remnants of supernovae, pulsars or neutron stars just to mention a few). The presence of dust in interstellar space has been known for a long time from star reddening, infrared emission or the existence of dark bands that block parts of various nebulae (Orion, Lagoon, Horsehead, Eagle, etc.). Dust-laden and dusty plasmas are therefore quite common in interstellar medium and they play a key role in the formation of dust clusters and structures that ultimately coalesce to form protostars and protoplanets [5]. The dust grains can be both dielectric (ices, silicates, carbonates, etc.) and metallic (graphite, magnetite, etc.) which explains why dusty and dust-laden plasma are considered responsible for such phenomena as instabilities of interstellar molecular clouds that lead to star formation or decoupling of magnetic fields from plasmas.

In laboratory conditions both dust-laden and dusty plasma are encountered some times as a source of contamination and instabilities and recently, more and more often as the object of studies. Dusty plasmas in laboratories differ significantly from those in nature since the discharges, in which they appear, are spatially more restricted (they have geometric boundaries) and have characteristics (structure, composition, temperatures, etc.) that strongly influence the formation and transport of the dust grains. At the same time, another difference comes from the installation that maintains and contains the dusty plasma as it imposes spatiotemporally varying boundary conditions on the dusty discharge [43].

Dust particles appear frequently in `dc` discharges but they become abundant under `rf` excitations. They mainly originate either from chemical reactions between the gases in the plasma or from sputtering of the electrodes. Their growth, charge, position and temperature depend on a myriad of factors and characteristics both external and internal to the experimental setup. The later usually consists of a low-pressure plasma processing reactor where dust grains produced in a discharge, are analyzed through laser light scattering and scanning electron microscopy(SEM). The low-voltage SEM allowed the resolving (without beam damage) of the surface texture of dust particulates which resemble cauliflowers [43].

It is long known that dust grains are present in fusion devices (tokamaks, spheramaks, stellaratons, etc.) yet only recently their influence on the plasma became the focus of different studies. The dust grains have size distributions as they are created through various processes like the desorption, arcing, sputtering, evaporation or sublimation of wall material or the spallation and flaking of thin films grown on the wall surface either for conditioning or gradually accumulated by discharge events. In the case of graphite wall components for instance, in addition to C atoms, clusters of C_n are liberated. Thus the size of dust particles varies from nanometers to a few micrometers while SEM images have shown shapes from flakes and metal cuttings to spheres of various radii or irregularly formed grains [41,43]. The nano-sized dust particles are usually in suspension and require about 2 hours of settlement. They are partly ferromagnetic and because of the existence of charged grains of opposite sign will form sub-micron agglomerates through coagulation which are not always closely packed but have rather a woolly open structure. However the nano-sized particles have the size and cauliflower-like structure which is consistent with those in processing plasmas. After a plasma discharge all dust particles will fall to the bottom of the tokamak chamber, however the lighter grains may be re-ejected into the fusion plasma either by magnetic effects or after electrostatic charging when they come in contact with the edge of the plasma. The material from which dust is created in tokamaks can also be radioactive, so radioactive, light dust particulates could play an important role in further understanding the transport processes in high density and low-temperature tokamak edges [42]. Therefore transport of dust particles is a possible mechanism of impurity deposition into the core plasma. Further on the dust grain charge fluctuates in response to plasma fluctuations and this may lead to stochastic heating of dust which in turn has an important effect on the coagulation and transport of dust particles. The ion drag force on dust particles was proposed as a way of removing dust but the effect of nonlinearities and collective effects in ion-dust Coulomb scattering can enhance the cross-section of this process.

The dispersion relation of plasma modes is also significantly influenced by ion/electron-dust collisions and dust charged fluctuations which affects the thresholds and growth rate of instabilities in the fusion plasma. Also the Thomson scattering cross section can be enhanced by a factor proportional to the square of the dust grain charge with respect to the normal cross sections for radiation scattering. Further enhancements in radiation scattering may come from alignment of magnetized dust particle momenta in external magnetic fields [4]. All these new phenomena prove that the presence of dust particles in fusion devices is an important issue which both future theories and experiments have to take into account.

The dust in plasma has a clear influence on semiconductor industry, in particular on plasma-enhanced deposition and etching processes. Electronic micro-devices are basically a series of patterned layers formed by deposition on which the patterns are created by etching. These processes are an integral part of the lithography steps as they are used to form the masks used to photo lithograph the device on silicon wafers. The study of dust grain induced phenomena in plasma is also important for plasma chemistry and nanotechnology. In reactive plasmas primary clusters of atoms form through a process of nucleation, up to a critical number density then the proto-particles undergo a process of agglomeration, as often observed in dusty plasmas, resulting macro-particles of about 50 nm which in turn grow to micrometer-sized grains by accretion of neutrals and ionic monomers, since further agglomeration is prevented by particle charging. Taking into account the dusty plasma nature, the agglomeration process theory has to take into account the charge of the nanometric particles which might reduce the agglomeration rate and set new maximum achievable limits for micronic particle sizes [43].

IV.1.2. Characteristics of Dusty Plasma

Loosely defined, a dusty plasma is a medium of normal electron-ion plasma with an additional charged component of micron- and submicron-sized (massive) dust grains, which are able to interact with each other, thus having a collective behavior. A dust-laden plasma is a dusty plasma in which the dust particles are completely screened by the surrounding plasma. In order to improve the previous definition, in this section there will be presented the main parameters characterizing a dusty plasma.

The presence of dust grains in plasma induces new properties for dusty plasma and enables new phenomena to manifest at different spatial and time scales and sometimes, in ways that are not characteristic to pure plasma. Therefore the chemical composition, size and shape of the dust particles are expected to greatly influence the properties of dusty plasmas.

Through-out this work, one will resume to consider unmagnetized, spherical dust grains characterized by their radius r_d and mass m_d. The medium distance between the dust grain will be denoted by a. In an isolated dusty plasma, the dust particulates gather electrical charge naturally which is usually negative because of the higher mobility of the electrons in comparison to that of the ions. As a plasma is globally neutral from electrical point of view, it follows that the charged dust grains influence the dusty plasma neutrality equation at equilibrium which, for a one single ionized ion species plasma, writes

$$Z_d n_d + n_{e0} = n_{i0} \tag{4.1}$$

Here Z_d is the number of elementary charges on the dust grain, n_d is the number density of the dust grains (number of particles in volume unit) and $n_{e0,i0}$ are the number densities at equilibrium of the electrons and the ions respectively. Since the dust grains are much more massive than any other component of the system, they can be considered, in a good approximation, at rest. At equilibrium the electron and ion number densities (n_e, n_i) are determined by the local potential φ and distributed according to Boltzmann's law

$$\begin{aligned} n_e &= n_{e0} \exp\left(\frac{e\varphi}{k_B T_e}\right) \\ n_i &= n_{i0} \exp\left(-\frac{e\varphi}{k_B T_i}\right) \end{aligned} \tag{4.2}$$

where e is the elementary electrical charge, k_B is the Boltzmann's constant and $T_{e,i}$ are the electron and ion gas temperatures, respectively. One of the characteristic properties of a plasma is the screening of static electric charges (charged particle or surface). This electrical screening is measured by the maximum distance (the Debye (shielding) length) to which the presence of the charged object influences the surrounding medium. The mechanism of Debye shielding in dusty plasmas is some-what similar to that in electron-ion plasma (see [8]). The charged dust grains gather plasma constituents of opposite charge around them. If not for the thermal agitation of the plasma gas, the number of electrons/ions surrounding the dust particle would equal its charge and the shielding would be perfect. Besides the cloud of charged plasma particles would collapse to a very thin layer (sheath). At a finite temperature the sheath of charged particles expands and the shielding is not perfect as particles at the edge of the cloud have enough thermal energy to escape the electrostatic attraction. The screening radius (for a spherical dust grain) has the value for which the potential energy is approximately equal to the thermal energy of the specific species of plasma constituents ($k_B T_s$ where $s = i, e$ indicates the species). Thus the field of the charged dust grain is shielded down to potentials of the order $k_B T_s/e$ that may act on the surrounding plasma. In order to derive an analytical expression for the Debye radius in dusty plasma let us consider the Poisson's equation [43]

$$\nabla^2 \varphi = \frac{1}{\varepsilon_0} \left(e n_e - e n_i - q_d n_d\right) \tag{4.3}$$

where $q_d = -eZ_d$ is the charge of the dust grain. At the edge of the cloud where $e\varphi/k_BT_{e,i} \ll 1$, one can expand the exponential functions in Taylor series and taking only the linear terms in φ, for an isolated dust grain, the equation (4.3) becomes

$$\nabla^2\varphi = \left(\frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2}\right)\varphi \tag{4.4}$$

where

$$\lambda_{De}^2 = \frac{\varepsilon_0 k_B T_e}{n_{e0}e^2} \quad \lambda_{Di}^2 = \frac{\varepsilon_0 k_B T_i}{n_{i0}e^2} \tag{4.5}$$

are the electron and ion Debye radii respectively. Of course this approach is not valid for the sheath where the potential of the dust grain is much bigger than the thermal energy of the plasma constituents; yet in the sheath, the potential rapidly decreases and the region does not contribute much to the thickness of the cloud. The solution of (4.4) can be written

$$\varphi = \varphi_0 \exp\left(\frac{-r}{\lambda_D}\right) \tag{4.6}$$

where φ_0 is the potential at the center of the cloud (on the surface of the dust grain) and

$$\lambda_D = \frac{\lambda_{De}\lambda_{Di}}{\sqrt{\lambda_{De}^2 + \lambda_{Di}^2}} \tag{4.7}$$

is the plasma Debye radius.

For a dusty plasma with negative charge on the dust grains the density of electrons is diminished (because of the electrons collected on the dust grains) and thus we have $n_{e0} \ll n_{i0} \longrightarrow \lambda_{De} \gg \lambda_{Di}$, $\lambda_D \simeq \lambda_{Di}$, therefore the shielding is due to the predominant, positive ions. When the dust particles become positively charged, the opposite happens $n_{i0}T_e \ll n_{e0}T_i \longrightarrow \lambda_{De} \ll \lambda_{Di}$, $\lambda_D \simeq \lambda_{De}$ and the shielding is done by electrons.

To improve the previous result one has to take into account the fluctuations of the charge on the dust grains. Then the expression for the Debye length in dusty plasma is

$$\frac{1}{\lambda_D^2} = \frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2} + \frac{1}{\lambda_{Dg}^2}$$

where λ_{Dg} contains the effect of the dust grain charge fluctuations

$$\frac{1}{\lambda_{Dg}^2} = \frac{\nu_2}{\nu_1 n_{d0} r_d}$$

and the expressions of ν_1 and ν_2 are given in [43] (see §2.6.1).

At this point one can give a more specific definition of the term dusty plasma as opposed to 'dust in plasma' or dust-laden plasma. Thus the situation when $r_d \ll \lambda_D < a$ corresponds to 'dust in plasma' where dust particles are considered a collection of isolated, screened grains and local plasma inhomogeneities must be taken into account. On the other hand when $r_d \ll a < \lambda_D$, the dust grains must be treated like massive, multiply charged particles which interact with each other and the surrounding plasma constituents and this is the situation encountered in a veritable dusty plasma [43]. There are also other conditions that dusty plasmas must comply with in order to behave as a true plasma. The Debye screening length must be much smaller than the dimensions characterizing the volume occupied by the dusty plasma and this is always satisfied as λ_D is of the order of micrometers. The requirement that the Debye sphere should contain a large number of plasma ions or electrons is also satisfied considering the values of the number densities of the plasma constituents. To understand the last criteria [8] which in a gas implies that the frequencies of the typical plasma oscillations be larger than the frequency of collisions with neutral atoms for the gas to behave as a plasma, one has to briefly discuss the plasmonic frequencies.

Another important property of any plasma is the stability of its electrical neutrality at macroscopic level. Any external perturbation will generate an electrical field which will tend to bring the charged constituents back to their equilibrium positions. But due to their inertia the charged particles will go past these positions generating an opposed electrical field. Thus collective oscillations will appear having a specific frequency called plasmonic frequency ω_p. In deriving an analytical expression for the plasmonic frequencies associated to each dusty plasma component let us consider an uniform, cold (no thermal motion taken into account), unmagnetized dusty plasma. Each type of particles satisfies the continuity equation

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \vec{v}_j) = 0 \tag{4.8}$$

where $j = (e, i, d)$ indicates the species (electrons, ion, dust grains), the momentum equation

$$\frac{\partial \vec{v}_j}{\partial t} + \vec{v}_j \cdot \nabla \vec{v}_j = -\frac{q_j}{m_j} \nabla \varphi \tag{4.9}$$

and the Poisson's equation

$$\varepsilon_0 \nabla^2 \varphi = -\sum_j q_j n_j . \tag{4.10}$$

In all these expressions $\vec{v}_j$ denotes the speed, q_j/m_j the specific charge and n_j the number density of each species while φ is the electrical field created by the external perturbation. For simplicity, sources or sinks as well as the forces due to pressure gradients were neglected. Let's assume that the amplitude of the oscillations is small enough so that a linear treatment can be applied and that at equilibrium (characterized by n_{j0}, $v_{j0} = 0$) no internal electrical field exists ($\varphi = 0$). Writing

$$n_j = n_{j0} + n_{j1}, \quad n_{j1} \ll n_{j0}, \tag{4.11}$$

the equations (4.8), (4.9) can be linearized

$$\frac{\partial n_{j1}}{\partial t} + n_{j0}\nabla \vec{v}_j = 0, \frac{\partial \vec{v}_j}{\partial t} = -\frac{q_j}{m_j}\nabla\varphi \tag{4.12}$$

and together with the expression of the Poisson law, one obtains

$$\frac{\partial^2}{\partial t^2}\nabla^2\varphi + 4\pi\sum_j \frac{n_{j0}q_j^2}{\varepsilon_0 m_j}\nabla^2\varphi = 0 \tag{4.13}$$

Integrating (4.13) twice over the space coordinates under the boundary condition $\varphi = 0$ at equilibrium, it becomes

$$\frac{\partial^2\varphi}{\partial t^2} + \omega_p^2\varphi = 0 \tag{4.14}$$

where

$$\omega_p^2 = \sum_j \frac{n_{j0}q_j^2}{\varepsilon_0 m_j} = \sum_j \omega_{pj}^2 \tag{4.15}$$

is the square of the plasma frequency. The oscillation process described earlier is specific to each plasma species. In the case of dusty plasmas these oscillations occur in completely different frequency ranges as the electrons oscillate around the ions at the frequency ω_{pe}, the ions around the dust grains at ω_{pi} and the dust particles around their equilibrium positions at the lowest frequency ω_{pd} which has values in the range of tenths of Hertz.

The collision frequencies of each of the dusty (complex) plasma species with the neutral atoms are other important characteristics of complex plasmas, given by

$$\nu_{jn} = n_n\sigma_{jn}v_{Tj} \tag{4.16}$$

where n_n is the number density of neutral atoms, σ_{jn} is the scattering cross section of neutral atoms in collisions with plasma particles of species j and $v_{Tj} = \sqrt{k_B T_j/m_j}$ is the thermal speed of species j of plasma particles. The result of the collisions with neutral atoms is a damping of the collective oscillations of each species of particles which gradually diminishes the corresponding oscillation amplitudes. Thus for this damping process to be weak the collision frequencies ν_{jn} must be smaller than the plasma frequency

$$\nu_{en}, \nu_{in}, \nu_{dn} < \omega_p$$

The Coulomb coupling parameter is a dusty plasma characteristic which measures the degree of interaction between neighboring dust grains and determines the possibility of the formation of ordered structures – dusty plasma crystals. If one takes into account the shielding effect, the Coulomb potential energy for two dust particles of charge q_d situated at distance a from each other is

$$E_c = \frac{q_d^2}{4\pi\varepsilon_0 a} \exp\left(-\frac{a}{\lambda_D}\right) \tag{4.17}$$

while their thermal energy is $k_B T_d$. The Coulomb coupling parameter, denoted by Γ_c, is defined as the ratio of the two energies

$$\Gamma_c = \frac{E_c}{k_B T_d} = \frac{Z_d^2 e^2}{4\pi\varepsilon_0 a k_B T_d} \exp\left(-\frac{a}{\lambda_D}\right) \tag{4.18}$$

A dusty plasma is weakly coupled when $\Gamma_c \ll 1$ and strongly coupled if $\Gamma_c \gg 1$, the coupling parameter value being determined by the dust fluid parameters (grain charge, dust temperature and inter-grain distance) and the Debye screening length of the dusty plasma. In laboratory conditions it is therefore easy to obtain very strong coupled dusty plasmas which undergo a phase transition from the disordered gas-like state to an ordered, crystalline phase – Wigner crystal (1938). For $\Gamma_c \geq 170$ such structures were predicted by Ikezi in 1986 [16], theoretically and observed experimentally 8 years later (1994) (see [43], chapter 8).

IV.2. Dust Grain Charging Process

The key to understanding the new phenomena due to the presence of dust particles in a plasma is the construction of an appropriate model to describe the charging of the dust grains as realistically as possible. When immersed into a plasma a dust grain will collect plasma particles and its charge variation will be determined by the sum of currents of various charged plasma particle species that fall onto it. At equilibrium this sum will be zero and the particle will be charged at a surface potential Φ_d. In a plasma without radiative background (especially photons) and in the absence of external fields the dust grains will charge naturally to a negative potential due to the higher mobility in the electron gas, the density of which will be reduced by this process.

The collection of plasma particles on the dust grain surface is, however, even in the case of isolated dust grains, accompanied by a series of other processes either as a side effect of the plasma particles collection (e.g. secondary electron emission due to electron and ion impact), as a consequence of the nature and composition of the ionized gas (impact ionization due to energetic neutral atoms hitting the dust particle and ionization of the dust particles due to radioactivity of elements in the composition of dust or molecules of the surrounding gas) or due to the external physical conditions like photoemission, thermionic emission or field emission.

When a charged plasma particle approaches the surface of an isolated dust grain in an unmagnetized plasma, it may be scattered (depending on the sign of the dust grain potential) before it reaches the surface or it may enter the dust grain and either stop immediately (stick onto the dust particle surface) or travel through the dust grain material losing energy while interacting with scattering centers until it stops or exists on the opposite side of the grain. While the reflection and absorption are part of the collection process that mainly interests the low energy charged particles and which will be presented in detail in the following subsection, the transmission/tunneling and secondary emission (due to interactions with the dust grain material) are characterized by threshold energies that only externally accelerated charged plasma particles may posses. If one considers the electron gas in the plasma, the energy of the electrons must exceed the value of the repulsive potential barrier when grains are charged negatively. The tunneling of this potential barrier may occur due to the presence of slow, positive ions with low energies (below 1keV) but this process mainly leads to the neutralization of the ions. Assuming that normal incident electrons loose energy according to the Thomson-Whiddington law (it applies for electrons up to 30keV), this energy loss is proportional to the yield of secondary electrons and the secondary electron flux decreases exponentially with the depth reached by the primary electron. An expression for the yield of secondary electrons was derived which closely matched the results found by Jonker (1952). Numerical calculations of the secondary yield showed that the secondary current increases with the electron temperature up to a maximum (corresponding to approximately 1keV) after which the current decreases since electrons pass through the whole body of the dust grain and exit on the opposite side (are transmitted through the dust grain). This "tunneling" process dominates when the primary electron initial energy (the electron energy at incident dust grain surface) is comparable to $(2D_W r_d)^{1/2}$, where D_W is Whiddington's constant for energy loss with distance. As already discussed low energy ions are neutralized by electrons and the energy released in this process may excite secondary electrons that may leave the dust grain surface. The number of such electrons is determined by the ionization potential energy W_i and the work function of the dust grain material W_f. When a conduction electron is captured by the incident ion it makes available a maximum energy of $W_i - W_f$; this must exceed the work function of the surrounding material to free another electron and therefore the following condition must be met $W_i > 2W_f$. For more energetic ions (above 10keV), the secondary electron yield will increase substantially as they loose energy in a smaller depth of material because of their larger mass. For this reason, the rate of re-emission/transmission of ions through the dust grains is negligible. The secondary electron yield from ion impact may be estimated using the same model as for electrons [43] (§2.2.2) though for relativistic ions (much higher energies – MeV) one should apply the Bethe-Bloch formula to compute the energy loss per distance.

If the dusty plasma is exposed to a flux of photons and the energy of these photons $\hbar\omega$ is greater than the photoelectric work function (W_f) of the dust grain material, the latter emit photo-electrons. This mechanism may determine the dust grain to charged positively and thus the photo-electrons will return to the grain unless the energy of the incoming photons is bigger to compensate for the attractive dust grain potential ($\hbar\omega > W_f + q_d e/r_d$). Thus the maximum dust grain charge can be estimated from

$$q_d = (\hbar\omega - W_f)\frac{r_d}{e}, \tag{4.19}$$

which is attained when the outgoing current of photo-electrons is balanced by the current of electrons returning to the grain surface. In case of negatively charged grains, the outgoing photo-electrons are accelerated and never return to the surface, yet there are other charging mechanisms to be considered that limit the grain charge.

In special conditions there are a number of other charging mechanisms to be considered important. For instance, thermionic emission of electrons or ions occurs when the dust grains are heated to high temperatures using lasers, infrared radiation or hot filaments surrounding the complex plasma. The tendency of this phenomenon is however to make the dust grain charge positively and therefore the outgoing thermionic current is limited by the attractive dust grain potential (as for the photo-emission). Rough estimations (see [43] §2.2.4) give a value of 300 W/cm^2 as the minimum needed intensity of a laser energy flux in order for the thermionic current of grains heated at $\sim$ 1700 K to become important. Another phenomenon that manifests in special circumstances when the micron-sized or nanometric dust particles acquire high potentials, is the field emission of electrons or ions depending on the sign of the dust grain charge. Field emission is a mechanism for limiting the dust grain potential and for common values of the work function of the dust grain material becomes important for dust radii of the order of a micron. Radioactivity of the dust grain materials is also a charging mechanism either because of the primary charged particles or because of the current of secondary electrons (excited by the passage of the primary radiation through the material) that leave the particle. The amount of ordinary radioactive isotopes in the dust material in the cosmic space is insignificant but there are however reports that dust created by novae and supernovae may have important radioactive levels due to abundance of long life $\beta^{+/-}$ emitter isotopes like Fe^{60}, Ni^{59}, Al^{26}, Na^{22}. Last but not least, at high density or high temperatures of neutral atoms gas in complex plasmas, the phenomenon of impact ionization can occur on the dust grain surface. It mainly consists in ionization of the incident neutral atom or atoms of the grain surface as result of collision at high kinetic energies of the projectile accompanied by release of electrons/ions depending of the grain potential.

IV.2.1. Orbit Limited Motion (OLM) Approximation

In the followings, the theory of the orbit limited motion approximation, as a model for the process of charged plasma particle collection, will be presented in detail [2].

Let us consider a plasma particle j (an electron or an ion) approaching a dust grain of radius r_d and charge q_d (usually negatively charged) on a ballistic trajectory. When it enters the Debye sphere it feels the influence of the electrostatic field of the charged dust particle being deflected. Let v_j denote the initial velocity of the charged particle (at large distance from the dust grain) and v_{gj} the velocity at grazing collision with the dust grain as indicated in figure IV.6, where b_j is the impact parameter corresponding

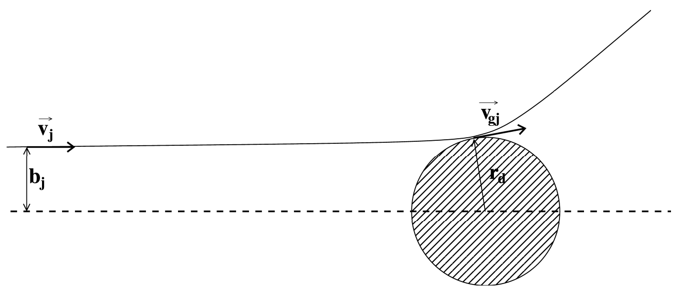

Figure IV.6: Schematic representation of the grazing collision between a plasma particle and a dust grain (repulsive case)

to the grazing collision. It is obvious that decreasing this parameter the plasma particle will hit the dust grain. Assuming a perfect plastic collision (any particle hitting the grain with eventually stick on it), the cross section for the charging process is $\sigma_j^d = \pi b_j^2$. In the OLM approximation, one neglects any details of the particle scattering in the central field of the dust grain and the secondary emission due to the plasma particle impact on the grain, taking into account only the momentum and energy conservation in the process, namely

$$\begin{aligned} m_j v_j b_j &= m_j v_{gj} r_d \\ \frac{1}{2} m_j v_j^2 &= \frac{1}{2} m_j v_{gj}^2 + \frac{1}{4\pi\varepsilon_0} \frac{q_j q_d}{r_d} \end{aligned} \tag{4.20}$$

where q_j is the charge of the plasma particle, $q_j = Z_i e$ for a Z_i ionized cation and $q_j = -e$ for electrons. From these relations we obtain

$$\begin{aligned} b_j &= \left(\frac{v_{gj}}{v_j}\right) r_d \\ \left(\frac{v_{gj}}{v_j}\right)^2 &= 1 - \frac{2}{4\pi\varepsilon_0} \frac{q_d q_j}{r_d m_j v_j^2} \end{aligned}$$

and the charging cross section becomes

$$\sigma_j^d = \pi r_d^2 \left(1 - \frac{2}{4\pi\varepsilon_0} \frac{q_j q_d}{r_d m_j v_j^2}\right) \tag{4.21}$$

assuming that the dust grain is negatively charged with charge $q_d = -Z_d e$. q_d is related to the floating potential of the dust grain $\Phi_d = \Phi_{fd} - \Phi_p$ (Φ_{fd} - the negative dust grain potential and Φ_p - the plasma potential) by $q_d = C\Phi_d$. Here C is capacitance of the spherical dust grain

$$C = 4\pi\varepsilon_0 r_d \exp\left(\frac{-r_d}{\lambda_D}\right) \simeq 4\pi\varepsilon_0 r_d, \tag{4.22}$$

if $r_d \ll \lambda_D$.

If one denotes by $f_j(v_j)$ the velocity distribution function for plasma particles of species j, the charging current carried by the plasma species j on the dust grain has the following expression

$$I_j = q_j \int_{v_{min}^{(j)}}^{\infty} v_j \sigma_j^d f_j(v_j) \mathrm{d}^3\, v_j \tag{4.23}$$

where $v_{min}^{(j)}$ is the minimum value of the plasma particle velocity for which the particle still hits the grain. There are two distinct situations $q_j\Phi_d < 0$ and $q_j\Phi_d > 0$. When $q_j\Phi_d < 0$, the plasma particle is attracted by the dust grain and then the integration extends over the whole velocity space ($v_{min}^{(j)} = 0$), while for $q_j\Phi_d > 0$ it is repelled and it has to surpass a repulsive potential barrier. In the later case

$$v_{min}^{(j)} = \sqrt{\frac{2q_j\Phi_d}{m_j}} \tag{4.24}$$

Usually the plasma species are considered at equilibrium so their velocities have Maxwellian distributions. For the plasma species j, one has

$$f_j(v_j) = n_j \left(\frac{m_j}{2\pi k_B T_j}\right)^{3/2} \exp\left(-\frac{m_j v_j^2}{2k_B T_j}\right) \tag{4.25}$$

where n_j is the number density and T_j the temperature of the plasma species j fluid. Performing the integration in (4.23) (in spherical coordinates), one gets

$$I_i = 4\pi r_d^2 n_i Z_i e \sqrt{\frac{k_B T_i}{2\pi m_i}} \left(1 + \frac{1}{4\pi\varepsilon_0} \frac{Z_i Z_d e^2}{r_d k_B T_i}\right) \tag{4.26}$$

for the ionic current (usually one takes $Z_i = 1$) and

$$I_e = -4\pi r_d^2 n_e e \sqrt{\frac{k_B T_e}{2\pi m_e}} \exp\left(-\frac{1}{4\pi\varepsilon_0} \frac{Z_d e^2}{r_d k_B T_e}\right) \tag{4.27}$$

for the electron current (or any single ionized anions with the corresponding mass substitution).

IV.2.2. Dust Grain Charging Time

As mentioned before, the charge on the dust grain reaches its equilibrium value when the ionic and electronic currents that fall on it equal each other. Therefore the time evolution of the dust grain charge is governed by the equation

$$\frac{\mathrm{d}q_d}{\mathrm{d}t} = I_e + I_i \tag{4.28}$$

and the equilibrium (stationary) charge of an isolated dust grain is determined from the flux balance

$$I_e + I_i = 0 \tag{4.29}$$

It is convenient to introduce the following dimensionless quantities

$$z = \frac{1}{4\pi\varepsilon_0}\frac{Z_d e^2}{r_d k_B T_i}, \quad \tau = \frac{T_e}{T_i} \tag{4.30}$$

Typically in gas discharge plasmas $\tau \sim 10 - 100$ and $z \sim 1$. With these notations, the equilibrium condition (4.29) becomes (note $n_{e,0} = n_{i,0} = n_0$)

$$\sqrt{\frac{m_e}{m_i}}(1+z) = \sqrt{\tau}\exp(-z/\tau) \tag{4.31}$$

Thus the (equilibrium) charge on an isolated dust grain is determined by the grain radius r_d and the temperatures T_e and T_i of the electronic and ionic fluids as the solution of the equation (4.31).

In a dusty plasma, for the previous discussion to be complete (and valid for non-isolated grains as well), one has to take into account the finite density of the dust grains in the plasma. This can be done by using the neutrality condition at equilibrium

$$Z_d n_{d,0} + n_{e,0} - n_{i,0} = 0 \tag{4.32}$$

As mentioned above, the negative charging of the dust grains determines a strong depletion of the electron density, thus making the ion density larger and strongly influencing the dusty plasma properties. Introducing the expressions (4.26) and (4.27) for I_i and I_e in the stationary condition (4.29) and using the neutrality condition (4.32), one obtains

$$\sqrt{\frac{m_e}{m_i}}(1+z) = \sqrt{\tau}\exp(-z/\tau)\left(1 - Z_d\frac{n_{d,0}}{n_{i,0}}\right) = \sqrt{\tau}\exp(-z/\tau)(1-Pz) \tag{4.33}$$

where $P = 4\pi n_{d,0} r_d \lambda_{Di}^2$ and λ_{Di} is the ionic Debye screening length. If $P \ll 1$ we regain the isolated grain case while in the opposite limit, $P \gg 1$, the dust grain charge is significantly reduced with respect to the single dust grain case. The figure IV.7 presents the dependence of z on $\log P$ for an argon plasma (with single ionized cations) for two different value of τ. It

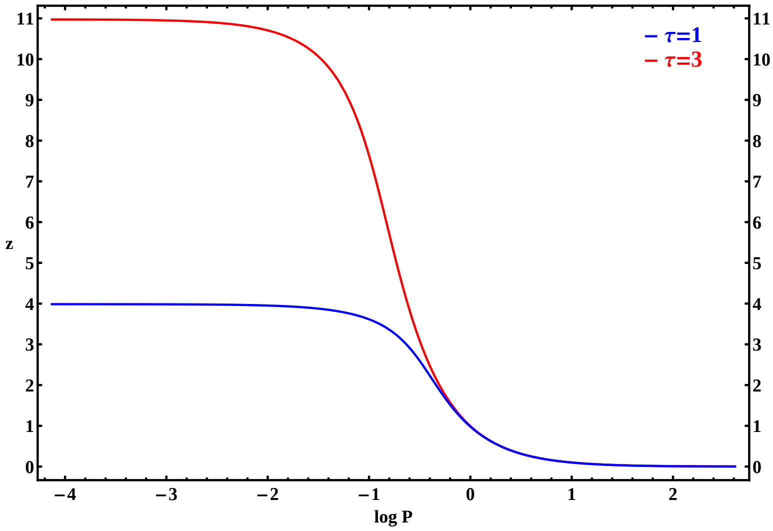

Figure IV.7: Dependence of z on $\log P$ for an Ar^+ plasma when $\tau = 1$ ($z \overset{P\to 0}{=} 3.985$) and $\tau = 3$ ($z \overset{P\to 0}{=} 10.976$).

is easily seen that the equilibrium potential energy for isolated grains (limit $P \to 0$) depends strongly on the temperature of the electrons.

In order to derive an expression that would allow the evaluation of the charging time, one has to consider small deviations from the equilibrium values and solve the linearized equation for the time evolution of the charge on a dust grain (4.28). In the case of an isolated grain, using the notations (4.30) and the expressions (4.26) and (4.27) for the ionic and electronic currents falling on the dust grain, it writes

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\beta \left[(1+z) - \frac{\sqrt{\tau}}{\alpha} \exp\left(-\frac{z}{\tau}\right) \right], \tag{4.34}$$

where

$$\beta = \frac{r_d v_{Ti}}{\sqrt{2\pi}\lambda_{Di}^2}, \quad \alpha = \sqrt{\frac{m_e}{m_i}}. \tag{4.35}$$

The equilibrium solution z_0 is computed by solving numerically the transcendental equation obtained by equaling the right hand side of (4.34) to zero. Considering small deviations from it and writing

$$z = z_0 + z_1, \quad z_1 \ll z_0 \tag{4.36}$$

the linearized equation satisfied by z_1 is

$$\frac{\mathrm{d}z_1}{\mathrm{d}t} = -\beta z_1 \left(1 + \frac{1+z_0}{\tau} \right). \tag{4.37}$$

The solution of this equation has the form

$$z_1 \sim \exp(-t/\tau_c), \quad \tau_c = \frac{\tau}{\beta(\tau + z_0 + 1)} = \frac{\lambda_{Di}^2 \sqrt{2\pi}}{r_d v_{Ti}} \frac{T_e}{T_e + (1+z_0)T_i}, \tag{4.38}$$

where τ_c is the charging time. Its value depends on the ionic temperature $T_i^{1/2}$ and less on T_e while it is decreases proportionally with the dust grain radius r_d and the equilibrium number density $n_{i0} = n_{e0} = n_0$.

For the case of non-isolated dust grains, using the notations (4.30), (4.35) and considering the equilibrium values for the ion and electron number densities, the time evolution equation for the charge on the dust particles has the form

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\beta\left[(1+z) - \frac{\sqrt{\tau}}{\alpha}(1-Pz)\exp\left(-\frac{z}{\tau}\right)\right] \tag{4.39}$$

Again considering small deviation from the equilibrium value z_0, solution of the transcendental equation

$$\alpha(1+z_0) = \sqrt{\tau}(1-Pz_0)\exp(-z_0/\tau),$$

the linearized equation for the time evolution of z_1 can be written

$$\frac{\mathrm{d}z_1}{\mathrm{d}t} = -\beta z_1\left[1 + \frac{1+z_0}{\tau}\left(1 + \frac{\tau P}{1-Pz_0}\right)\right]. \tag{4.40}$$

Thus the expression of the charging time for non-isolated grains is

$$\frac{1}{\tau_c} = \beta\left[1 + \frac{1+z_0}{\tau}\left(1 + \frac{\tau P}{1-Pz_0}\right)\right], \tag{4.41}$$

which has a correction to charging time that characterizes an isolated dust grain. One may easily notice that the charging time for non-isolated dust particles τ_{cn} is usually smaller than for isolated grains τ_{c0}

$$\frac{\tau_{c0}}{\tau_{cn}} = 1 + \frac{P}{1-Pz_0}\frac{1+z_0}{1+\dfrac{1+z_0}{\tau}} \tag{4.42}$$

A careful evaluation of the charging time correction factor will yield values for τ_{cn} that are a few orders of magnitude below τ_{c0}, the later being in the range of a few microseconds(7 μs – see [43], pp. 59). However it should be noted as well that these extremely small values are obtained when $Pz_0 \simeq 1$ and since $1 - Pz = 1 - Z_d\frac{n_d}{n_{i,0}} = \frac{n_{e,0}}{n_{i,0}} \to 0$ for a given grain radius r_d and ion temperature T_i, the situation corresponds to a near complete depletion of the electron gas (see figure IV.8). The figure also shows that the electron gas is never completely depleted and there is a minimum saturation electron number density which is attained quicker as the electron temperature increases (when the ion temperature is constant). This happens because the electrons have larger thermal kinetic energy and overcome the dust grain reflective potential in greater number contributing to the increase of the charge on the dust particle. For a given dust grain radius r_d the parameter P is proportional to the dust particle number density. As n_d increases the equilibrium potential of the dust grain (z) decreases and, as the potential

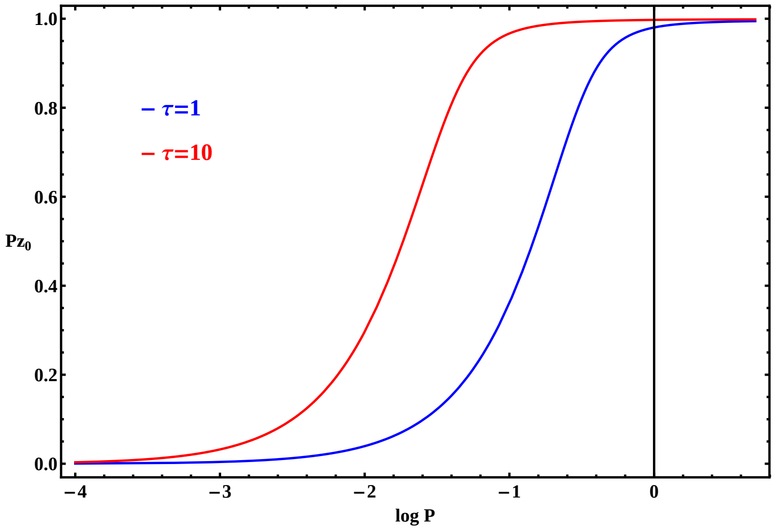

Figure IV.8: The parameter Pz_0, a measure of the depletion of the electron gas due to dust grain charging, as a function of P for two electron temperatures $\tau = 1$ and $\tau = 10$.

is proportional to the number of elementary charges on the dust grain, so does Z_d. This effect is natural because the increase in the dust particle number density is at the expense of the electrons having the same Maxwellian thermal speed distribution, for a given electron temperature, and thus the same flux of incident electrons is distributed among an bigger number of dust grains. These processes are reflected in the drop of the charging time of non-isolated grains, τ_{cn}, with respect to the charging time of an isolated dust particle, τ_{c0}, shown in figure IV.9. When the near depletion electron number density is reached, the charging time magnitude grows almost linearly with the dust particle density number as the equilibrium charge on the dust grains decreases. Yet in this region a new charging model should be applied as the ion gas controls the process, given that $\lambda_D \simeq \lambda_{Di}$ (see §IV.1).

A less ideal dusty plasma contains several types of dust particulates of different shapes and dimensions. In the followings we shall consider only the case of a dusty plasma with spherical dust grains of different radii. From electrostatic equilibrium arguments we expect that all the dust particulates, no matter their type, will get charged at the same grain potential. This is easy to be observed if a single dust grain of radius r_2 is immersed into a dusty plasma containing only dust grains of radius r_1 and number density N_1. The dust grains of the host complex plasma are charged according to the formula (4.33)

$$\begin{aligned} \sqrt{\frac{m_e}{m_i}}(1+z_1) =& \sqrt{\tau}\exp(-z_1/\tau)\frac{n_{e,0}}{n_{i,0}} \\ \frac{n_{e,0}}{n_{i,0}} =& 1 - Z_1\frac{N_1}{n_{i,0}} \end{aligned} \tag{4.43}$$

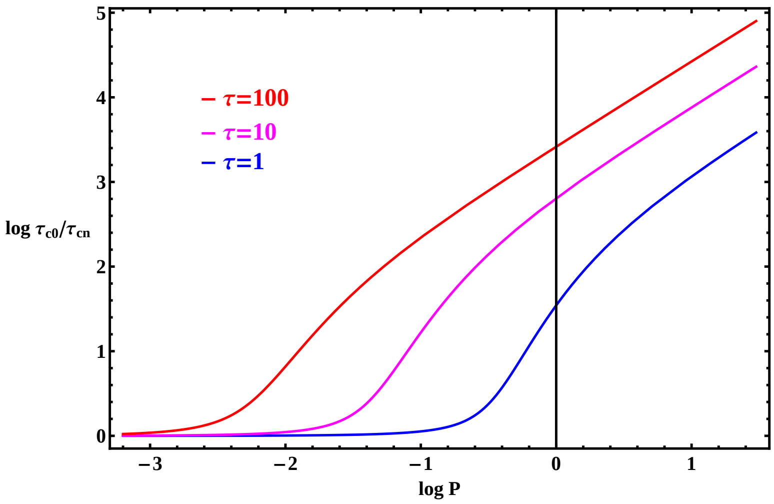

Figure IV.9: The magnitude of the charging time decrease as a function of P for different electron temperatures.

where the last relation reflects the depletion of the electron gas due to the charging of the dust particles. When the new dust grain of radius r_2 is introduced in this dusty plasma its equilibrium charge will be determined from a formula similar to (4.31), generalized to take into account the existing depletion of the electron component, namely

$$\sqrt{\frac{m_e}{m_i}}(1+z_2) = \sqrt{\tau}\exp(-z_2/\tau)\frac{n_{e,0}}{n_{i,0}} \tag{4.44}$$

Comparing the two formulas, it is clear that $z_2 = z_1$, i. e. the new added dust grain is charged at the same potential as the host dust grains.

Using these arguments, one can easily extend the treatment to a complex dusty plasma containing several types $j = \overline{1,p}$ of spherical dust grains of radius r_j and number densities $N_j = c_j N_0$, where $\sum_{j=1}^{p} c_j = 1$. The depletion of the electron gas can be now evaluated from

$$\frac{n_{e,0}}{n_{i,0}} = 1 - \sum_{j=1}^{p} Z_j \frac{N_j}{n_{i,0}} = 1 - \frac{N_0}{n_{i,0}} \sum_{j=1}^{p} c_j Z_j \tag{4.45}$$

All the dust grains (no matter their radius) will be charged at the same potential

$$z_j = \frac{1}{4\pi\varepsilon_0}\frac{Z_j e^2}{r_d k_B T_i} = z \tag{4.46}$$

and z is the solution of the equation (4.33) where

$$P = 4\pi N_0 \langle r \rangle \lambda_{Di}^2 \tag{4.47}$$

and the *mean radius* $\langle r \rangle$ is defined as

$$\langle r \rangle = \sum_{j=1}^{p} c_j r_j \tag{4.48}$$

Further on, the method can be applied to a continuum distribution of grain radii as well. Denoting by $f(r)$ the radius distribution function for the dust grain species, the number of dust grains, having their radius in the range $(r, r + \mathrm{d}r)$, is

$$\mathrm{d}N(r) = N(r)\mathrm{d}r = N_0 f(r)\mathrm{d}r$$

Then the dimensionless potential (in absolute value) of any dust grain

$$z(r) = \frac{1}{4\pi\varepsilon_0}\frac{Z(r)e^2}{rk_BT_i}, \tag{4.49}$$

is determined from the same equation (4.33) with the parameter P defined by (4.47) where the mean value $\langle r\rangle$ has the following expression

$$\langle r\rangle = \int_0^\infty rf(r)\mathrm{d}r \tag{4.50}$$

Several distribution functions $f(r)$ have been considered by different authors [7, 29, 30]

- Gaussian or normal distribution

  $$f(r) = K\exp\left(-\frac{(r-r_0)^2}{2\sigma^2}\right),$$
  $$K = \frac{1}{\sqrt{\pi/2}\sigma}\frac{1}{1+\Phi\left(r_0/\sqrt{2}\sigma\right)},$$

  where r_0 is the most probable radius value, and

  $$\Phi(y) = \frac{2}{\sqrt{\pi}}\int_0^y \exp(-x^2)\mathrm{d}x$$

  is the partition function.

- log normal or Galton distribution

  $$f(r) = \frac{1}{r\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln r-\mu)^2}{2\sigma^2}\right),$$

  where μ and σ are the mean value and standard deviation of the natural(/decimal) logarithm of the radius (the logarithm of the dust grain radius is normally distributed).

- power law distribution (widely accepted as appropriate for space plasmas)

  $$f(r) = \begin{cases} Kr^{-\beta} & r \in [r_{min}, r_{max}] \\ 0 & \text{outside} \end{cases}$$
  $$K = \frac{(\beta-1)(r_{min}r_{max})^{\beta-1}}{r_{max}^{\beta-1} - r_{min}^{\beta-1}}$$

  where $\beta = 4.5$ for Saturn's F ring, 6–7 in Saturn's G ring and $\beta = 3.4$ for cometary environment.

IV.3. Hydrodynamic Model for Dusty Plasma

A great variety of collective wave phenomena arises due to the coherent motion of the plasma constituents. They can be both longitudinal and transverse waves. In an unmagnetized plasma the transverse waves are purely electromagnetic. On the other hand the longitudinal waves are accompanied by density and potential fluctuations, and can be either linear or nonlinear. In the followings, we'll consider only this type of waves.

The presence of charged dust particulates can modify the wave propagation and even introduce new oscillation modes and phenomena which are absent in an usual plasma. These new types of waves appear in the low frequency range and usually are associated with the dust fluid motion (dust acoustic waves – DAW). In a higher frequency domain the ion-acoustic wave of a pure plasma are strongly influenced (via the neutrality condition) by the presence of the dust particulates resulting a new kind of collective waves (the dust ion-acoustic waves – DIAW). They were predicted theoretically by Rao *et al.* (1990) – DAW [40], and Silin and Shukla (1992) – DIAW [44] and later observed experimentally by Barkan *et al.* (1995) [6] and Nakamura *et al.* [33, 34] respectively.

There are two main approaches to discuss the collective waves in a complex plasma. The first implies a hydrodynamic description when the dusty plasma is considered as a mixture of different fluids (electrons, different types of positive an negative ions and different kinds of charged dust particles), each characterized by a number density n_j (or mass density $\rho_j = m_j n_j$), a fluid velocity v_j, the charge q_j of the individual j plasma component and, at equilibrium, by a temperature T_j of the fluid ($j = (i, e, d)$). For dust particles we need to distinguish between grains of different sizes and charges.

For each of these components we can write down a continuity equation

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \vec{v}_j) = S_j \tag{4.51}$$

and an equation of motion

$$m_j \left(\frac{\partial}{\partial t} + \vec{v}_j \cdot \nabla \right) \vec{v}_j = \vec{F}_j \tag{4.52}$$

In the equation (4.51) S_j is a source/sink term describing the change in the population of the j-species through interaction with other plasma species, or with the external environment for open systems. For electrons and ions, this source term is needed because the dust grains capture/release plasma particles, depending on the details of the charging mechanism. Even in an isolated plasma, the source term corresponding to the dust grains may describe the agglomeration process of small dust particles into bigger ones.

The F_j terms in the right-hand side of equation (4.52) is the force acting on the j complex plasma component due to the interaction with plasma fields and other plasma species. Some of these forces will be briefly discussed below [20, 21, 36, 42, 43]:

a. Electromagnetic force

$$\vec{F}_{el} = q_j \left(\vec{E} + \frac{1}{c} \vec{v}_j \times \vec{B} \right), \tag{4.53}$$

where $\vec{E} = -\nabla\varphi$ is the electrical field (φ – the plasma potential) and $\vec{B}$ the magnetic field. If $\vec{E}$ is usually generated by the collective motion of all the plasma components (through the Poisson's equation), the magnetic field $\vec{B}$ is, in most cases, of external origin (in space physics it is, for instance, the magnetic field of a rotating planet which is orbited by the complex plasma cloud).

b. Gravitational force, acting especially on the dust particles, can be decomposed in two parts, namely *(i)* the attraction of the dust grains in the external gravitational field (from a nearby planet or star) and *(ii)* the attraction between dust grains. For micron-sized, charged dust particles present in celestial bodies (such as the Saturn rings) both the electromagnetic and gravitational force have the same order of magnitude so both their effects have to be taken into account at the same time (one speaks of *gravito-electrodynamics*)

c. Pressure force: In a fluid of density ρ_j the force generated by a variable pressure is $-\nabla p_j/\rho_j$. But for an ideal, isothermal gas $p_j = k_B T_j n_j$ so the force term becomes

$$f_{pj} = -\frac{1}{n_j} \nabla \left(\frac{k_B T_j}{m_j} n_j \right) = -v_{Tj}^2 \frac{\nabla n_j}{n_j}. \tag{4.54}$$

d. Drag forces defined as the time rate of the momentum transfer from dust particles to the other plasma components, particularly to the ions and the neutral atoms, or from the plasma components to the dust particles as a result of collisions. Considering only the ion drag force, it can be written $f_{id} = -m_d \nu_{di} v_d$, where ν_{di} is a characteristic momentum transfer frequency from ions to the dust grains. A balance equation $m_d n_d \nu_{di} = m_i n_i \nu_{id}$ relates ν_{di} to ν_{id}, the characteristic momentum transfer frequency from dust grains to ions, at equilibrium, and the expression of ν_{id} depends on the ion-grain scattering model. For cold dust grains the expression obtained by Khrapak *et al.* (2002) is [19]

$$\nu_{id} = \frac{4}{3} \sqrt{2\pi} r_d^2 n_d v_{Ti} z^2 \tau \Gamma,$$

where $z = \frac{1}{4\pi\varepsilon_0} \frac{Z_d e^2/a}{k_B T_e}$ is the surface potential energy of the dust grain expressed in units $k_B T_e$, $\tau = T_e/T_i$ and Γ is a modified Coulomb factor.

Other types of forces will not be considered in the present work.

To the equations (4.51) and (4.52) we have to add the Poisson's equation

$$\varepsilon_0 \nabla^2 \Phi = -\sum_j n_j q_j, \tag{4.55}$$

where the summation is extended on all the charged species of the plasma (electrons, different type of ions and dust particles), and also an equation describing the time variation of the charge of the dust grain. Of course, depending on the kind of collective excitations we are interested in, simplifications of this set of coupled equations are possible.

The second approach is based on a kinetic theory of the dusty plasmas, where each plasma component is described by a distribution function $f^{\alpha}_{\vec{p}}(\vec{r}, t)$ in the phase space $(\vec{r}, \vec{p})$. In the case of the dust particles, the distribution function depends also on the new dynamical variable which is the charge q collected on the dust grain, $f^{d}_{\vec{p}}(\vec{r}, t, q)$. The equations satisfied by the distribution functions are obtained from the generalized Bogoliubov-Klimontovich scheme by including the effect of the dust charge variation. A linear theory is completely discussed in a series of papers by Tsytovich and de Angelis [22, 46–50]. The equations satisfied by the distribution functions of the electrons and ions write

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{r}} + q_{\alpha} \vec{E} \cdot \frac{\partial}{\partial \vec{p}}\right) f^{\alpha}_{\vec{p}} = S_{\alpha} - \int \sigma_{\alpha}(q, v) v f^{q}_{\vec{p'}}(q) \mathrm{d}q \frac{\mathrm{d}^3 \vec{p'}}{(2\pi)^3} f^{\alpha}_{p}, \tag{4.56}$$

where S_{α} describes any external source of plasma particles, $\vec{E}$ is the total electric field (the electrostatic field plus the field generated by all the plasma components), $\sigma_{\alpha}(q, v)$ is the collision cross section for collisions between the plasma particle with velocity v and the dust grain characterized by charge q and $f^{q}_{\vec{p'}}(q)$ is the distribution function of dust grains of momentum $\vec{p'}$ and charge q. For the dust distribution function $f^{d}_{\vec{p}}(\vec{r}, t, q)$ one obtains

$$\left[\frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{r}} + q\vec{E} \cdot \frac{\partial}{\partial \vec{p}} + \frac{\partial}{\partial q}\left(I_{ext} + \sum_{\alpha} I_{\alpha}\right)\right] f^{d}_{\vec{p}}(\vec{r}, t, q) = 0, \tag{4.57}$$

where I_{ext} is an external source current, and $I_{\alpha}(\vec{r}, t, q)$ is the current of plasma particles of type α collected by the grain at time t and point $\vec{r}$ in space. We shall not comment any more on this approach as it will not be used in the present paper.

In the next sections, the author's and his collaborators' contributions in the study of (nonlinear) collective waves in dusty plasma will be presented [13, 14]. In section §IV.5 the problem of dust acoustic waves, taking into account the charge variation on the dust grain surface in a very simple model (local equilibrium approximation), will be discussed. Next the influence of dust grain dimension on DAW will be considered in a model with two types of spherical dust grains of different radii. Further on, the influence of a finite density of (static) dust grains on the dust ion-acoustic waves (DIAW) in a plasma with positive and light negative ions will be investigated. Finally a short section of conclusions, summarizing the original results of the author, will close this chapter.

IV.4. Collective Waves in a Dusty Plasma

The most important collective phenomenon in a dusty plasma is the existence of very low frequency waves, of the order of a few Hz, the so-called dust acoustic waves, which involve the fluctuation of the density of charged dust particles. Let us consider a complex plasma, with no external fields or radiation fluxes, composed of electrons, one species of single ionized positive ions and only one type of spherical, unmagnetized dust grains. As the mass of the dust grains is several orders of magnitude larger than the mass of the other plasma constituents, their motion is also several orders of magnitude slower than the electron and ion motion. Then in the presence of the slow motion of a dust wave both the electrons and the ions can be in permanent thermal equilibrium with the local plasma potential φ. Therefore, the first approximation is to consider isothermal electrons (of temperature T_e) and ions (of temperature T_i), their number densities being given by Boltzmann distributions

$$\begin{aligned} n_e &= n_{e,0} \exp\left(\frac{e\varphi}{k_B T_e}\right), \\ n_i &= n_{i,0} \exp\left(-\frac{e\varphi}{k_B T_i}\right), \end{aligned} \tag{4.58}$$

$n_{e,0}$ and $n_{i,0}$ being, respectively, the equilibrium electronic and ionic number densities. The movement of the dust particles is described by a continuity equation

$$\frac{\partial n_d}{\partial t} + \nabla\left(n_d \vec{v}_d\right) = 0 \tag{4.59}$$

$n_d(\vec{r}_d, t)$ being the number density of the dust particles and $\vec{v}_d$ the dust fluid velocity (one neglects any source/sink term), and by a momentum equation

$$m_d\left(\frac{\partial}{\partial t} + \vec{v}_d \cdot \nabla\right)\vec{v}_d = -q_d \nabla\varphi - \frac{k_B T_e}{n_d}\nabla n_d. \tag{4.60}$$

Here $q_d = -Z_d e$ is the negative charge of the dust grain of mass m_d and T_d is the temperature of the dust fluid. The second term in the right hand side of (4.60) represents the pressure force of the dust fluid at temperature T_d. To these equations one has to add the Poisson equation

$$\nabla^2\varphi = \frac{e}{\varepsilon_0}\left(n_e + Z_d n_d - n_i\right) \tag{4.61}$$

and the equation describing the charge fluctuations on the dust grain. As the hydrodynamic time associated with the motion of the dust wave is proportional to ω_{pd}^{-1}, where ω_{pd} is the plasma frequency of the charged dust fluid, $\omega_{pd}^2 = \frac{n_{d,0}(eZ_{d,0})^2}{\varepsilon_0 m_d}$, and it is several orders of magnitude larger than the charging time of the dust grains, τ_c, one can assume that the dust grain charge has enough time to accommodate itself with the slow varying plasma potential generated by the fluctuation in the plasma density when dust waves propagate. In other words, the formula (4.33), giving the equilibrium dust charge, remains valid if the corresponding quantities, constant at equilibrium, are replaced by the local slowly varying ones. Denoting by the index zero the equilibrium values, z_0, P_0, of the adimensional parameters z, P (see §IV.2) which satisfy the equation

$$\sqrt{\frac{m_e}{m_i}}(1+z_0) = \sqrt{\tau}\exp(-z_0/\tau)(1-P_0 z_0), \tag{4.62}$$

small perturbation will be taken $z_1 \ll z_0$, $P_1 \ll P_0$, so that $z = z_0 + z_1$ and $P = P_0 + P_1$ satisfy a similar equation

$$\sqrt{\frac{m_e}{m_i}}(1+z) = \sqrt{\tau}\exp(-z/\tau)(1-Pz). \tag{4.63}$$

It is easily shown that

$$P_1 = P_0\left(\frac{n_{d,1}}{n_{d,0}} - \frac{n_{i,1}}{n_{i,0}}\right), \tag{4.64}$$

where $n_d = n_{d,0} + n_{d,1}$ and $n_i = n_{i,0} + n_{i,1}$ with $n_{d,1} \ll n_{d,0}$ and $n_{i,1} \ll n_{i,0}$ respectively. Introducing the series expansions around the equilibrium values in (4.63) and linearizing, one obtains

$$\left(\frac{1}{1+z_0} + \frac{1}{\tau} + \frac{P_0}{1-P_0 z_0}\right) z_1 = \frac{P_0 z_0}{1-P_0 z_0}\left(\frac{n_{i,1}}{n_{i,0}} - \frac{n_{d,1}}{n_{d,0}}\right) \tag{4.65}$$

and finally z_1 is related to the first order fluctuations of the ion and dust densities $n_{i,1}$ and $n_{d,1}$ through the expression

$$z_1 = \alpha\left(\frac{n_{i,1}}{n_{i,0}} - \frac{n_{d,1}}{n_{d,0}}\right) z_0, \tag{4.66}$$

where

$$\alpha = \frac{P_0}{P_0 + (1-P_0 z_0)\left(\frac{1}{\tau} + \frac{1}{1+z_0}\right)}. \tag{4.67}$$

For a given ionic temperature T_i, the fluctuation of the dust grain charge is

$$Z_{d,1} = Z_{d,0}\frac{z_1}{z_0}$$

or using (4.66), it writes

$$Z_{d,1} = Z_{d,0}\alpha\left(\frac{n_{i,1}}{n_{i,0}} - \frac{n_{d,1}}{n_{d,0}}\right). \tag{4.68}$$

In the followings this approximation will be referred to as the "*local equilibrium approximation*" (LEA). Similar arguments to LEA were used by Ma and Liu in their study of the dust-acoustic waves [23, 24].

Further on using the local equilibrium approximation, the linear dust acoustic and dust ion-acoustic waves will be discussed and then, extending these arguments also in higher orders, the nonlinear (solitary) dust-acoustic waves will be studied.

IV.4.1. Dust-Acoustic Waves

To study the linear dust acoustic waves, the equations (4.58)-(4.61) need to be linearized around the equilibrium values. Writing $n_e = n_{e,0} + n_{e,1}$, $n_i = n_{i,0} + n_{i,1}$, $n_d = n_{d,0} + n_{d,1}$ and $Z_d = Z_{d,0} + Z_{d,1}$, one gets

$$\begin{aligned} n_{e,1} &= n_{e,0}\frac{e\varphi}{k_B T_e} \\ n_{i,1} &= -n_{i,0}\frac{e\varphi}{k_B T_i} \\ \frac{\partial n_{d,1}}{\partial t} &+ n_{d,0}\nabla\vec{v_d} = 0 \\ m_d\frac{\partial \vec{v_d}}{\partial t} &= eZ_{d,0}\nabla\varphi - \frac{k_B T_d}{n_{d,0}}\nabla n_{d,1} \\ \nabla^2\varphi &= \frac{e}{\varepsilon_0}\left(n_{e,1} - n_{i,1} + Z_{d,0}n_{d,1} + Z_{d,1}n_{d,0}\right) \end{aligned} \tag{4.69}$$

Using (4.68) and the first two equations (4.69), the Poisson equation writes

$$\nabla^2\varphi = \frac{e}{\varepsilon_0}\left[\left(1 - \alpha Z_{d,0}\frac{n_{d,0}}{n_{i,0}} + \frac{1}{\tau}\frac{n_{e,0}}{n_{i,0}}\right)\frac{en_{i,0}}{k_B T_i}\varphi + Z_{d,0}(1-\alpha)n_{d,1}\right] \tag{4.70}$$

Looking for plane wave solutions

$$\begin{aligned} \vec{v_d} &= \vec{V}\exp\left[i(\vec{k}\cdot\vec{r} - \omega t)\right] \\ n_{d,1} &= N\exp\left[i(\vec{k}\cdot\vec{r} - \omega t)\right] \\ \varphi &= \Phi\exp\left[i(\vec{k}\cdot\vec{r} - \omega t)\right] \end{aligned} \tag{4.71}$$

from the other equations (4.69), one gets

$$\begin{aligned} &\vec{k}\cdot\vec{V} = \omega\frac{N}{n_{d,0}} \\ &m_d\omega\left(\vec{k}\cdot\vec{V}\right) = -eZ_{d,0}k^2\Phi + \frac{k_B T_d}{n_{d,0}}k^2 N \\ &-k^2\Phi = \frac{e}{\varepsilon_0}\left[\left(1 - \alpha Z_{d,0}\frac{n_{d,0}}{n_{i,0}} + \frac{1}{\tau}\frac{n_{e,0}}{n_{i,0}}\right)\frac{en_{i,0}}{k_B T_i}\Phi + Z_{d,0}(1-\alpha)N\right] \end{aligned} \tag{4.72}$$

From the first two equation (4.72), it results

$$\left(\omega^2 - v_{Td}^2k^2\right)N = -\frac{eZ_{d,0}n_{d,0}}{m_d}k^2\Phi, \quad v_{Td} = \left(\frac{k_B T_d}{m_d}\right)^{1/2}$$

and v_{Td} is the thermal speed of the dust particle fluid. Introducing this into the third equation (4.72), one finds

$$-k^2 = \frac{1}{\lambda_{Di}^2}\left(1 - \alpha Z_{d,0}\frac{n_{d,0}}{n_{i,0}} + \frac{1}{\tau}\frac{n_{e,0}}{n_{i,0}}\right) - (1-\alpha)\frac{\omega_{pd}^2 k^2}{\omega^2 - v_{Td}^2 k^2}$$

(λ_{Di} is the ionic Debye length) and the following dispersion relation is obtained

$$\omega^2 = v_{Td}^2 k^2 + (1-\alpha)\frac{\omega_{pd}^2\left(\lambda_{Di}k\right)^2}{1 + \frac{1}{\tau} - \left(\alpha + \frac{1}{\tau}\right) Z_{d,0}\frac{n_{d,0}}{n_{i,0}} + \lambda_{Di}^2 k^2}, \tag{4.73}$$

where the equilibrium neutrality condition was used

$$\frac{n_{e,0}}{n_{i,0}} = 1 - Z_{d,0}\frac{n_{d,0}}{n_{i,0}}$$

The case of constant charge on the dust grain ($Z_d = Z_{d,0}$) is formally obtained from (4.73) by taking $\alpha = 0$

$$\omega^2 = v_{Td}^2 k^2 + \frac{\omega_{pd}^2\left(\lambda_{Di}k\right)^2}{1 + \frac{1}{\tau}\left(1 - Z_{d,0}\frac{n_{d,0}}{n_{i,0}}\right) + \left(\lambda_{Di}k\right)^2}, \tag{4.74}$$

Actually $z_1 = 0$ implies that the relative fluctuations of the ionic and dust fluid are the same, $\frac{n_{d,1}}{n_{d,0}} = \frac{n_{i,1}}{n_{i,0}}$, and the result (4.74) is derived omitting in the Poisson equation the term $Z_{d,1}n_{d,0}$.

IV.4.2. Dust Ion-Acoustic Waves

The dust ion-acoustic waves were predicted theoretically by Shukla and Silin in 1992 as oscillation modes in complex plasmas with phase velocity larger than the ion and dust particle thermal speed but much smaller than the electron thermal speed. Consequently, the ions are no longer in equilibrium with the local potential and the ionic fluid is described by a continuity and momentum equation

$$\begin{aligned} &\frac{\partial n_i}{\partial t} + \nabla\left(n_i \vec{v}_i\right) = 0 \\ &m_i\left(\frac{\partial}{\partial t} + \vec{v}_i \cdot \nabla\right)\vec{v}_i = -q_i \nabla\varphi - \frac{k_B T_i}{n_i}\nabla n_i, \end{aligned} \tag{4.75}$$

where v_i is the ionic fluid velocity, m_i the ion mass and T_i the ionic fluid temperature. In case of single ionized positive ions and uni-directional movement, these equation give the following relation for the first order ion number density perturbation

$$\begin{aligned} &\frac{\partial n_{i,1}}{\partial t} + n_{i,0}\nabla\vec{v}_i = 0 \\ &m_i\frac{\partial \vec{v}_i}{\partial t} = -e\nabla\varphi - \frac{k_B T_i}{n_{i,0}}\nabla n_{i,1}, \end{aligned} \tag{4.76}$$

which combined give

$$\left(\frac{\partial^2}{\partial t^2} - v_{Ti}^2\nabla^2\right) n_{i,1} = \frac{n_{i,0}e}{m_i}\nabla^2\varphi \tag{4.77}$$

Dust ion-acoustic waves were observed experimentally by Barkan *et al* (1996) and Nakamura *et al* (1999) having frequencies in the range of tens of kHz [43]. Therefore the Local Equilibrium Approximation theory, presented before, may still hold even if the two orders of magnitude between the characteristic dust particle charging time and the period of the dust ion-acoustic oscillations seem to stress its limits. Besides, due to inertial effects, one may consider that the dust grain fluid isn't perturbed by the propagation of the dust ion-acoustic wave and consequently, in the equations (4.68) and the rest of relations in (4.69) one may take $n_{d,1} \simeq 0$. Finally one obtains

$$\begin{aligned} Z_{d,1} &= \alpha Z_{d,0}\frac{n_{i,1}}{n_{i,0}} \\ n_{e,1} &= n_{e,0}\frac{e\varphi}{k_B T_e} \\ \varepsilon_0\nabla^2\varphi &= e\left(n_{e,1} - n_{i,1} + Z_{d,1}n_{d,0}\right) \end{aligned} \tag{4.78}$$

which, together with (4.77) describe the ion-acoustic wave. Looking for plane wave solutions (4.71), it is easy to derive the following dispersion relation

$$\omega^2 = k^2\omega_{pi}^2\left[\lambda_{Di}^2 + \frac{\lambda_{De}^2}{1+k^2\lambda_{De}^2}\left(1 - \alpha\frac{n_{d,0}Z_{d,0}}{n_{i,0}}\right)\right] \tag{4.79}$$

If the case of constant charge on the dust grain is considered, the term $Z_{d,1}n_{d,0}$ vanishes from the Poisson equation (equivalent to taking $\alpha = 0$) and the dispersion relation writes

$$\omega^2 = k^2\left(v_{Ti}^2 + \frac{c_S^2}{1+k^2\lambda_{De}^2}\right), \tag{4.80}$$

where $c_S = \omega_{pi}\lambda_{De}$. This turns into the same result in [43] (§4.2.2) when the condition $\omega \gg kv_{Ti}, kv_{Te}$ is imposed.

IV.5. Influence of Dust Charge Variation on Dust-Acoustic Solitary Waves

The complexity of the dusty plasma medium arises from considering the dynamical character of the charge collected on the dust grains as it depends on the local properties of the surrounding plasma. To study the influence of the dust charging process on the solitary waves in the (dust) acoustic range, let us consider the simple model of a complex plasma composed from single ionized cations, electrons and one type of spherical, unmagnetizable dust particles of radius r_d, which is isolated from external fields and radiation fluxes. We'll use the *orbit limited motion approximation* to derive the currents of charged particles falling on the dust grains and thus estimate numerically the characteristic charging time τ_c. As this time is in the range of microseconds, for oscillations in the acoustic domain and especially for the associated nonlinear phenomena that manifest at event large time and space scales, a new approximation can be applied: *the local equilibrium approximation*(LEA). It states that the fluctuating charge on the dust grain during wave propagation satisfies the same equation as the neutrality condition obtained for non-isolated grains at equilibrium because of the time scale difference between the particle charging and the wave propagation phenomenon. Further on, the plasma fluids will be considered at the same temperature $T_e = T_i = T_d = T$ and dimensionless variables and quantities will be introduced [13]. Thus the space unit will be the dust fluid Debye length, the time will be scaled in units related to the dust fluid plasmonic frequency (ω_{pd}^{-1}), the dust fluid velocity will be expressed in units of the dust particle thermal velocity v_{Td} and the electrostatic potential in units of $k_B T/e$. The number of elementary charges Z_d will be in units of its equilibrium value Z_{d0} and the number densities of the different plasma constituents n_j $(j = i, e, d)$ in units of their equilibrium values respectively (n_{j0}).

Using the model described above, the electrons and ions will be in equilibrium with the local plasma potential and, together with the dust continuity and momentum equations, the Poisson equation and the local neutrality relation given by LEA, the complex plasma is described by

$$n_e = \exp(\Phi), \quad n_i = \exp(-\Phi) \tag{4.81}$$

$$\frac{\partial n_d}{\partial t} + \frac{\partial}{\partial x}(u_d n_d) = 0 \tag{4.82}$$

$$\frac{\partial u_d}{\partial t} + u_d \frac{\partial u_d}{\partial x} = Z_d \frac{\partial \Phi}{\partial x} \tag{4.83}$$

$$\frac{\partial^2 \Phi}{\partial x^2} = -\mu_i n_i + \mu_e n_e + Z_d n_d, \quad \mu_j = \frac{n_{j0}}{n_{d0} Z_{d0}}, \quad j = e, i \tag{4.84}$$

$$(1 + Y) = \sqrt{\frac{m_i}{m_e}}(1 - PY)\exp(-Y) \tag{4.85}$$

where the local dynamical Y and P are related to their equilibrium values

$$\begin{aligned} Y &= Y_0 Z_d \quad Y_0 = \frac{e^2/r_d}{k_B T} Z_{d0} \\ P &= P_0 \frac{n_d}{n_i} \quad P_0 = \frac{k_B T}{e^2/r_d} \frac{n_{d0}}{n_{i0}} \end{aligned} \tag{4.86}$$

As the effects of nonlinearity manifest at large time and space scales, an appropriate asymptotic method must be used to describe nonlinear phenomena. Such a method is the multiple scales method which is used to construct uniformly valid approximations to the solutions of perturbation problems. It introduces fast or slow-scale variables for independent variables which are then considered as independent. The additional freedom, thus introduced, leads to secular terms when the problem is reformulated that need to be eliminated. The process of elimination puts constraints of the approximate condition, called solvability conditions, which in the case of nonlinear problems lead to hierarchies of evolution equations in increasing orders of approximation.

For the case of the dusty plasma when dust grain charge variation is considered, let us introduce the stretched variables

$$\xi = \varepsilon^{1/2}(x - v_0 t), \quad \tau = \varepsilon^{3/2} t, \tag{4.87}$$

where ε is a small parameter ($\varepsilon \ll 1$) and v_0 is the speed of the space-time frame where the nonlinearity manifests. Then the quantities of interest, n_d, u_d, Φ, Z_d, can be expanded in power series of ε

$$\begin{aligned} n_d &= 1 + \varepsilon n_d^{(1)} + \varepsilon^2 n_d^{(2)} + \ldots \\ u_d &= \quad \varepsilon u_d^{(1)} + \varepsilon^2 u_d^{(2)} + \ldots \\ \Phi &= \quad \varepsilon \Phi^{(1)} + \varepsilon^2 \Phi^{(2)} + \ldots \\ Z_d &= 1 + \varepsilon Z_d^{(1)} + \varepsilon^2 Z_d^{(2)} + \ldots \end{aligned} \tag{4.88}$$

and introducing these expansions into (4.85), one gets

$$\begin{aligned} &Y = Y_0 \left(1 + \varepsilon Z_d^{(1)} + \varepsilon^2 Z_d^{(2)} + \ldots\right), \\ &P = P_0 \left\{1 + \varepsilon \left(n_d^{(1)} - n_i^{(1)}\right) + \varepsilon^2 \left[n_d^{(2)} - n_i^{(2)} + \left(n_i^{(1)}\right)^2 - n_d^{(1)} n_i^{(1)}\right] + \ldots\right\}. \end{aligned} \tag{4.89}$$

In the zero-th order of ε, from the Poisson equation one recovers the neutrality condition at equilibrium

$$\mu_i = 1 + \mu_e. \tag{4.90}$$

The first order constraints allow one to express all the first order quantities with respect to the first order approximation of the local potential and considering (4.85) in the local equilibrium approximation, one gets

$$v_0 = \left[\frac{1 - B}{\mu_i + \mu_e - B}\right]^{1/2}, \quad \frac{1}{B} = 1 + \frac{(2 + Y_0)(1 - P_0 Y_0)}{P_0(1 + Y_0)}, \tag{4.91}$$

which for $B = 0$ becomes the well-known result in the case when the dust particle charge is kept constant [43].

In the order ε^2, expressing the second order quantities with respect to the first order quantities (especially the first order approximation of the local potential) and eliminating them, one can derive the following KdV equation (see [13])

$$\frac{\partial \Phi^{(1)}}{\partial \tau} + \frac{v_0^3}{2(1 - B)} \frac{\partial^3 \Phi^{(1)}}{\partial \xi^3} - M \frac{v_0^3}{2} \Phi^{(1)} \frac{\partial \Phi^{(1)}}{\partial \xi} = 0, \tag{4.92}$$

where

$$\begin{aligned} M &= \frac{3}{v_0^4} - \frac{B}{v_0^2}\left(1 - \frac{1}{v_0^2}\right) - \frac{K}{1-B} \\ K &= 1 + 2B\left[\frac{1}{v_0^2}\left(1 - \frac{1}{v_0^2}\right) + D\right] \\ D &= -\frac{1}{2} + \left(1 - \frac{1}{v_0^2}\right)\left[1 + B\left(1 - \frac{1}{v_0^2}\right)\left(1 + BY_0\frac{(1-P_0Y_0)(3+Y_0)}{2P_0(1+Y_0)}\right)\right] \end{aligned} \tag{4.93}$$

Neglecting the effect of the dust particle charge variation by taking $B = 0$, (4.92) transforms into the result in [43]

$$\begin{gathered} \frac{\partial \Phi^{(1)}}{\partial \tau} - a_s \Phi^{(1)} \frac{\partial \Phi^{(1)}}{\partial \xi} + b_s \frac{\partial^3 \Phi^{(1)}}{\partial \xi^3} = 0 \\ a_s = \frac{v_0^3}{2}\left(\frac{3}{v_0^4} - 1\right) > 0, \qquad b_s = \frac{v_0^3}{2}. \end{gathered} \tag{4.94}$$

In (4.94), $a_s > 0$ because $\mu_i > 1$ and then $v_0 < 1$.

Taking into account the definition of the parameter P_0, (4.91) and the limited range of variation for P_0Y_0 (the ratio of dust charge density to ion charge density) in a dusty plasma, one concludes that the B parameter has a very weak dependence on the dust charge density at equilibrium over a large interval and a tendency to decrease as the dust number density increases. Thus the previous conclusions remain valid when the dust particle charge variation is considered and denoting

$$a_s = \frac{v_0^3}{2} M > 0, \qquad b_s = \frac{v_0^3}{2(1-B)} \tag{4.95}$$

the one soliton solution of (4.92) writes

$$\begin{gathered} \Phi^{(1)}(\xi, \tau) = -\,\Phi_m^{(1)} \operatorname{sech}^2\left[(\xi - u_0\tau)/\Delta\right] \\ \Phi_m^{(1)} = \frac{3u_0}{a_s}, \qquad \Delta = \sqrt{\frac{4b_s}{u_0}} \end{gathered} \tag{4.96}$$

describing a propagating soliton with velocity u_0. As $\Phi^{(1)}(\xi, \tau) < 0$, from multiple scale analysis [13], we have

$$n_d^{(1)}(\xi, \tau) = -\frac{1}{v_0^2}\Phi^{(1)}(\xi, \tau) > 0 \tag{4.97}$$

and the soliton is a compressive solitary wave.

The problem of the influence of dust grain charge fluctuations on the dusty plasma properties was discussed in many papers. Only few examples shall be presented here, namely those related to the influence of dust charge variation on the dust acoustic waves.

In their paper [27] Melandsø *et al.* considered a physical situation close to the complex plasma condition in many planetary rings, when the wave frequency of the DAW excitation is of the same order, or less than the charging frequency of the dust particles ($1/\tau_c$). This is completely different than the basic assumptions of the local equilibrium approximation used in this paper. Taking into account the fluid equation for the charging of dust particles they found, in the first order of approximation

$$\frac{\partial q_{d,1}}{\partial t} = -\Omega_1 q_{d,1} - \Omega_2 \varphi_1,$$

where $q_{d,1}$ and φ_1 are the first order corrections to the equilibrium values of the dust particle charge and local plasma potential, respectively. The effect on the linear DAW is the introduction of an imaginary part of the angular pulsation leading to the damping of the wave. From physical point of view this damping arises because the first order dust charge perturbation is out of phase with the dust number density perturbation provoked by the passing wave. Similar conclusions have been obtained in [17, 51].

Extending these arguments to the nonlinear case, Rao and Shukla [39] found in the case of DAW excitations a KdV equation with a damping term (eq. (34)) describing the evolution of the plasma potential first order approximation. The damping rate almost the same as the damping rate predicted in the linear theory. They emphasize that their result is valid only for time-scales much shorter than the dust charging time.

A different point of view was adopted by Ma and Liu [23, 25], similar to the LEA presented herein. Considering situations when the charging time is several orders of magnitude smaller than the corresponding hydrodynamic time characterizing the DAW ($\tau_c \sim$ ns, $\tau_h \sim \omega_{pd}^{-1} \sim$ 2ms), they assume that the dust charge is determined by the local electrostatic potential and find the influence of dust grain charge variation on a Sagdeev's potential.

A more detailed discussion of the effect of dust grain charge variation on (linear) ion-acoustic waves was done by Vladimirov *et al.* [53] taking into account the ionization and recombination processes in plasma as well.

Several other papers deal with the influence of dust charge variation upon the dusty plasma properties and here [10, 35, 37, 38, 55, 56] is a selected list where more references on the subject can be found.

IV.6. Nonlinear Dust Acoustic Modes in a Dusty Plasma with Dust Grains of Different Sizes

Dust grain material, shape and sizes have a great influence on the behavior of complex plasmas and implicitly on their oscillation modes, especially in the "acoustic" frequency range. In order to study the influence of the dust grain size distribution on the dust-acoustic modes for instance, let use consider a very simple model of a non-magnetizable dusty plasma, composed of isothermal electrons and ions and two species of spherical dust particles of radii r_1 and r_2 and number densities N_1 and N_2. We'll assume that the complex plasma is not exposed to any external fields or radiation fluxes so that the dust grains will charge negatively. At equilibrium $N_{10} = cN_0$, $N_{20} = (1-c)N_0$, N_0 being the constant number density of all the dust particles in the system. The equilibrium charges on the two types of dust particles will be denote by $-eZ_1$ and respectively $-eZ_2$ and their variation will be neglected. According to the previous discussions, the charging process occurs in such a way as to obtain, at equilibrium, the same potential on each dust grain regardless its radius (type). Therefore $Z_1/r_1 = Z_2/r_2 = \langle Z\rangle / \langle r\rangle$, where $\langle r\rangle = cr_1 + (1-c)r_2$, and the equilibrium value $\langle Z\rangle$ is found as the solution of the equation

$$\begin{aligned} \sqrt{\frac{m_e}{m_i}}(1+z) &= \sqrt{\tau}\exp\left(-z/\tau\right)\left(1-Pz\right) \\ z &= \frac{1}{4\pi\varepsilon_0}\frac{\langle Z\rangle e^2}{\langle r\rangle k_B T_i} \\ P &= 4\pi N_0 \langle r\rangle \lambda_{Di}^2 \end{aligned} \tag{4.98}$$

The inertial effects of the electron and ion fluids will be neglected so that one can assume Boltzmann distributions for the electron and ion number densities

$$\begin{aligned} n_e &= n_{e0}\exp\left(\frac{e\varphi}{k_B T_e}\right) \\ n_i &= n_{i0}\exp\left(-\frac{e\varphi}{k_B T_i}\right) \end{aligned} \tag{4.99}$$

φ being the plasma electric potential and T_e, T_i the temperatures of the electron and ion fluid respectively. The motion of the dust fluids is described by the equations

$$\begin{aligned} \frac{\partial N_j}{\partial t} + \frac{\partial}{\partial x}\left(N_j v_j\right) &= 0 \\ m_j\left(\frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x}\right)v_j &= eZ_j\frac{\partial\varphi}{\partial x} \end{aligned} \tag{4.100}$$

where $j = 1, 2$ and v_j are the flowing velocities of the two dust fluids. To complete the suite of relations describing the fluid dynamics, one has to consider the Poisson equation

$$\frac{\partial^2\varphi}{\partial x^2} = \frac{e}{\varepsilon_0}\left(n_e - n_i + Z_1N_1 + Z_2N_2\right) \tag{4.101}$$

and the neutrality condition at equilibrium

$$n_{e0} + Z_1N_{10} + Z_2N_{20} = n_{i0} \tag{4.102}$$

which can be written as

$$\begin{aligned} \frac{n_{e0}}{n_{i0}} &= 1 - \langle Z \rangle \frac{N_0}{n_{i0}} \\ \langle Z \rangle &= cZ_1 + (1-c)Z_2 \end{aligned} \tag{4.103}$$

In formulating these expressions only one-dimensional movement was considered and in the equations of motion the fluid pressure and other forces were neglected. For simplicity the same temperature will be taken for all the plasma components $T_e = T_i = T_d = T$.

As in the previous calculations, it is convenient to use dimensionless variables by considering appropriate units. Therefore the space coordinate will be expressed in units of the Debye screening length $\lambda_d^2 = \frac{\varepsilon_0 k_B T}{e^2 \langle Z \rangle N_0}$, the time will be measured in units ω_{pd}^{-1}, the inverse of the dust fluid plasmonic frequency $\omega_{pd}^2 = \frac{e^2 N_0 \langle Z \rangle^2}{\varepsilon_0 \langle m \rangle}$, where $\langle m \rangle = cm_1 + (1-c)m_2$, the fluid velocities in units of the dust fluid mixture thermal speed $v_{Td}^2 = \frac{k_B T \langle Z \rangle}{\langle m \rangle}$, $v_{Td} = \lambda_d \omega_{pd}$, the electrostatic potentials in units of $k_B T/e$, the thermal kinetic energy in eV, and the plasma species number densities n_e, n_i, N_j, $(j = 1, 2)$ in units of their respective equilibrium values n_{e0}, n_{i0}, N_0. Then the equations (4.99)-(4.103) write

$$n_e = \exp(\varphi), \quad n_i = \exp(-\varphi) \tag{4.104}$$

$$\left.\begin{aligned} &\frac{\partial N_j}{\partial t} + \frac{\partial}{\partial x}(N_j v_j) = 0 \\ &\left(\frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x}\right) v_j = Q_j \frac{\partial \varphi}{\partial x} \end{aligned}\right. \quad j = 1, 2 \tag{4.105}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \mu_e n_e - \mu_i n_i + \zeta_1 N_1 + \zeta_2 N_2 \tag{4.106}$$

$$\mu_i = \mu_e + 1 \tag{4.107}$$

where the following notations were introduced

$$\begin{aligned} &\mu_k = \frac{n_{k0}}{N_0 \langle Z \rangle}, \qquad k = (e, i) \\ &\zeta_1 = \frac{cZ_1}{\langle Z \rangle}, \ \zeta_2 = \frac{(1-c)Z_2}{\langle Z \rangle}, \ \zeta_1 + \zeta_2 = 1, \\ &Q_j = \frac{\langle m \rangle Z_j}{\langle Z \rangle m_j}, \ j = 1, 2 \end{aligned} \tag{4.108}$$

To study the effect of the nonlinearities, a multiple scale analysis will be performed using the stretched variables

$$\xi = \varepsilon^{1/2}(x - u_0 t), \quad \tau = \varepsilon^{3/2} t, \tag{4.109}$$

where u_0 is the speed of the space-time frame where the nonlinearity manifests. Expanding n_e, n_i, N_j, v_j, φ in power series in ε

$$\begin{aligned} n_k &= 1 + \varepsilon n_k^{(1)} + \varepsilon^2 n_k^{(2)} + \dots \quad (k = e, i) \\ N_j &= 1 + \varepsilon N_j^{(1)} + \varepsilon^2 N_j^{(2)} + \dots, \\ v_j &= \varepsilon v_j^{(1)} + \varepsilon^2 v_j^{(2)} + \dots \quad (j = 1, 2) \\ \varphi &= \varepsilon \varphi^{(1)} + \varepsilon^2 \varphi^{(2)} + \dots \end{aligned} \tag{4.110}$$

in the zero-th order from the Poisson equation one recovers the neutrality condition (4.107). In the following two orders, the equations (4.104) give

$$\begin{aligned} n_e^{(1)} = \varphi^{(1)}, \quad n_i^{(1)} = -\varphi^{(1)}, \\ n_e^{(2)} = \varphi^{(2)} + \frac{1}{2}\left(\varphi^{(1)}\right)^2, \quad n_i^{(2)} = -\varphi^{(2)} + \frac{1}{2}\left(\varphi^{(1)}\right)^2, \end{aligned} \tag{4.111}$$

which used in the Poisson equation (4.106) lead to

$$\zeta_1 N_1^{(1)} + \zeta_2 N_2^{(1)} + (\mu_i + \mu_e)\varphi^{(1)} = 0 \tag{4.112}$$

$$\frac{\partial^2 \varphi^{(1)}}{\partial \xi^2} = (\mu_i + \mu_e)\varphi^{(2)} - \frac{1}{2}(\mu_i - \mu_e)\left(\varphi^{(1)}\right)^2 + \zeta_1 N_1^{(2)} + \zeta_2 N_2^{(2)} \tag{4.113}$$

in order ε and ε^2 respectively. For the same orders of the small expansion parameter, from the continuity equations in (4.105) one gets

$$\begin{aligned} v_j^{(1)} = u_0 N_j^{(1)}, \\ -u_0 \frac{\partial N_j^{(2)}}{\partial \xi} + \frac{\partial N_j^{(1)}}{\partial \tau} + \frac{\partial\left(N_j^{(1)} v_j^{(1)}\right)}{\partial \xi} + \frac{\partial v_j^{(2)}}{\partial \xi} = 0, \end{aligned} \tag{4.114}$$

while from the equations of motion one obtains, successively

$$\begin{aligned} Q_j \varphi^{(1)} = -u_0 v_j^{(1)}, \\ Q_j \frac{\partial \varphi^{(2)}}{\partial \xi} = -u_0 \frac{\partial v_j^{(2)}}{\partial \xi} + \frac{v_j^{(1)}}{\partial \tau} + v_j^{(1)} \frac{\partial v_j^{(1)}}{\partial \xi}. \end{aligned} \tag{4.115}$$

The first equations in (4.114) and (4.115) allow us to express the first order approximations $N_j^{(1)}$ and $v_j^{(1)}$ with respect to $\varphi^{(1)}$ and using these new expressions for $N_j^{(1)}$ in (4.112) the following relation is obtained for the velocity u_0

$$u_0^2 = \frac{\zeta_1 Q_1 + \zeta_2 Q_2}{\mu_i + \mu_e}. \tag{4.116}$$

Combining the second equations in (4.114) and (4.115) and eliminating $\partial v_j^{(2)}/\partial \xi$, the following relation results

$$\frac{\partial N_j^{(2)}}{\partial \xi} = -\frac{Q_j}{u_0^2}\frac{\partial \varphi^{(2)}}{\partial \xi} - 2\frac{Q_j}{u_0^3}\frac{\partial \varphi^{(1)}}{\partial \tau} + \frac{3Q_j^2}{u_0^4}\varphi^{(1)}\frac{\partial \varphi^{(1)}}{\partial \xi}. \tag{4.117}$$

Derivating (4.113) with respect to ξ and using (4.117) to eliminate the second order approximations, one derives the following Korteweg-de Vries equation

$$\frac{\partial\varphi^{(1)}}{\partial\tau} - a_s\varphi^{(1)}\frac{\partial\varphi^{(1)}}{\partial\xi} + b_s\frac{\partial^3\varphi^{(1)}}{\partial\xi^3} = 0,$$
$$b_s = \frac{u_0^3}{2(\zeta_1 Q_1 + \zeta_2 Q_2)}, \tag{4.118}$$
$$a_s = \frac{u_0^3}{2(\zeta_1 Q_1 + \zeta_2 Q_2)}\left(\frac{3(\zeta_1 Q_1^2 + \zeta_2 Q_2^2)}{u_0^4} - 1\right) > 0.$$

If one considers a single type of dust grains, the well-known result (4.110) is recovered (with v_0 being replaced by u_0).

To study the effect of the dust grain size in a comparison with the case of one dust species complex plasma, let us consider that the first type of dust grains has a larger radius $r_1 > r_2$ which consequently means a larger mass $m_1 > m_2$ (if one takes the dust grains made of the same material) and a larger equilibrium charge $Z_1 > Z_2$ (since at equilibrium $Z_1/r_1 = Z_2/r_2$). Let's denote the ratio of the radii $\delta = \frac{r1}{r2} \geq 1$. Then

$$\zeta_1(c) = c\delta/(c\delta + 1 - c),$$
$$\zeta_2(c) = (1 - c)/(c\delta + 1 - c)$$

will variate between $\zeta_1 = 0$, $\zeta_2 = 1$ when $c = 0$ and $\zeta_1 = 1$, $\zeta_2 = 0$ for $c = 1$ as in the figure IV.10. Besides $Z_1(c) = \delta Z_2(c)$, $m_1(c) = \delta^3 m_2(c)$ and

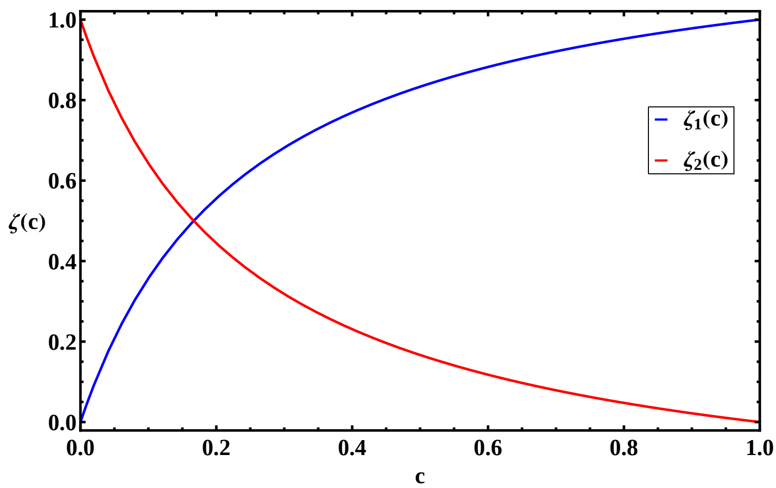

Figure IV.10: The dependence ζ_1 and ζ_2 on the concentration c for a complex plasma with $\delta = 5$.

consequently $Q_1(c) = Q_2(c)/\delta^2$ where $Q_2(c) = (c\delta^3 + 1 - c)/(c\delta + 1 - c)$. The parameters Q_1 and Q_2 are in fact the specific charge of each type of dust grains normalized to the mean value of the specific charge of dust fluid particles. As ζ_1, ζ_2 are always positive and less than unity but satisfying the relation $\zeta_1 + \zeta_2 = 1$, they can be interpreted as probabilities. Then the following mean values can be introduced

$$\langle Q \rangle_\zeta = \zeta_1 Q_1 + \zeta_2 Q_2$$
$$\langle Q^2 \rangle_\zeta = \zeta_1 Q_1^2 + \zeta_2 Q_2^2,$$

which for the limiting situation when only one type of dust grains is present ($c = 0$, $c = 1$), are equal to 1. Considering the definitions (4.108) and (4.98), for a given concentration c, one gets the following relation

$$\mu_i + \mu_e = 2\frac{n_{i,0}}{N_0 \langle Z \rangle} - 1 = \frac{2}{Pz} - 1.$$

By solving numerically the charge neutrality equation in the local equilibrium approximation, the parameter u_0 can be determined as a function of c and consequently the dependence of the coefficients in the KdV equation (4.118), a_s, b_s, on the concentration of larger dust grain is given in figure IV.11. It is obvious that none of the coefficients is zero or changes sign.

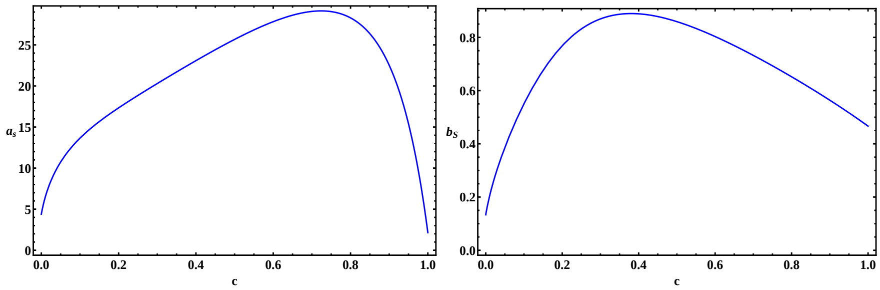

Figure IV.11: The dependence $a_s(c)$ (*left*) and $b_s(c)$ (*right*) for a complex plasma with $\delta = 5$.

In constructing the graphics IV.10 and IV.11, a complex plasma with $\delta = 5$ was considered in which the smaller dust grains of radius $r_2 = 1\mu$m where gradually substituted by larger ones (c is the concentration of dust type with larger radius) so that the ratio of dust particles (regardless their size) to ions is kept at $n_{i,0}/N_0 = 10^{-4}$ and the electronic and ionic fluids have the same temperature $T_e = T_i = T = 3 \times 10^4$K.

A simple solitary wave solution of equation (4.118) (the 1-soliton solution) is easily found assuming that $\varphi^{(1)}(\xi, \tau)$ depends only on $X = \xi - V_0\tau$, where V_0 is the soliton velocity. Then integrating twice the differential equation with respect to the new variable X with vanishing conditions at infinity, one obtains

$$\left(\frac{\mathrm{d}\varphi^{(1)}}{\mathrm{d}X}\right)^2 = \frac{V_0}{b_s}\left(\varphi^{(1)}\right)^2 + \frac{a_s}{3}\left(\varphi^{(1)}\right)^3$$

Looking for a solution of the form

$$\varphi^{(1)} = \varphi_m \,\mathrm{sech}^2\, X/\Delta$$

one gets

$$\Delta = 2\sqrt{\frac{b_s}{V_0}}, \quad \varphi_m = -\frac{3V_0}{a_s}$$
$$\varphi^{(1)}(\xi - V_0\tau) = -\frac{3V_0}{a_s}\frac{1}{\cosh^2\left[\frac{1}{2}\sqrt{\frac{V_0}{b_s}}(\xi - V_0\tau)\right]} \tag{4.119}$$

The DAW soliton is always associated with a negative plasma potential and since from (4.76), (4.77), one gets $n_j^{(1)} = -Q_j/u_0^2\varphi^{(1)}$, it is a compressive wave.

IV.7. Dust Ion-Acoustic Solitons in a Dusty Plasma with Positive and Negative Ions

More than twenty years ago it was shown that a small amount of light negative ions affects strongly the plasma behavior, namely that there is a critical concentration of the negative ions for which the equation describing the propagation of a nonlinear wave (soliton) is the modified Korteweg-de Vries equation, and not the usual KdV one [9, 18, 52, 54]. The same problem will be discussed here for a dusty plasma, then, in the next section, the basic equation describing the dusty plasma will be presented. In the following, the non-critical and critical case will be investigated.

Let us consider an unmagnetized dusty plasma composed of single ionized positive and light, negative ions, isothermal electrons and negatively charged dust particles [14]. Due to the mass difference and frequency range of the studied waves, the dust particles will be considered at rest and will consequently only contribute to the equilibrium neutrality condition. We'll denote the equilibrium number densities by $n_i^{(0)}$ $(i =, e, +, -, d)$ and the $+, -$ subscripts will be used for quantities characterizing the positive, respectively the negative ions. For simplicity we'll consider that $T_+ = T_- \simeq T_e$ and the ratio of the negative ion mass to the positive ion mass will be denoted by $Q = m_-/m_+$. Also adimensional variables will be used so that the distances will be measured in units of plasma Debye length λ_D, the time in units of ω_p^{-1}, where ω_p is the plasmonic frequency of the predominant, positive ions, the potential in units of k_BT_e/e, the velocities in units of the thermal speed of the positive ions and the densities of the plasma species in units of the positive ion density at equilibrium $n_+^{(0)}$.

In a hydrodynamic description the positive and negative ion fluids will satisfy each a continuity equation and the following equations of motion

$$\frac{\partial u_+}{\partial t} + u_+\frac{\partial u_+}{\partial x} = -\frac{\partial \Phi}{\partial x},$$
$$\frac{\partial u_-}{\partial t} + u_-\frac{\partial u_-}{\partial x} = -\frac{1}{Q}\frac{\partial \Phi}{\partial x}, \tag{4.120}$$

where Φ is the local plasma potential, $u_{+,-}$ denote the positive and negative ion fluid velocities. Only the electrical forces have been taken into account, the others, such as pressure forces, being neglected. The electrons are considered in equilibrium with the local potential, therefore

$$n_e = n_e^{(0)} \exp(\Phi). \tag{4.121}$$

To these equations one has to attach the Poisson equation

$$\frac{\partial^2 \Phi}{\partial x^2} = n_e + n_- + Z_d n_d^{(0)} - n_+ \tag{4.122}$$

and the complex plasma neutrality condition

$$1 = n_e^{(0)} + n_-^{(0)} + Z_d n_d^{(0)}, \tag{4.123}$$

which introduce the contribution of the dust grains into the problem. In both equations (4.122), (4.123), the product $Z_d n_d^{(0)}$ can be regarded as the average value of a certain size distribution of dust particles like in the previous section.

The effects of nonlinearities are cumulative in space and time, therefore appropriate asymptotic methods need to be used in order to study them. One of these is the multiple scales method which introduces stretched space and time variables and expansions of the important quantities in series of a small parameter ε which determines these new variables (see [14]). Let us consider the stretched variables of the form

$$\begin{aligned} \xi &= \varepsilon^{1/2}(x - vt), \\ \tau &= \varepsilon^{3/2} t, \end{aligned} \tag{4.124}$$

where v is the velocity of the space frame where the nonlinearity develops. Then, the complex plasma fluid quantities are expanded in power series

$$\begin{aligned} n_+ &= 1 + \varepsilon n_+^{(1)} + \varepsilon^2 n_+^{(2)} + \dots & u_+ &= \varepsilon u_+^{(1)} + \varepsilon^2 u_+^{(2)} + \dots \\ n_- &= n_-^{(0)} + \varepsilon n_-^{(1)} + \varepsilon^2 n_-^{(2)} + \dots & u_- &= \varepsilon u_-^{(1)} + \varepsilon^2 u_-^{(2)} + \dots \\ n_e &= n_e^{(0)} + \varepsilon n_e^{(1)} + \varepsilon^2 n_e^{(2)} + \dots & \Phi &= \varepsilon \Phi^{(1)} + \varepsilon^2 \Phi^{(2)} + \dots, \end{aligned} \tag{4.125}$$

satisfying the boundary conditions $|x| \to \infty$. Introducing the expansions (4.125) and using the stretched variables (4.124) into the system of equations that describe the plasma fluids, it has to be satisfied in each order of ε. In the zero-th order the neutrality condition (4.123) is recovered, while in the first order one gets

$$n_e^{(1)} + n_-^{(1)} - n_+^{(1)} = 0 \tag{4.126}$$

$$n_e^{(1)} = n_e^{(0)} \Phi^{(1)} \tag{4.127}$$

$$u_+^{(1)} = \frac{1}{v} \Phi^{(1)}, \quad u_-^{(1)} = -\frac{1}{Qv} \Phi^{(1)} \tag{4.128}$$

$$n_+^{(1)} = \frac{1}{v^2} \Phi^{(1)}, \quad n_-^{(1)} = -\frac{n_-^{(0)}}{Qv^2} \Phi^{(1)} \tag{4.129}$$

Using (4.127) and (4.129) in (4.126) an expression for the velocity v is obtained

$$\frac{1}{v^2} = \frac{n_e^{(0)}}{1 + \dfrac{n_-^{(0)}}{Q}}, \tag{4.130}$$

as a function of $n_-^{(0)}$, Q and $Z_d n_d^{(0)}$ (if one considers the neutrality condition at equilibrium to replace $n_e^{(0)}$).

In the second order of the expansion, a series of relations are obtained and the second order approximations of the plasma parameters can be eliminated from them using the above relations (4.126)-(4.129) [14]. Finally, all the quantities are related to $\Phi^{(1)}$ which has to satisfy the following Korteweg-de Vries (KdV) equation

$$\frac{2}{v^3}\left(1 + \frac{n_-^{(0)}}{Q}\right)\frac{\partial \Phi^{(1)}}{\partial \tau} + \frac{\partial^3 \Phi^{(1)}}{\partial \xi^3} + \\ + \left[\frac{3}{v^4}\left(1 - \frac{n_-^{(0)}}{Q^2}\right) - \left(1 - z_d n_d^{(0)} - n_-^{(0)}\right)\right]\Phi^{(1)}\frac{\partial \Phi^{(1)}}{\partial \xi} = 0. \tag{4.131}$$

It is easily seen that there is a critical concentration of negative ions at equilibrium $n_-^{(0)}$ for which the coefficient of the nonlinear term in the KdV equation (4.131) vanishes. Its value can be obtained by solving a second order algebraic equation

$$\left(n_-^{(0)}\right)^2 - \left[\frac{3}{2}\left(1 - Z_d n_d^{(0)} + Q^2\right) + Q\right] n_-^{(0)} + \left(1 - \frac{3}{2} Z_d n_d^{(0)}\right) Q^2 = 0, \tag{4.132}$$

Its discriminant is always positive so the number of positive roots depends on the sign of the free term. If $Z_d n_d^{(0)} < 2/3$ there are two positive roots from which only one is physically acceptable as it satisfies the inequality $n_-^{(0)} < 1 - Z_d n_d^{(0)}$ imposed by the charge neutrality condition (4.123). For small values of Q, light negative ions, the critical negative ion density may be approximated as

$$n_-^{(crit)} = \frac{\frac{2}{3} - Z_d n_d^{(0)}}{1 - Z_d n_d^{(0)}} Q^2 + \mathcal{O}(Q^3). \tag{4.133}$$

The value $Z_d n_d^{(0)} = \frac{2}{3}$ is actually the value of the dust charge density for which the critical condition is realized (the nonlinear term vanishes) in the absence of negative ions while for higher values the condition is no longer satisfied. Therefore the critical condition is obtained, in fact, for small concentrations of light negative ions but only when $Z_d n_d^{(0)} < \frac{2}{3}$ and this is a new feature of the phenomenon induced by the presence of dust grains in plasma. Let us introduce the following notation for the coefficient of the nonlinear term in the KdV equation (4.131)

$$\gamma = \frac{3}{v^4}\left(1 - \frac{n_-^{(0)}}{Q^2}\right) - n_e^{(0)}. \tag{4.134}$$

When $n_-^{(0)} < n_-^{(crit)}$, $\gamma > 0$ and the KdV equation has compressing soliton solution, while if $n_-^{(0)} > n_-^{(crit)}$, $\gamma < 0$ and one has rarefracting solitons as solutions of (4.131). Since the critical negative ion density has small values, the domain where compressing solitons may appear is very narrow. The equation has only rarefracting soliton solutions when $Z_d n_d^{(0)} > \frac{2}{3}$.

When the concentration of the negative ions is around the critical value, the analysis above no longer applies so higher order nonlinear effects have to be considered. In this "critical" region, the stretched variables that will be used are [14]

$$\xi = \varepsilon(x - vt), \quad \tau = \varepsilon^3 t \tag{4.135}$$

Applying the same algorithm as before but using the stretched variables (4.135), one finds the neutrality condition in the zero-th order. In the next order, corresponding to ε^2, one recovers the equations (4.126)-(4.130) and in the following order the equations for the second order approximations of the quantities write

$$n_e^{(2)} + n_-^{(2)} - n_+^{(2)} = 0 \tag{4.136}$$

$$n_e^{(2)} = n_e^{(0)}\left[\Phi^{(2)} + \frac{1}{2}\left(\Phi^{(1)}\right)^2\right] \tag{4.137}$$

$$\begin{aligned} -v\frac{\partial n_+^{(2)}}{\partial \xi} + \frac{\partial u_+^{(2)}}{\partial \xi} + \frac{\partial\left(n_+^{(1)}u_+^{(1)}\right)}{\partial \xi} &= 0 \\ -v\frac{\partial n_-^{(2)}}{\partial \xi} + n_-^{(0)}\frac{\partial u_-^{(2)}}{\partial \xi} + \frac{\partial\left(n_-^{(1)}u_-^{(1)}\right)}{\partial \xi} &= 0 \end{aligned} \tag{4.138}$$

$$\begin{aligned} v\frac{\partial u_+^{(2)}}{\partial \xi} - u_+^{(1)}\frac{\partial u_+^{(1)}}{\partial \xi} &= \frac{\partial \Phi^{(2)}}{\partial \xi} \\ v\frac{\partial u_-^{(2)}}{\partial \xi} - u_-^{(1)}\frac{\partial u_-^{(1)}}{\partial \xi} &= -\frac{1}{Q}\frac{\partial \Phi^{(2)}}{\partial \xi} \end{aligned} \tag{4.139}$$

Integrating (4.138) and (4.139) and expressing the first order approximations of the quantities in terms of the first order approximation of the potential using (4.128), (4.129), the second order quantities are expressed in terms of

$\Phi^{(1)}$ and $\Phi^{(2)}$

$$\begin{aligned} n_+^{(2)} &= \frac{3}{2v^4}\left(\Phi^{(1)}\right)^2 + \frac{1}{v^2}\Phi^{(2)} \\ n_-^{(2)} &= \frac{3n_-^{(0)}}{2Q^2v^4}\left(\Phi^{(1)}\right)^2 - \frac{n_-^{(0)}}{Qv^2}\Phi^{(2)} \\ u_+^{(2)} &= \frac{1}{2v^3}\left(\Phi^{(1)}\right)^2 + \frac{1}{v}\Phi^{(2)} \\ u_-^{(2)} &= \frac{1}{2Q^2v^3}\left(\Phi^{(1)}\right)^2 - \frac{1}{Qv}\Phi^{(2)} \end{aligned} \tag{4.140}$$

It is easy to verify that (4.136) is satisfied if $n_-^{(0)}$ is the critical negative ion concentration (solution of (4.132)), therefore from here on $n_-^{(0)} = n_-^{(crit)}$. In the next order of ε one gets

$$\begin{aligned} & -v\frac{\partial n_+^{(3)}}{\partial \xi} + \frac{\partial n_+^{(1)}}{\partial \tau} + \frac{\partial}{\partial \xi}\left[u_+^{(3)} + n_+^{(1)}u_+^{(2)} + n_+^{(2)}u_+^{(1)}\right] = 0 \\ & -v\frac{\partial n_-^{(3)}}{\partial \xi} + \frac{\partial n_-^{(1)}}{\partial \tau} + \frac{\partial}{\partial \xi}\left[n_-^{(0)}u_-^{(3)} + n_-^{(1)}u_-^{(2)} + n_-^{(2)}u_-^{(1)}\right] = 0 \\ & -v\frac{\partial u_+^{(3)}}{\partial \xi} + \frac{\partial u_+^{(1)}}{\partial \tau} + \frac{\partial}{\partial \xi}\left(u_+^{(1)}u_+^{(2)}\right) = -\frac{\partial \Phi^{(3)}}{\partial \xi} \\ & -v\frac{\partial u_-^{(3)}}{\partial \xi} + \frac{\partial u_-^{(1)}}{\partial \tau} + \frac{\partial}{\partial \xi}\left(u_-^{(1)}u_-^{(2)}\right) = \frac{1}{Q}\frac{\partial \Phi^{(3)}}{\partial \xi} \\ & n_e^{(3)} = n_e^{(0)}\left[\Phi^{(3)} + \Phi^{(1)}\Phi^{(2)} + \frac{1}{6}\left(\Phi^{(1)}\right)^3\right] \\ & \frac{\partial^2 \Phi^{(1)}}{\partial \xi^2} = n_e^{(3)} + n_-^{(3)} - n_+^{(3)} \end{aligned} \tag{4.141}$$

The third order approximation of the physical parameters can be eliminated from the set of equations above and using the expressions (4.126) - (4.130) for the first order and (4.140) for the second order quantities, after careful calculations, one finds a modified Korteweg-de Vries equation satisfied by $\Phi^{(1)}$

$$\begin{aligned} & \frac{2}{v^2}\left(1 + \frac{n_-^{(0)}}{Q}\right)\frac{\partial \Phi^{(1)}}{\partial \tau} + \frac{\partial^3 \Phi^{(1)}}{\partial \xi^3} + \\ & + \left[\frac{15}{2v^6}\left(1 + \frac{n_-^{(0)}}{Q^3}\right) - \frac{1}{2}\left(1 - Z_d n_d^{(0)} - n_-^{(0)}\right)\right]\left(\Phi^{(1)}\right)^2\frac{\partial \Phi^{(1)}}{\partial \xi} = 0 \end{aligned} \tag{4.142}$$

Thus the final results correspond to the initial expectations. The novelty of the present calculations is that the soliton parameters (the coefficients in the equations) depend on the dust charge density and the negative ion mass to positive ion mass ratio. If $Q < 1$ and only if the dimensionless dust charge density is 2/3, one can find a critical region for small concentrations of light, negative ions. This critical region is the boundary between the region where the KdV equation for the potential has rarefracting soliton solutions (and the coefficient of the nonlinear term is negative, $\gamma < 0$) from the region with compressing soliton solutions ($\gamma > 0$). Effectively, the region with compressing solitons is found to be in a very narrow range of the negative ion number density.

Finally, one should note that the previous results were obtained in the situation when the electron and ion fluids have the same temperature and the charge on the dust grain is considered constant. An improvement of this outcome will be to take different ionic and electronic temperatures ($T_+ = T_- \neq T_e$) which is not expected to influence the complete integrability of the final result in any way. Considering the charge on the dust particle as a dynamic variable will also lead to results with better physical meaning.

IV.8. Conclusions and Perspectives

In this section the main results of the author will be briefly summarized and the directions of further investigations will be pointed out.

Recognizing the importance of the dust grain charging process for complex plasma properties, a detailed discussion on this subject was presented. Though limited to the orbit limited motion approximation, the effect of higher electron temperatures in the charging of isolated dust particles was emphasized. when the dust number density $n_{d,0}$ increases, the electrons' influence diminishes and the charge on the dust grain decreases asymptotically to the same value regardless of the electron gas temperature. This is due to electron gas depletion as more and more of them are caught on the surface of the dust grains and the increasing electronic temperature favors this phenomenon.

An important parameter, useful in analyzing different processes in a dusty plasma, is the charging time, defined as the time needed for the charge on a dust grain to return to the equilibrium value after a small fluctuation. It was shown that for usual plasma conditions in laboratory experiments, this parameter is of the order of microseconds. Using this fact, and only in these conditions, the effect of charge variation on DAW and DIAW was investigated. As characteristic times for dust-acoustic and dust ion-acoustic waves are of order 10^{-2} s and 10^{-4} s respectively, it is reasonable to assume that the charge on any dust particle has enough time yo adapt itself to the local plasma conditions (local plasma potential) during the propagation of such oscillations through the medium. This approximation was called "*local equilibrium approximation*" (LEA), and using it the fluctuations of the dust grain charge were expressed in terms of the fluctuations of the plasma quantities. As mentioned in the text these conditions ($\omega_{pd}, \omega_{pi} \gg 1/\tau_c$) are not fulfilled in spatial conditions (planetary rings for instance), where the opposite situation is realized, and therefore the equation (4.28) has to be employed in determining the charge variation. The effect of the charge variation on DAW properties is different in these two approximations, namely using LEA one gets a shift of the dust acoustic wave characteristics while in the other limit (in cosmic space conditions) the effect is a damping of the wave.

In the local equilibrium approximation, the effect of the dust grain charge variation on the linear and nonlinear dust acoustic waves and on the linear dust ion-acoustic oscillations was determined. The effect is a shift in the parameters of the DAW excitations.

In order to observe the influence of the dust fluid composition on DAW, a model with two types of spherical grains of different radii ($r_1 > r_2$) and same material(s) was considered. Applying a multiple scale analysis method, a Korteweg-de Vries equation was derived for the local plasma potential perturbation with its coefficients depending on the concentration of the two components and on the ratio $\delta = r_1/r_2$. As a general rule, the 1-soliton solution has a decreasing amplitude as it grows wider up to a certain dust mixture composition and it narrows as one of the dust types becomes the majority while its amplitude approaches the value corresponding to a complex plasma with only one type of dust grains (the predominant one, see figure IV.11).

Finally, a dusty plasma with a small amount of light negative ions was considered. As expected from previous plasma research [14], a critical concentration of negative ions exists for which the coefficient of the nonlinear term of the resulting KdV equation vanishes. In the near vicinity of this critical concentration, new stretched variables must be used and the multiple scale analysis leads, this time, to a modified Korteweg-de Vries equation (mKdV). The dependence of the coefficients, in both equations, on the dusty plasma characteristic parameters is also a novelty in the results of this investigation.

Several perspectives to continue and improve the previous results can be envisaged.

First, a better analysis of the charging process taking into account a better approximation of the electron and ion scattering in the screened field of the negative charged dust grains is necessary. Also the influence of trapped ions in the negative field of the dust grain should be studied. As a result a better approximation for the equilibrium charge and the charging time will result.

The influence of several species of dust particles of various radii on the dust acoustic wave properties, and not only, is easily achievable by extending the calculations in §IV.6 or applying similar methods to other problems. Also a continuous distribution of dust grain radii deserves to be investigated as well as taking dust grains of shapes other than spherical.

A far-off perspective could be the extension of the linear kinetic theory developed by Tsytovich and De Angelis to the nonlinear regime.

The original results presented in the current chapter are published in:

a. "*Dust Acoustic Solitons in a Dusty Plasma with Dust Particle Charge Variation*" **A. Grecu**, D. Grecu, *Annals of the University of Craiova: Physics*, **18**, pp. 178–187 (2008).

b. "*Nonlinear Excitations in a Dusty Plasma with Dust Particle Charge Variation*" **A. T. Grecu**, D. Grecu, *Romanian Journal of Physics*, **53**(9-10), pp. 1131–1138 (2008).

c. "*Dust Ion Acoustic Solitons in a Dusty Plasma with Positive and Negative Ions*" D. Grecu, **A. T. Grecu**, *Journal of Optoelectronics and Advanced Materials*, **10**(1), pp. 80–84 (2008).

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