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Hadronization of the quark-gluon plasma

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Abstract

The quark-gluon plasma formed in atomic nuclei by high-energy nuclear collisions is analyzed through its various stages of development. The threshold energy for igniting the nuclear quark-gluon plasma is derived, the subsequent expansion and cooling of the plasma are described, and the condensation mechanism of the quarks into hadrons is presented. It is shown that the hadronization process is a phase transition of the first kind, dominated by hadrons with the simplest structure. The transition temperature is derived, and the phase transition is characterized. A few introductory notes are given, concerning the excitation of heavy atomic nuclei, and **Appendixes** are included, of relevance on these matters.

The Generic Nucleus. The nucleon in the atomic nuclei has a radius $a = 1.5\times 10^{-15}m$ $(= 1.5fm)$ and an average binding energy $\varepsilon \simeq 8MeV$ (denoted usually by $-q$, see **Appendix 1**). On the other hand, it has a rest energy $E_b = Mc^2 \simeq 1GeV$. It follows that the nucleon extends over the Compton wavelength $\lambda = \hbar c/E_b \sim 10^{-16}m = 0.1fm$, and, consequently, it may move over distance a with energy of the order $\varepsilon \simeq 8MeV$. It has a momentum $p \sim \hbar/a$ and a velocity $v \sim \varepsilon/p = \varepsilon a/\hbar \simeq 2\times 10^7 m/s$, such that $v^2/c^2 \sim 10^{-3}$, which indicates that the nucleon moves non-relativistically (as expected from the ratio of the two characteristic energies ε and E_b).

The Atomic Nucleus is Cold. The nucleons may be brought into statistical equilibrium in time $\tau_{eq} = \hbar/\varepsilon$, providing energy ε is shared among a large number of energy levels (see **Appendix 2**). This is not the case for the atomic nucleus with mean-field nucleons, the "shell-model" included. Indeed, the momentum of free fermions is given by $p = \hbar n/R$, where $R = aN^{1/3}$ is the radius of the nucleus, and the Fermi momentum is $p_F \sim \hbar n_F/R = \hbar n_F/aN^{1/3}$, hence the Fermi number $n_F \sim N^{1/3} \sim 6$ for $N \sim 200$. The energy levels are given by $\varepsilon_n = (\hbar^2/MR^2)n^2 = (\hbar^2/Ma^2)n^2/N^{2/3}$, and for $n = n_F$ we get the Fermi energy $\varepsilon_F \sim \hbar^2/Ma^2$ $(\sim 15MeV)$.[1] We see that only a few energy levels are occupied ($n_F \sim 6$), as a consequence of the spatial degeneracy. The energy separation is $\delta\varepsilon \sim (\hbar^2/MR^2)n$, and $\delta\varepsilon_F \sim (\hbar^2/Ma^2)/N^{1/3} \sim \varepsilon_F/6$, which is comparable with the Fermi energy. Consequently, we cannot have a statistical equilibrium. The free nucleons in a square potential well are purely a quantal ensemble, unable to sustain thermalization.

A self-consistent potential well of a mean field does not change the situation. The nucleons may accommodate to each other through mutually correlated motions over the entire volume of the nucleus, such as to produce a mean field acting as an external potential. It is usually a central-force field, like an oscillator potential, and it explains satisfactorily the nuclear shells and magic numbers.

[1] Actually, this value of the Fermi energy is changed to somewhat extent by specific numerical factors, see **Appendix 1**.

The energy separation is then reduced to somewhat extent $(1-2MeV)$, but the degeneracy is still present, as indicated by the ~ 7 nuclear shells. The equilibrium is still unattainable. Even if, ideally, we distribute all the nucleons uniformly over an energy of the order ε_F, and get an energy separation $\delta\varepsilon \sim \varepsilon_F/N$, this separation is still insufficient for a consistent statistical equilibrium, in the sense that we would have then large fluctuations ($\sim 7\%$ for $N \sim 200$).

The atomic nucleus is too small to have a statistics of quasi-independent particles. It is cold, and there is no nuclear temperature, as long as such a gas-like ground-state is maintained. In order to get a thermodynamics, the atomic nucleus must change its ground-state.[2]

The Nuclear Solid. In an excited nucleus the short-range strong interaction between the nucleons spoils any mean field, and the motion passes from a quantal, global one, over the entire nucleus, to a local movement, involving distinctly each nucleon. This is a liquid state, and one may think that the atomic nucleus under excitations is a nuclear liquid. However, the nuclear excitation energies are comparatively high (for instance, the lower threshold is precisely the binding energy per nucleon $\varepsilon \simeq 8MeV$), and they would lead to the vaporization of the nuclear liquid. Consequently, for stability, the nucleus adopts a rigid, solid state, similar with an amorphous, finite-size solid. The thermodynamics of such a state is stable, it is attainable, and the nuclear solid can sustain high excitation energies and "hot" temperatures.[3]

The Nuclear Quark-Gluon Plasma. Ignition Threshold Energy. According to the asymptotic freedom of the quantum chromodynamics, for energies E per nucleon higher than the binding energy E_b, quarks and gluons may be released in such nuclear collisions, and they may form a quark-gluon plasma. In the initial stage, this quark-gluon plasma may be viewed as consisting of radiation (gluons) and ultrarelativistic fermions (the quarks uud and udd, corresponding to the nucleon states, $m_u \sim 4MeV$, $m_d \sim 8MeV$). If the nuclear collision process is such that only a few nucleons are destroyed, *i.e.* the total energy E_{tot} given to the nucleus is slightly greater than the binding energy of a few nucleons only, then the number of released quarks and gluons is small, and they may be delocalized as wave packets over the entire volume of the nucleus. Consequently, their density is low, and such a rarefied plasma may attain equilibrium in a very long time only, of the order of $\tau_{eq} \sim \hbar/(E-E_b)$, where $E = E_{tot}/N$ is the average energy imparted to each nucleon among those N destroyed nucleons. The characteristic scale energy of this ensemble of a few quarks and gluons is comparable with the spacing of their quantal energy levels, which indicates that equilibrium is not reached in fact for such an ensemble. It is a cold plasma, in non-equlibrium, and the original nucleons may in fact be quickly recovered, as the large delocalization may nullify in fact the asymptotic freedom. One may say that the quark-gluon plasma has not yet been ignited in this case.

In order to be fully developed, the hadronization process requires a quark-gluon plasma as dense as possible, and as hot as possible. It is desirable therefore, first, to unbind as many nucleons in the nucleus as possible. It is also worth noting in this case that if, conceivably, the nuclear collison process is such as to impart to each nucleon in the nucleus an energy slightly greater than its binding energy, then, again, the equilibrium cannot be reached, and the original nucleons are again quickly recovered. It is obvious, therefore, that there is a threshold energy ε_{thr} (leaving aside the nucleon binding energy) for igniting the quark-gluon plasma. It corresponds to the characteristic scale energy of a degenerate ideal gas of ultrarelativistic identical fermions (Fermi energy), with density $n_q \sim N/V \sim 1/a^3$, where V is the volume of the nucleus and N is of the

[2] The mean field can still work for special excitations, like the radiative capture of slow neutrons, where the neutron is gently accommodated. On the other hand, as it is well-known, one-particle nuclear models describe satisfactorily the nuclear shells, magic numbers, and even the mass formula.

[3] The detailed, quantitative arguments for the nuclear solid are given in J. Theor. Phys. **125** (2006).

order of the number of nucleons in nucleus. This threshold energy is then given by

$$\varepsilon_{thr} \sim \hbar c/a = 125MeV \ . \tag{1}$$

It corresponds to an average energy $(3/4)\varepsilon_{th}$ per quark, or, since we may take 3 quarks released per nucleon, to an average energy $(9/4)\varepsilon_{th}$ per nucleon (beside the binding energy E_b).[4] The ensemble of quarks may then reach equilibrium in time $\tau_{eq} \sim \hbar/\varepsilon_{thr} \sim 10^{-23}s$. However, the spacing between their quantal levels is of the order of $\delta\varepsilon_q \sim \hbar c/aN^{1/3} = \varepsilon_{thr}/N^{1/3}$, and, again, one can see that a lot of quantal fluctuations are expected (because N is small, of the order of the number of nucleons in the nucleus). We assume that the nuclear quark-gluon plasma is fully developed at the scale of the entire nucleus, and its energy per quark is far above the ignition threshold energy given by equation (1) (in fact much far above, as it is shown below).

Hot and Dense Quark-Gluon Plasma. The energy of the quark-gluon plasma can be written as

$$E_p = E_q + E_g \ , \tag{2}$$

where $E_q$denotes the energy of the quarks and E_g stands for the energy of the gluons. For low temperatures, the energy of the quarks reads

$$E_q = \frac{3}{4}N_q\varepsilon_F + \frac{3\pi^2}{2}N_q\frac{T^2}{\varepsilon_F} + ..., \tag{3}$$

where N_q is the number of quarks, $\varepsilon_F = (6\pi^2/2g_q)^{1/3}\hbar c \cdot n_q^{1/3}$ is their Fermi energy, g_q denotes the statistical weight of their multiplicities, and T stands for temperature. Equation (3) corresponds to a degenerate gas of ultrarelativistic fermions with density $n_q = N_q/V$ at temperature $T \ll \varepsilon_F$. The Fermi energy ε_F is comparable with the threshold energy ε_{thr}, for the threshold density ($N_q \sim N$). The energy of the gluons

$$E_g = (\pi^2 g_g/15)VT^4/(\hbar c)^3 \tag{4}$$

is that of a black-body radiation in volume V at temperature T, where g_g stands for the statistical weight of the gluons multiplicities. The number of gluons is also given by $N_g = 0.244 g_g V(T/\hbar c)^3$. It is easy to see that the quark-gluon plasma is dominated by the gluon energy, since the number of gluons increases appreciably with increasing temperature.

As long as the number of quarks is fixed, even for very high excitation energies (when the quarks may form a classical gas of ultrarelativistic fermions) the quark-gluon plasma is dominated by gluons, and the hadronization process is not expected to have a rich output. Actually, the strong interactions in the hot quark-gluon plasma lead to the production of a large number of quarks, of various species, antiquarks included, like, for instance, by pair production. These quarks are in equlibrium with the gluons, so they have a vanishing chemical potential, their number is not fixed, and for sufficiently high energies they may be viewed as an ultrareletivistic gas of fermions. The energy of such a quark gas is given by

$$E_q = (7\pi^2 g_q/240)VT^4/(\hbar c)^3 \ , \tag{5}$$

and one can see that, up to some immaterial numerical factors, it is the same as the energy of the gluons given by (4). Similarly, the number of these quarks is given by $N_q = (1.8g_q/2\pi^2)V(T/\hbar c)^3$, which is equal to the number of gluons, except for some immaterial numerical factors. Therefore,

[4] The energy of a degenearte gas of N ultrarelativistic fermions is $E = (3/4)N\varepsilon_F$. If we take $n_q = 1/r^3 = 3N/V = 3/a^3$, then the threshold energy is $\varepsilon_{thr} \sim \hbar c/r = \sqrt[3]{3}\hbar c/a \sim 180MeV$.

leaving aside such numerical factors, the energy of the hot and dense quark-gluon plasma can be represented as

$$E_p \simeq VT^4/(\hbar c)^3 \ . \tag{6}$$

According to (6), for the nuclear volume $V = Na^3$, we get the temperature $T = [10^6 E_p(MeV)/N]^{1/4}$, and for $E_p/V \sim 10^3 GeV/fm^3$ the temperature is $T \sim 1GeV$. The number of ultrarelativistic quarks in the quark-gluon plasma (or the number of gluons) is given by $N_q = N[T(MeV)/100]^3$, and for $T \sim 1GeV$ we get $N_q \sim 10^3 N$, where N is the number of nucleons in nucleus.

We can see that for such temperatures ($1GeV$) it is unlikely to have massive quarks in the quark-gluon plasma (temperature should be of the order of their rest energy mc^2 at least). Beside u and d quarks, only the s quark is expected ($m_s \sim 150MeV$), which may also be viewed as being in the ultrarelativistic limit. In general, if massive quarks are present in this process ($m_c \sim 1.5GeV$, $m_b \sim 4.7GeV$, $m_t \sim 176GeV$), their number and their energy are much lower than the values given here, so the process may be viewed as being dominated by gluons and ultrarelativistic quarks in equilibrium.[5]

This hot and dense ultrarelativistic quark-gluon plasma, extended over the whole volume of the nucleus, reaches equlibrium very quickly (in time $\sim \hbar/T \sim 10^{-24}s$), expands, get cool, and hadronizes. The thermalization condition $T \gg \delta\varepsilon_q$ is much better fulfilled now.

Hadronization. Classical statistics. The quark-gluon plasma expands with light velocity. Its radius increases from the radius R_0, which may be taken as the radius $R_0 = aN^{1/3}$ of the original nucleus,[6] to $R = R_0 + ct$ for time t, so we can write

$$R = R_0(1 + ct/aN^{1/3}) \ . \tag{7}$$

Making use of equation (6), with $V = R^3$ (and $V_0 = R_0^3$), we get that plasma temperature decreases according to

$$T = T_0(1 + ct/aN^{1/3})^{-3/4} \ , \tag{8}$$

where $T_0 = E_p^{1/4}(\hbar c/R_0)^{3/4}$ is the original plasma temperature. Similarly, the number of quarks (or gluons) increases in time during this expansion according to

$$N_q = N_{q0}(1 + ct/aN^{1/3})^{3/4} \ , \tag{9}$$

where $N_{q0} = (R_0T_0/\hbar c)^3 = N(T_0 a/\hbar c)^3$ is their initial number. The expansion of the quark-gluon plasma is a non-equilibrium, irreversible, process, with increase of entropy.[7] However, the plasma

[5]Usually, the chemical potential μ for relativistic particles of mass m includes the rest energy mc^2, $\mu = \mu_0 + mc^2$, and it is this potential that vanishes at equlibrium with gluons. The relativistic energy $\sqrt{m^2c^4 + c^2p^2}$ in the exponent of the statistical distributions makes the corresponding number and energy of particles much smaller in comparison with the ultrarelativistic limit (which formally corresponds to $m \to 0$). For instance, in the limit $T \ll mc^2$, these quantities are exponentially small ($\sim e^{-mc^2/T}$).

[6]In the center-of-mass reference frame the plasma is at rest, so there is no Lorentz contraction anymore of the volume in the colliding-beam direction (in contrast with the nucleon-meson plasma). Similarly, a hydrodynamic regime loses its validity for high-energy radiation and ultrarelativistic quarks. At the same time, the adiabatic expansion would imply a fixed number of particles, which may hold in the later stages of expansion, close to the hadronization stage, as suggested below. In general, the picture of many-mesons "multiple" production in high-energy proton-proton (or proton-nucleus, nucleus-nucleus) collision plasma (Fermi-Landau theory of "pronged-stars" production) is different from the hadronization mechanism, at least in two respects: first, the mesons are massive (in contrast with quarks and gluons), and this may render plausible a hydrodynamic picture, at least in later stages of expansion, where interaction weakens; and this latter aspect is another great difference with respect to hadronization, where interaction does come into play precisely in later stages of expansion.

[7]The entropy of the quark-gluon plasma is $S \sim (4/3)E_p/T \sim N_q$. Similarly, its pressure p is given by $pV \sim E_p/3$.

is in equilibrium at any instant of time, since, for instance, the inequality $T > \delta\varepsilon_q \sim \hbar c/R$ is satisfied for any $t > 0$, according to (7) and (8).

According to the "asymptotic freedom", the process of hadronization begins with the quarks in the outer shells of the plasma. Let $N'_q \ll N_q$ be the number of these quarks at some moment. It is given by $N'_q = N_q(\Delta R/R)$, where ΔR is the thickness of the outer shell of the plasma. Both ΔR and R have the same time dependence, so we may take $N'_q = N_q(\Delta R_0/R_0) = N_q/N_{q0}^{1/3}$. We denote by $f \ll 1$ the fraction $1/N_{qo}^{1/3}$ and write $N'_q = fN_q$. These "surface" quarks are the first in time that begin to feel the effect of interaction. Consequently, they are gradually decoupled from the rest of the quark-gluon plasma, and can be viewed as a gas of ultrarelativistic fermions with a fixed number of particles, moving uniformly in the plasma volume and in equlibrium with plasma at temperature T.[8] Their Fermi energy is given by $\varepsilon_F = \hbar c/r'$, where r' is given by $N'_q r'^3 = R^3$, whence $r' = R/(fN_q)^{1/3} = \hbar c/Tf^{1/3}$. We can see that $\varepsilon_F = Tf^{1/3}$ and $T/\varepsilon_F = 1/f^{1/3} = 1/N_{q0}^{1/9} \gg 1$, *i.e.* this ultrarelativistic gas may be viewed as obeying approximately the classical statistics. In the limit of very hot and dense plasma this condition is much better fulfilled. The energy of such a classical gas is given by $E'_q = 3N'_qT = 3fVT^4/(\hbar c)^3$, and one can see that the time dependence of the temperature as given by (8) is maintained (up to some minor numerical factors), according to energy conservation $VT^4/(\hbar c)^3 + E'_q = E_p$.

A similar conclusion applies to the resulting hadrons, because the condition for classical statistics for relativistic particles with mass m reads $\sqrt{m^2c^4 + (\hbar c/r)^2} - mc^2 \ll T$, where r is the mean inter-particle distance, and, since $\sqrt{m^2c^4 + (\hbar c/r)^2} - mc^2 < \hbar c/r$, one can see that it is satisfied if the same condition is satisfied for a gas of ultrarelativistic particles with the same density.[9]

We note that, though very likely, the condition for the hadronizing gas of quarks (or the resulting hadronic gas) to be in the classical limit is not essential for the mechanism of quark condensation which is described below. In general, it is very likely that the hadronization of the quarks begins with those placed at some moment in the outer shells of the plasma, and their number is a fraction f of the total number of quarks at that moment. Fraction f may differ from the one given above, and may even have a time dependence, as depending on the particularities of the "asymptotic freedom" mechanism of interaction. Time (and space) evolution of this interaction may change the time dependence given by equations (7)-(9) of the plasma expansion. All these particularities do not affect essentially the condensation mechanism of quarks into hadrons given further herein.

Hadronization. Transition temperature and the hadronic yield. A classical gas of N relativistic particles enclosed in volume V is described by the usual distribution $dN = [gV/(2\pi\hbar)^3]e^{\mu/T}e^{-\varepsilon/T}d\mathbf{p}$, where $\varepsilon = \sqrt{m^2c^4 + c^2p^2}$, μ is the chemical potential and g is the corresponding weight factor. For an ultrarelativistic classical gas of quarks the chemical potential is given by

$$\mu_q = -T\ln(g_qT^3/3\pi^2\hbar^3c^3n_q) \ , \tag{10}$$

where $n_q = N_q/V$ is the density of quarks. We introduce a scale temperature $T_q = \hbar c n_q^{1/3}$ (Fermi temperature), and write approximately

$$\mu_q \simeq -3T\ln(T/T_q) \tag{11}$$

for $T \gg T_q$. The energy is given by $E_q = 3N_qT$ and the thermodynamic potential $\Omega_q = -p_qV = -N_qT$, where p_q is the pressure of the quark gas. The number of quarks N_q in (10) is in fact number

[8]Under the action of the attraction they do not rest on the surface, but move in the whole volume.

[9]Temperature $\sqrt{m^2c^4 + (\hbar c/r)^2} - mc^2 \ll mc^2$ is also a scale temperature for the Bose-Einstein condensation of relativistic bosons.

N_q' of "surface" quarks introduced above, and the Fermi temperature $T_q = Tf^{1/3}$. Similarly, the quark pressure p_q is in fact a partial pressure in the quark-gluon plasma.

We label the hadron species by $j = 1, 2, 3, ...$, and characterize each species by its number of quarks $n_j = 2, 3, ...$, its mass m_j and momentum $\mathbf{p}_j$, the later two being related by energy $\varepsilon_j = \sqrt{m_j^2 c^4 + c^2 p_j^2}$. There may exist also a relationship between number of quarks n_j and mass m_j, but we let m_j to be an independent parameter, as, for instance, to account for resonances in hadron spectra. Other quantal numbers may be introduced similarly, according to the desired classification of the hadrons, and subjected to various conservation laws or selection rules. We impose the conservation of the number of quarks

$$N_q = \sum_{j-states} n_j p_j \ , \tag{12}$$

and the conservation of hadronic energy

$$E_h = \sum_{j-states} \varepsilon_j p_j \ , \tag{13}$$

where p_j is the probability of states. It follows then straightforwardly the hadron distribution

$$dN_j = \frac{g_j V_h}{(2\pi\hbar)^3 m_0} e^{\mu_h n_j/T} e^{-\varepsilon_j/T} dn_j dm_j d\mathbf{p}_j \ , \tag{14}$$

where g_j is the statistical weight of the multiplicity of the species j, m_0 is a scale of minimal mass, μ_h is the chemical potential and V_h is the volume of the hadronic gas (it differs from the original volume of the quarks, as a result of the hadronic condensation). Allowing for m_j to extend continuously to infinite, and replacing the summation over mass spectrum by integration (with m_0 the mean mass inter-spacing) we get straightforwardly from (14)

$$dN_j = \frac{g_j V_h}{(2\pi\hbar)^3 m_0} \cdot \frac{6\pi(\pi-1)}{c^5} T^4 e^{\mu_h n_j/T} dn_j \ . \tag{15}$$

In order to estimate the summation over j we introduce the mean hadronic weight g_h by

$$\sum_j g_j e^{\mu_h n_j/T} = g_h \sum_{n=s}^{\infty} e^{\mu_h n/T} \ , \tag{16}$$

and starts the summation with $n = s \geq 2$, as for the smallest composite hadrons. We get the number of hadrons[10]

$$N_h \simeq \frac{g_h V_h}{(2\pi\hbar)^3 m_0} \cdot \frac{6\pi(\pi-1)}{c^5} T^4 \cdot e^{\mu_h s/T} \ , \tag{17}$$

and, according to (12), the number of quarks

$$N_q \simeq \frac{s g_h V_h}{(2\pi\hbar)^3 m_0} \cdot \frac{6\pi(\pi-1)}{c^5} T^4 \cdot e^{\mu_h s/T} \ . \tag{18}$$

We can see that $N_h = N_q/s$, *i.e.* the hadronic condensate is dominated by the smallest composite hadrons (corresponding to the smallest s, as, for instance $s = 2$), *i.e.* by hadrons with the simplest

[10] We note that the summation over n_j does not necessarily follow the sequence of all natural integers, but must obey the sequence corresponding to the defined (observed) hadron species.

structure. The number of hadrons made of $s+1$, $s+2$, etc quarks is smaller by exponential factors $e^{\mu_h/T}$, $e^{2\mu_h/T}$, etc than this number.

Equation (17) or (18) determines the (large, negative) chemical potential of the hadronic gas. It is approximately given by

$$\mu_h \simeq -(1/s)T\ln[3g_h(\pi-1)T^4/4\pi^2\hbar^3c^5m_0n_h] \ , \tag{19}$$

where $n_h = N_h/V_h$ is the density of hadrons. Similarly, the energy of the hadronic gas is given by[11]

$$E_h \simeq \frac{g_hV_h}{(2\pi\hbar)^3m_0}\cdot\frac{24\pi(\pi-1)}{c^5}T^5\cdot e^{\mu_hs/T} = 4N_hT \ . \tag{20}$$

It is easy to see that the thermodynamic potential $\Omega_h = -p_hV_h$ is given by $\Omega_h = -N_hT = -E_h/4$, hence the equation of state $p_hV_h = N_hT$, where p_h is the pressure of the hadronic gas. For equlibrium this pressure equals the one of the quark gas, given by $p_qV_q = N_qT$. It follows that concentrations n_h and n_q must be equal at equlibrium, and, since $N_h = N_q/s$, it follows $V_h = V_q/s$, as expected for condensation.

Experimentally, μ_h, T and m_0 are fit parameters for hadron distributions given by (14). By measuring the latter we may characterize the hadronic output, as well as the original gas of quarks that hadronizes. It is worth noting here that the mass spectrum is discrete, and it does not extend to infinite, in contrast to the estimations made above. A similar note applies also to the hadron structure defined by sets of integers n_j. It follows that the energy and the chemical potential above should be computed according to the empirical statistical ensemble analyzed. The experimental temperature T determined from the hadronic output is the transition temperature.

Indeed, according to the above description the hadronization process is a phase transition of the first kind.[12] The critical temperature is given by

$$\mu_q = \mu_h \ , \tag{21}$$

where μ_q is given by (11) and μ_h is given by (19), for the same pressure, *i.e.* the same density $n_q = n_h$. Under these circumstances, equation (19) can also be written as $\mu_h = -(1/s)T\ln(T^4/T_q^3T_m)$, where $T_m = [4g_qm_0c^2/3sg_h(\pi-1)] \sim m_0c^2$. By (21), we get then the critical temperature of hadronization

$$T_c = T_q(T_q/T_m)^{1/(3s-4)} \tag{22}$$

It is the temperature below which the hadron distributions given by (14) are observed.[13] The latent heat Q involved in the hadronization process is given by the jump in heat functions $W_q = E_q+p_qV_q = 4N_qT$ and $W_h = 5N_hT = 5N_qT/s$ at equilibrium, which leads to $Q = (5/s-4)N_qT_c = (5/s-4)E_q/3$.[14] One can see that it is negative, which means that the energy (and temperature) of the quark-gluon plasma increases slightly in the hadronization process. The latent heat is released in the hadronization process.

The critical temperature of hadronization T_c must be much higher than the characteristic quark temperature T_q in order to use the classical statistics. A similar condition $T_c^4 \gg T_q^3T_m$ holds also

[11] The difference between the prefactor 4 in hadronic energy and the prefactor 3 in the energy of the quark gas comes from the additional degree of freedom of the mass.

[12] For more details on the mechanism of matter condensation see J. Theor. Phys. **123** (2006).

[13] For a discrete mass spectrum equation (22) gives the scale temperature T_m.

[14] It corresponds to the extra degree of freedom due to the mass, originates in quark interaction, and accounts for the remanent entropy of the hadronic gas.

for the hadronic gas. Both conditions are satisfied providing $T_q \gg T_m$. Making use of $T_q = T_c f^{1/3}$ we get from (22) $T_c = T_m/f^{s-1} \gg T_m$.[15] It is worth noting that the experimental hadronic distributions are not continuous in mass spectrum, nor in the quark constituency, as it is assumed in the estimations given herein. Accordingly, the parameters like T_q or T_m, that might be derived from the analysis of the empirical distributions of hadrons, can be different from their expressions given here. In addition, it must also be noted that the mechanism of hadronization described above through the condensation of the quark gas is not restricted to classical statistics. Quantal statistics can be used, if necessary, both for the hadronizing gas of quarks and for the resulting hadrons, which change the expressions given above for the chemical potential and for the critical temperature.

Functions $\mu_q(T)$ and $\mu_h(T)$ as given by (11) and (19) for the same $n_q = n_h$ are such that $\mu_q < \mu_h$ for $T > T_c$ and $\mu_q > \mu_h$ for $T < T_c$, which means that the phase diagram favours the quarks for $T > T_c$, and hadrons for $T < T_c$, as expected. The hadronization of the first N'_q "surface" quarks can be viewed as the first stage in the hadronization process. After this stage is completed the number of remaining quarks is diminished, as it is the radius of the remaining plasma. The temperature of the remaining plasma is increased to some extent, as due to the released latent heat, but it quickly reaches again the value of the critical temperature by expansion, and the first-stage process of hadronization is repeated. However, it is very likely that at some moment in its expansion the quark-gluon plasma ceases to sustain an equilibrium between quarks and gluons, as a result of its cooling (in any case the rate of this equilibrum slows down on cooling the plasma). Under this circumstance, the number of quarks in plasma becomes fixed, the temperature decreases at a higher rate, and the cool and more rarefied plasma favours the condensation of heavier, more complex hadrons. It may be said, therefore, that in the hadronization process there appear first hadrons with a simpler structure (which are, very likely, lighter) at higher T_c, and, gradually, more complex hadrons (which, likely, are heavier) at various slightly lower values of temperature, such that a temporal analysis of the hadronization might indicate a succession of "phase transitions", or a cascade of hadronization processes, in the order indicated here. It is worth noting that this order corresponds to the energy (mass)-time uncertainty relationship. However, such a temporal series of hadronization occurs in a very rapid succession, which is beyond the observational means.

Let us include finally a numerical estimation, in order to get a feeling of relevant figures. Suppose that $N \sim 50$, which makes the fraction $f = 1/N_{q0}^{1/3} = 1/10N^{1/3} \simeq 0.03$, and the critical temperature $T_c \simeq 30T_m$. Then it is conceivable that the minimal mass parameter m_0 may correspond to the ligtest quark, say, $m_0 \sim 4MeV$, *i.e.* $T_m \sim 4MeV$, so that $T_c \sim 120MeV$ for $s = 2$. Making use of (8), this temperature is reached in time $t \sim 10^{-22}s$ ($ct/aN^{1/3} \sim 20$), for an initial temperature $T_0 \sim 1GeV$, a lapse of time during which the quark-gluon plasma expands its radius by a factor of 20. The energy of the condensed quarks $E'_q = 3N'_q T = 3fE_p$ is the fraction $3f \sim 10\%$ of the plasma energy, which represents the efficiency coefficient of hadronization in the first stage. The corresponding latent heat amounts to $(5/s-4)E'_q/3 \sim -0.5E'_q$ (for $s = 2$), which indicates a rather high remanent entropy, as expected for this gas of light hadrons. The number of hadronized quarks in the first stage of hadronization is given by $N'_q = fN_{qo}(1+ct/aN^{1/3})^{3/4} \sim 0.03 \cdot 10^3 N \cdot 10 \sim 300N$, and the corresponding number of hadrons is $N_h \sim N'_q/s \sim 150N$ for $s = 2$.

Appendix 1. The virial, free nucleons, and mass formula

Let $\dot{q}\partial T/\partial \dot{q} = 2T$ be twice the kinetic energy for a generic motion of coordinates q. Integrating

[15] Condition $T_q \gg T_m$ implies $f \gg f^{3(s-1)}$, which is satisfied for $f \ll 1$. This condition is better fulfilled than the condition for the classical behaviour of the quark gas because the classical statistics is favoured for massive hadrons.

by parts over motion time t we get

$$2T = qp_t/t - q\dot{p} = qp_t/t + q\partial U/\partial q \tag{23}$$

for averages, where p denotes the momenta. For a bound motion the momenta are finite at time $t \to \infty$, so we are left with $2T = q\partial U/\partial q$, which integrated again by parts over the motion volume enclosed by surface S, gives

$$2T + U = (qU)_S/q \ . \tag{24}$$

This is the well-known "theorem" of the virial.

If the surface is a virtual one in the bulk, the term $(qU)_S/q$ is negative. We denote it by $-E_s$ and have the binding energy for the bulk

$$E_b = T + U = -T - E_s < 0 \ . \tag{25}$$

For a real surface the term $(qU)_S/q$ vanishes in (24) and we get the binding energy

$$E = T + U = -T = E_b + E_s < 0 \ , \tag{26}$$

on account of the same value of the kinetic energy in both cases. One can see that the binding energy of a body is higher than the binding energy of the bulk by the surface term E_s (indeed, in order to break down a body we have to supply the fracture with its surface energy).

It is easy to see from (24) that the surface energy goes like number $N^{2/3}$ of surface particles, since $(qU)_S/q$ involves a summation over those particles only (or the integration over the surface), while the bulk energy goes like N, so that we can write $E_s = u_s N^{2/3}$ and $E_b = -u_b N$, where the coefficients u_s and u_b are close in value to each other. Averaging over large N the surface energy may be neglected with respect to the bulk energy. Indeed, $\langle E_s \rangle / \langle E_b \rangle = 6/5N_c^{1/3} \sim 0.2$ for a cutoff $N_c = 200$. It is worth noting that the usual mass formula for atomic nuclei implies a fit to empirical binding energies, which amounts to such averaging procedures with respect to N.

It is customary to view the nucleons as free fermions, embedded in a square potential well $U = -N\varepsilon_0$, and write down $N = gVp_F^3/6\pi^2\hbar^3$, or $p_F = (6\pi^2/g)^{1/3}\hbar/a$, where p_F is the Fermi momentum and g is a statistical weight (for instance, $g = 4$, spin and isotopic spin included). Then we get the Fermi energy $\varepsilon_F = p_F^2/2M$ and the kinetic energy $T = 3N\varepsilon_F/5$. It is worth noting that in employing such formulae, the thermodynamic limit $N \to \infty$ is assumed, so that the surface energy is vanishing. According to (25) we get the binding energy of the bulk $E = -N\varepsilon_0 + 3N\varepsilon_F/5 = -3N\varepsilon_F/5$. We are interested in estimating the change q in energy for a change $\delta N = 1$ in number of particles at constant concentration. It is easy to see that it is given by $q = -3\varepsilon_F/5$. We emphasize that q differs from the chemical potential $\mu = \varepsilon_F$. Using $a = 1.5 \cdot 10^{-15}m$ we get $\varepsilon_F = \mu = 46MeV$ (for $g = 4$), the potential depth $-\varepsilon_0 = -6\varepsilon_F/5 = 55.2MeV$ and $q = -3\varepsilon_F/5 = 27.6MeV$. (The fermions have also a pressure $p = 2\varepsilon_F/5a^3$, compensated by the pressure produced by the potential well ε_0). In order to compare these results with the empirical mass formula we must average over number of nucleons, which amount to take half of the above figures. We get therefore the (average) Fermi energy $\langle \varepsilon_F \rangle = 23MeV$, the depth of the potential well $-\langle \varepsilon_0 \rangle = 27.6MeV$ and $u_b \simeq -u_s = -q = 3\langle \varepsilon_F \rangle/5 = 13.8MeV$, a figure which compares well with the experimental fits. It is worth noting that $q = -13.8MeV$ differs from the empirical binding energy per particle $q = E/N \simeq -8MeV$ on account of additional energy contributions, especially the Coulomb repulsion, not included herein.

The Coulomb interaction for atomic nuclei can be written as $E_c \sim Z(Z-1)e^2/2R = Z(Z-1)N^{-1/3}e^2/2a$, where e is the electron charge (the factor 2 has been introduced in the denominator in order to account for the average value of $1/R$). Writing $E_c = u_c Z(Z-1)N^{-1/3}$ we get the coefficient $u_c = e^2/2a = 0.48MeV$, which agrees with the empirical value.

The last contribution to the mass formula comes from the symmetry effect, which is a consequence of the exclusion principle. It should increase the energy from a $g = 4$-degenerate energy level under a transform which replaces a proton by a neutron, or conversely, a neutron by a proton. This energy contribution, denoted E_r, should read $E_r = u_r(Z-N)^2/A$, where Z denotes the number of protons, N denotes now the number of neutrons and $A = Z + N$ is the mass number. In the limit $A \to \infty$ this term should compensate the bulk contribution, so that we get a value $u_{r1} \simeq -u_b = 13.8MeV$. Actually, this value should be somewhat larger, because E_r can also be written as $E_r = u_r(A-2Z)^2/A = u_r A x^2 < u_r A$, where $x = 1 - 2Z/A$. We average x^2 around $x = 1$, over the range described by the tangent to x^2 for $x = 1$. We get $\langle x^2 \rangle = 7/12$ and the second value $u_{r2} \simeq 23MeV$. Finally, we get the mean value $u_r = (u_{r1} + u_{r2})/2 = 18.4MeV$, which agrees well with the empirical value.

Appendix 2. Statistical equilibrium and thermalization

For a consistent description of statistical equilibrium of an ensemble of particles a series of inequalities of the type

$$\varepsilon_{eq} > T > \delta\varepsilon_f > \delta\varepsilon_{ex} \gg \delta\varepsilon_q > \delta\varepsilon_{obs} \tag{27}$$

should be satisfied, where ε_{eq} is a mean (scale) energy, T is the temperature, $\delta\varepsilon_f = T(\partial\varepsilon/\partial T)^{1/2}$ is the thermal fluctuation energy, $\delta\varepsilon_{ex}$ is the uncertainty in the energy of the elementary excitations, $\delta\varepsilon_q$ is the spacing between the quantal energy levels and, finally, $\delta\varepsilon_{obs}$ is the uncertainty in the observed (measured) energy, all per particle. For a large number of particles such inequalities are fulfilled, in general, but for small numbers of particles they may not be satisfied, which means that the ensemble is not in equilibrium, since, for instance, ε_{eq} may be comparable with $\delta\varepsilon_q$ in this case. The meaning of such inequalities resides in the succession of time intervals

$$\tau_{eq} < \tau_{th} < \tau_f < \tau_{life} \ll \tau_q < \tau_{obs} \tag{28}$$

required for measuring consistent mean values of various quantities, according to the generic uncertainty relationship $\tau \sim \hbar/\delta\varepsilon$. In (28) $\tau_{th} = \hbar/\varepsilon_{th}$ is the time needed to establish the thermal equilibrium, and τ_{life} is the lifetime of the elementary excitations.

For fermions at zero temperature ε_{eq} is of the order of the mean energy per particle, or Fermi energy ε_F, the next two terms in (27) do not appear, while the rest of inequalities in (27) keep their meaning. It is interesting to note that even in the absence of the thermal equilibrium we may still have a statistical equilibrium. Indeed, the mean energy is $\varepsilon_{eq} = 3\varepsilon_F/5$, while its mean square is $\overline{\varepsilon^2} = 3\varepsilon_F^2/7$, which is comparable with ε_{eq}^2. It shows how effective the establishing of the statistical equilbrium can be in this case, by exchanging energy during collisions. The fact that the statistical equilbrium may be independent of temperature originates in describing ensembles by probabilities, which is unavoidable when talking about such ensembles of particles in terms of particles. For a degenerate gas of fermions the discussion is similar, and for high temepratures the gas behaves classically. In both cases the meaning of (27) is defined.

For bosons at low temperature the scale energy ε_{eq} is the temperature $T_0 \sim \hbar^2/mr^2$ of the Bose-Einstein condensation, where r is the mean inter-particle distance and m is the particle mass. Above the condensation temperature the role of the ε_{eq} is played by the chemical potential (its absolute value), which for high temperatures becomes again that of a classical gas. A similar discussion holds also for other ensembles of particles (of an academic interest in this context might be the Bose-Einstein condensation of relativistic particles, more exactly relativistic corrections to the Bose-Einstein condensation), the black-body radiation included.

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