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The basis of the Fermi liquid theory

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Abstract

The basis of the Fermi liquid theory is shown, and the electronic liquid is briefly discussed.

Interaction may affect drastically the many-particle ensembles; for instance, an attraction, even weak, between electrons, binds them up in pairs, leading to superconductivity; interacting fermions in one dimension get bosonized; anisotropic fermions with "nested" Fermi surfaces become non-homogeneous, when interacting, and develop charge- or spin-density waves. All these are different phases, and appear as symmetry breakings, spontaneous or induced; they are also termed as instabilities of the many-body systems, under interaction. Hints toward their nature are often obtained through studying the interacting two-particle problem, scattering included.

Leaving aside these cases, switching on the interaction may preserve the nature of the particles, their statistics, the symmetries, and the (repulsive) interaction may behave perturbationally. However, even in this case, the ground-state is to be treated distinctly from the low-energy excitations; indeed, the low energy of the latter may be comparable with the weak interaction effects, which may result in new kinds of elementary excitations, as compared with the non-interacting system; the most common case is probably provided by the quantum sounds in both interacting Bose and Fermi ensembles. The general perturbational scheme for the ground-state is based on the observation that, to the lowest order of perturbation theory, the wavefunction does not change, while the energy changes by the average of the interaction over the unchanged (non-perturbed) ground-state,

$$\psi = \psi_0 + ... \ , \quad E = E_0 + (\psi_0, U\psi_0) + ... \ . \tag{1}$$

The Fermi sea (as well as the one-particle plane waves), or the boson condensed ground-state, are not changed, therefore, by interaction in this scheme, while a constant energy is acquired by each particle, as if each of them were moving in a constant, external potential; in addition, the individual fermions behave as if they would have, approximately, a different, effective mass. Indeed, the only free parameters in the hamiltonian of a free quantum particle are the particle mass and a constant potential (up to redefining momentum, occasionally). As such, this perturbational scheme may be viewed as an even lower approximation than the quasi-classical approximation. For the boson condensed ground-state the calculation of the energy correction within this scheme proceeds by means of the particle operators turned c-numbers; a distinct particularity appears for the Fermi sea, namely the quantum exchange effect between the fermion states, seen in what is usually called the "Hartree-Fock approximation"; in both cases the particles react as a whole to the interaction, a feature called the "random phase approximation"; this reaction is either static

or dynamic, the latter implying virtually excited states, retardation and damping effects; all these features have actually the aspect of an interacting mean-field; the difference between fermions and bosons originates, trivially, in their distinct statistics, of course.

Naturally, the question of the validity of this perturbational scheme arises, or, as sometimes termed, the "convergence" of the perturbation theory for these "normal" many-body systems. Typically, the various orders of the perturbation theory may be viewed as containing factors of the form

$$\sum_s \frac{v(\mathbf{q})/V}{\varepsilon} \quad ; \tag{2}$$

here the summation is extended over all the s states allowed by statistics, coupled by interaction to the ground-state; their "excitation" energy is denoted by ε in (2), where V stands for the volume of the ensemble; of particular importance in (2) is the Fourier transform

$$v(\mathbf{q}) = \int d\mathbf{r} \cdot v(\mathbf{r}) e^{-i\mathbf{qr}} \tag{3}$$

of the two-particle interaction potential $v(\mathbf{r})$; the "excited" states in (2) are connected to the ground-state through the momentum ($\hbar\mathbf{q}$) and energy conservation, according to the invariance under the space and time translations; in addition, the states s are real, not virtual "excited" states, *i.e.* their energies are related to their momenta through the free-particle Galilean relationship, as for non-relativist particles. The estimation of (2) is different for bosons and fermions, as well as for various space dimensions, but before proceeding one should emphasize the general model-like assumptions for many-body systems: particles are point-like and the potentials are Fourier-transformable; *i.e.*, distances shorter than the typical atomic length (Bohr radius) are meaningless, and the highly-repulsive "hard-core" atomic potentials are replaced by delta-type potentials, at most. Now, the most dangerous contribution to (2) comes from small $\mathbf{q}$ and ε, and for three-dimensional fermions this means $\varepsilon \sim \hbar^2 k_F q/m$, while the density of states is of the form $V \cdot q^2 dq$; the angular factors are rendered ineffective by the Fermi statistics, as one can see easily; k_F is the Fermi wavevector, and m denotes the fermion mass. In addition, $k_F \sim 1/a$, where a is the mean inter-particle distance, the interaction may be represented as $v(\mathbf{q} \sim 0) = \overline{v}a^3$, where $\overline{v}$ is a characteristic average interaction energy per particle, and the integration may be extended up to k_F; under these circumstance (2) is $\overline{v}/(\hbar^2/ma^2)$ at most, and it leads to

$$\overline{v}/(\hbar^2/ma^2) \ll 1 \tag{4}$$

i.e. the average interaction energy per particle must be much smaller than the particle localization energy over inter-particle distance. This is the typical condition for the perturbation scheme of the "normal" many-body systems. The condition holds for two-dimensional fermions too, while it diverges logarithmically for fermions in one dimension; as known, the perturbation scheme is no longer valid for the latter. For bosons $\varepsilon \sim \hbar^2 q^2/2m$, naturally, and the condition (4) holds in three dimensions; the condensed ground-state of the bosons is stable under perturbations in three dimensions. On the contrary, (2) diverges for bosons in two and one dimensions, and the corresponding condensed ground-state is destabilized by fluctuations in these cases. As in the one-dimensional case of fermions, the ground-state in these situations is the vacuum of the corresponding elementary excitations; these systems are not termed anymore as being "normal" systems, though the term "normal" is not too usually used for bosons in three dimensions either, in view of their Bose-Einstein condensation.

An ensemble of N identical particles of mass m and interacting through a two-particle potential

v is described by the hamiltonian

$$H = \sum_i \mathbf{p}_i^2/2m + \frac{1}{2}\sum_{i\neq j} v(\mathbf{r}_i - \mathbf{r}_j) \quad , \tag{5}$$

where i, j label the particles. For long-range potentials the stability of the system must be ensured, as, for instance, in the case of Coulomb interacting electrons, where a uniform, neutralizing background of positive charges must be added (which may have its own dynamics, as well; as known, this is often called the "jellium" model of interacting electrons). The corresponding interaction

$$\frac{1}{2}nN\int d\mathbf{r}\cdot v(\mathbf{r}) = \frac{1}{2}nN\cdot v(\mathbf{q}=0) \tag{6}$$

must then be subtracted from hamiltonian, which amounts to removing the $\mathbf{q}=0$ Fourier component $v(0)$ of the potential; $n = N/V$ above is the particle concentration. The quantum-mechanical counterpart of (5) is written by means of the field operators

$$\psi(\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} c_{\mathbf{k}} e^{i\mathbf{kr}} \quad , \tag{7}$$

where the second-quantization particle operators (of creation and annihilation) satisfy the commutation (anticommutation) relations

$$\left[c_{\mathbf{k}}, c_{\mathbf{k}'}^+\right] = \delta_{\mathbf{kk}'} \quad , \quad \left\{c_{\mathbf{k}}, c_{\mathbf{k}'}^+\right\} = \delta_{\mathbf{kk}'} \quad , \quad etc \quad , \tag{8}$$

corresponding to Bose and, respectively, Fermi statistics, such that

$$\left[\psi(\mathbf{r}), \psi^+(\mathbf{r}')\right] = \delta(\mathbf{r}-\mathbf{r}') \quad , \quad \left\{\psi(\mathbf{r}), \psi^+(\mathbf{r}')\right\} = \delta(\mathbf{r}-\mathbf{r}') \quad , \quad etc \quad . \tag{9}$$

The spin label is to be introduced, or tacitly accepted, together with the position $\mathbf{r}$ and the wavevector $\mathbf{k}$. The number of particles is then given by

$$N = \int d\mathbf{r}\cdot\psi^+(\mathbf{r})\psi(\mathbf{r}) = \sum_{\mathbf{k}} c_{\mathbf{k}}^+ c_{\mathbf{k}} \quad , \tag{10}$$

and the particle density

$$n(\mathbf{r}) = \sum_i \delta(\mathbf{r}-\mathbf{r}_i) \tag{11}$$

becomes

$$n(\mathbf{r}) = \int d\mathbf{r}_i\cdot\psi^+(\mathbf{r}_i)\delta(\mathbf{r}-\mathbf{r}_i)\psi(\mathbf{r}_i) = \psi^+(\mathbf{r})\psi(\mathbf{r}) = \frac{1}{V}\sum_{\mathbf{q}} n_{\mathbf{q}} e^{i\mathbf{qr}} \quad , \tag{12}$$

where the Fourier components

$$n_{\mathbf{q}} = \sum_{\mathbf{k}} c_{\mathbf{k}}^+ c_{\mathbf{k}+\mathbf{q}} \tag{13}$$

are also called the particle-density fluctuations, for $\mathbf{q}\neq 0$. The hamiltonian (5) is then writen as

$$\begin{aligned} H &= \int d\mathbf{r}\cdot\psi_\alpha^+(\mathbf{r})(\mathbf{p}^2/2m)\psi_\alpha(\mathbf{r}) + \\ &+\frac{1}{2}\int d\mathbf{r}d\mathbf{r}'\cdot\psi_\alpha^+(\mathbf{r})\psi_\beta^+(\mathbf{r}')v_{\alpha\beta}(\mathbf{r}-\mathbf{r}')\psi_\beta(\mathbf{r}')\psi_\alpha(\mathbf{r}) \quad , \end{aligned} \tag{14}$$

where $\mathbf{p} = -i\hbar\partial/\partial\mathbf{r}$ and the spin labels have been written explicitly; summations are to be understood over these labels, while the potential may, in general, depend on the spin, with the natural

symmetry properties; for instance, for one-half spin fermions the potential can be represented as $v_{\alpha\beta} = v + \alpha\beta \cdot u$, where $\alpha, \beta = \pm 1$. Noteworthy, the interaction can also be written as

$$U = \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \cdot v_{\alpha\beta}(\mathbf{r} - \mathbf{r}') n_\alpha(\mathbf{r}) n_\beta(\mathbf{r}') - \frac{1}{2} v_{\alpha\alpha}(\mathbf{r} = 0) N_\alpha \quad , \tag{15}$$

where the self-interaction is redundantly introduced; one may agree to ignore it, occasionally, and use (15) as a more convenient form. Without spin labels the kinetic energy can also be written as

$$K = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} c^+_{\mathbf{k}} c_{\mathbf{k}} \quad , \tag{16}$$

where $\varepsilon_{\mathbf{k}} = \hbar^2 k^2 / 2m$, while the interaction can be represented as

$$U = \frac{1}{2V} \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} v(\mathbf{q}) c^+_{\mathbf{k}+\mathbf{q}} c^+_{\mathbf{k}'-\mathbf{q}} c_{\mathbf{k}'} c_{\mathbf{k}} = \frac{1}{2V} \sum_{\mathbf{q}} v(\mathbf{q}) n_{\mathbf{q}} n_{-\mathbf{q}} - \frac{1}{2} v(\mathbf{r} = 0) N \quad , \tag{17}$$

where the self-interaction has been written again. All this formalism being set up, one can continue now with the normal fermions (in three dimensions). Similar considerations hold also for bosons, care being taken of their distinct statistics, and of their condensed ground-state.

Assuming that the perturbation condition (4) is satisfied, *i.e.* the interaction is weak and short-range, one may take the average of the hamiltonian given, for instance, by (16) and (17) over the fermion ground-state, leading to the ground-state energy

$$E = \sum_{\mathbf{k}\alpha} \varepsilon_{\mathbf{k}} n_{\mathbf{k}\alpha} + \frac{1}{2V} \sum_{\mathbf{k}\mathbf{k}'\alpha\beta} \left[v_{\alpha\beta}(0) - v_{\alpha\alpha}(\mathbf{k} - \mathbf{k}') \delta_{\alpha\beta} \right] n_{\mathbf{k}\alpha} n_{\mathbf{k}'\beta} \quad , \tag{18}$$

where $n_{\mathbf{k}\alpha}$ is the fermion occupancy, and the explicit summation over the spin labels is restored. This is the Hartree-Fock approximation, the first interacting term being the Hartree, or direct, contribution, while the second one is the Fock contribution; obviously, the latter is due to the exchange effects. The equation of motion can also be written for a one-particle state described by $c^+_{\mathbf{k}\alpha}$, and averaging again over the ground-state one obtains

$$\varepsilon^{HF}_{\mathbf{k}\alpha} c^+_{\mathbf{k}\alpha} = \varepsilon_{\mathbf{k}} c^+_{\mathbf{k}\alpha} + \frac{1}{V} \sum_{\mathbf{k}'\beta} \left[v_{\alpha\beta}(0) - v_{\alpha\alpha}(\mathbf{k} - \mathbf{k}') \delta_{\alpha\beta} \right] n_{\mathbf{k}'\beta} c^+_{\mathbf{k}\alpha} \quad , \tag{19}$$

where $\varepsilon^{HF}_{\mathbf{k}\alpha}$ may be seen as the single-state energy. The same energy can be obtained by taking the variation of the average of the hamiltonian (5) over the antisymmetrized wavefunction $A\varphi_1(\mathbf{r}_1)\varphi_2(\mathbf{r}_2)...$, where the φ-labels denote the one-particle states (this wavefunction is sometimes called a Slater determinant); one obtains

$$\begin{aligned} & -\frac{\hbar^2}{2m} \Delta \varphi_{\mathbf{k}\alpha} + \sum_{\mathbf{k}'\beta} \left[\left(\varphi_{\mathbf{k}'\beta}, v_{\alpha\beta} \varphi_{\mathbf{k}'\beta} \right) \varphi_{\mathbf{k}\alpha} - \delta_{\alpha\beta} \left(\varphi_{\mathbf{k}'\alpha}, v_{\alpha\alpha} \varphi_{\mathbf{k}\alpha} \right) \varphi_{\mathbf{k}'\alpha} \right] \\ = \; & \varepsilon^{HF}_{\mathbf{k}\alpha} \varphi_{\mathbf{k}\alpha} \quad , \end{aligned} \tag{20}$$

which are called the Hartree-Fock equations, and which, obviously, are identical with (19); and no wonder that the solutions to (20) are plane waves, as long as one starts with orthogonal one-particle wavefunctions and the hamiltonian is translationally invariant. The single-state energies $\varepsilon^{HF}_{\mathbf{k}\alpha}$, and the corresponding plane waves, do not, obviously, describe independent particles; each of them depends on the whole rest of one-particle states, and, for this reason, the Hartree-Fock equations

are said to describe self-consistently an interacting mean-field. In particular, the ground-state energy (18) is not the sum of the one-particle energies $\varepsilon^{HF}_{\mathbf{k}\alpha}$, but, on the contrary, a factor one-half must be introduced in the interacting contribution, for not counting twice the same state (this observation is sometimes referred to as Koopman's "theorem"). The ground-state energy must be minimal not only under variations of the one-particle wavefunctions, but also under the variations of the fermion occupancy, of course, under the constraint of a fixed number N of particles. With the notation

$$f_{\alpha\beta}(\mathbf{k},\mathbf{k}') = v_{\alpha\beta}(0) - v_{\alpha\alpha}(\mathbf{k}-\mathbf{k}')\delta_{\alpha\beta} \quad , \tag{21}$$

this leads to the variation of

$$E = \sum_{\mathbf{k}\alpha} \varepsilon_{\mathbf{k}} n_{\mathbf{k}\alpha} + \frac{1}{2V} \sum_{\mathbf{k}\mathbf{k}'\alpha\beta} f_{\alpha\beta}(\mathbf{k},\mathbf{k}') n_{\mathbf{k}\alpha} n_{\mathbf{k}'\beta} - \mu \left(\sum_{\mathbf{k}\alpha} n_{\mathbf{k}\alpha} - N \right) \quad , \tag{22}$$

where μ is the chemical potential, which yields

$$\mu = \varepsilon_{\mathbf{k}} + \frac{1}{V} \sum_{\mathbf{k}'\beta} f_{\alpha\beta}(\mathbf{k},\mathbf{k}') n_{\mathbf{k}'\beta} \quad ; \tag{23}$$

here, $\mathbf{k}$ is on the Fermi surface, where the small variations $\delta n_{\mathbf{k}\alpha}$ are localized, and the symmetry of the f-function has been used. Equation (23) is an extremely important equation; it must be fulfilled together with the conservation of the number of particles,

$$\sum_{\mathbf{k}\alpha} n_{\mathbf{k}\alpha} = N \quad ; \tag{24}$$

in fact, it defines the Fermi surface, and shows, together with (24), that both the volume and the shape of the Fermi sea are preserved, *i.e.* the Fermi sea is preserved, as expected; this statement is also known as Luttinger's "theorem", and $\mu = \partial E/\partial N$ for a change in the Fermi distribution at the Fermi surface is also known as van Hove's "theorem". The full variation of E with respect to the Fermi sea so determined gives the elementary excitations; this variation is

$$\begin{aligned} \delta E \;\; = \;\; & \sum_{\mathbf{k}\alpha} \varepsilon_{\mathbf{k}} \delta n_{\mathbf{k}\alpha} + \frac{1}{V} \sum_{\mathbf{k}\mathbf{k}'\alpha\beta} f_{\alpha\beta}(\mathbf{k},\mathbf{k}') n_{\mathbf{k}\alpha} \delta n_{\mathbf{k}'\beta} + \\ & + \frac{1}{2V} \sum_{\mathbf{k}\mathbf{k}'\alpha\beta} f_{\alpha\beta}(\mathbf{k},\mathbf{k}') \delta n_{\mathbf{k}\alpha} \delta n_{\mathbf{k}'\beta} \quad , \end{aligned} \tag{25}$$

where the important remark is to be made that the $n\delta n$-term vanishes; indeed, the δn-variations are just outside the Fermi sea, the excited states being thus "orthogonal" to the ground-state. Therefore,

$$\delta E = \sum_{\mathbf{k}\alpha} \varepsilon_{\mathbf{k}} \delta n_{\mathbf{k}\alpha} + \frac{1}{2V} \sum_{\mathbf{k}\mathbf{k}'\alpha\beta} f_{\alpha\beta}(\mathbf{k},\mathbf{k}') \delta n_{\mathbf{k}\alpha} \delta n_{\mathbf{k}'\beta} \quad , \tag{26}$$

and the energy of an elementary excitation is

$$\tilde{\varepsilon}_{\mathbf{k}\alpha} = \varepsilon_{\mathbf{k}} + \frac{1}{V} \sum_{\mathbf{k}'\beta} f_{\alpha\beta}(\mathbf{k},\mathbf{k}') \delta n_{\mathbf{k}'\beta} \quad ; \tag{27}$$

these are quasi-particles, lying just outside the Fermi sea (and may also be classified as if they had a one-half spin). This is the basis of Landau's theory of the Fermi liquid, and the f-function given by (21) is the scattering amplitude of this theory, to the first approximation. It is worth

remarking here the harmonic oscillators structure of the quasi-particle energy (26). As known, while the ordinary sound can not propagate as local density oscillations through the ideal, gas of quasi-particles with infinite viscosity at zero temperature, the liquid may oscillate as a whole through its quasi-particles, which now support a quasi-classical dynamics; in the sense that their energy, as described by (27), depends now both on momentum and position, the latter through the variations in the Fermi distribution; the time changes in the δn-coordinates are described by the energies $\widetilde{\varepsilon}_{\mathbf{k}\alpha}$, which play now the role of the hamiltonian either in the Poisson brackets, leading thus to the Boltzmann equation, or in the quantum commutators, as for a quantum dynamics; yielding in either case, beside the particle-hole excitations, the collective oscilations of the zero sound, which is the quantum guise the ordinary sound takes in Fermi liquids at zero temperature. In principle, these modes add their contribution to energy (and to the wavefunction), which, however, is insignificant in this case.

For strong, or long-range, interactions the validity condition for the perturbation scheme expressed by (4) may not be fulfilled. While the former case is rather unphysical, the typical example for the latter is provided by the Coulomb interacting electrons. The perturbation scheme may not be valid in terms of the original coordinates of the individual particles, but the many-body systems possess an additional type of coordinates, called collective coordinates, which describe the variations of particle density, and the motion of the ensemble as a whole. These collective modes, which corresponds to the zero sound discussed above, and which in the case of the electrons are called plasmons, screen the long-range interaction, whose strong effects are spent on the plasmon zero-point oscillations. The remaining screened potential is of a short range, and the perturbation scheme may be valid for it. This particular effect is contained in what is called the "random phase approximation", and, obviously, it is not a perturbation effect; though it can be obtained perturbationally, by suming up a sub-series of perturbations, which, however, is not formally convergent. Properly, the "random phase approximation" means the reduction of sums like

$$\sum_i e^{i\mathbf{q}\mathbf{r}_i} \quad , \tag{28}$$

where the summation is over all the randomly distributed particles, to

$$\sum_i e^{i\mathbf{q}\mathbf{r}_i} \cong N\delta_{\mathbf{q}0} \quad ; \tag{29}$$

as such, the "approximation" is valid for high concentrations, affects the long-range components of the motion, and has the aspect of a mean-field (to be self-consistently determined), as it does not depend anymore on the particle coordinates. A derivation of the static random phase approximation for fermions can readily be obtained as follows. Suppose that when the interaction is switched on the particle density changes from n_0 to n by the amount δn; the corresponding change in the kinetic energy is

$$\delta K = \int d\mathbf{r} \cdot \mu\delta n \quad , \tag{30}$$

where μ is the chemical potential. According to the random phase approximation this change may be written as

$$\mu\delta n_\alpha = (3/2)n_\alpha\delta\varepsilon_\alpha \quad , \tag{31}$$

since $\mu \sim \varepsilon \sim k_F^2$ and $n \sim k_F^3$; noteworthy, n_α in (31) is the concentration, and, though not very relevant, the spin labels are introduced. The change in the kinetic energy may be thought of as arising from the variation $-\varphi$ of a potential energy (mean-field),

$$\delta K = \int d\mathbf{r} \cdot \mu\delta n = -\varphi = \int d\mathbf{r} \cdot n_{0\alpha}\delta\varepsilon_\alpha \quad , \tag{32}$$

where now $n_{0\alpha}$ includes the density variations, and spin summations are understood. The potential φ spends interaction, and the total potential energy

$$\varphi + U = \varphi + \frac{1}{2}\int d\mathbf{r}d\mathbf{r}' \cdot v_{\alpha\beta}(\mathbf{r}-\mathbf{r}')n_{0\alpha}(\mathbf{r})n_{0\beta}(\mathbf{r}') \tag{33}$$

should be minimized (the form (15) is used for interaction, ignoring the self-interaction). The variation of (33) leads to

$$-\delta\varepsilon_\alpha + \int d\mathbf{r}' \cdot v_{\alpha\beta}(\mathbf{r}-\mathbf{r}')n_{0\beta}(\mathbf{r}') = 0 \ ; \tag{34}$$

obviously, the total energy $K + \delta K + \varphi + U = K + U$ is left unchanged; using (31) one obtains

$$-\frac{2\mu}{3n_\alpha}\delta n_\alpha + \int d\mathbf{r}' \cdot v_{\alpha\beta}(\mathbf{r}-\mathbf{r}')n_{0\beta}(\mathbf{r}') = 0 \quad , \tag{35}$$

or

$$-\frac{2\mu}{3n_\alpha}\delta n_\alpha + \int d\mathbf{r}' \cdot v_{\alpha\beta}(\mathbf{r}-\mathbf{r}')\left[n_\beta(\mathbf{r}') - \delta n_\beta(\mathbf{r}')\right] = 0 \ . \tag{36}$$

The two equations above are easily solved for δn_α by Fourier transforms; though not very realistic, one may assume, for the sake of some generality, $v_{\alpha\beta} = v + \alpha\beta \cdot u$, where $\alpha, \beta = \pm 1$, as for one-half spin fermions; then one obtains

$$\delta n_{\mathbf{q}\alpha} = \frac{3n_\alpha}{2\mu}v_{\alpha\beta}(\mathbf{q})n_{0\mathbf{q}\beta} = \frac{3n_\alpha}{2\mu}\tilde{v}_{\alpha\beta}(\mathbf{q})n_{\mathbf{q}\beta} \quad , \tag{37}$$

where

$$\tilde{v}(\mathbf{q}) = \frac{v(\mathbf{q})}{1 + 3nv(\mathbf{q})/2\mu} \tag{38}$$

and

$$\tilde{u}(\mathbf{q}) = \frac{u(\mathbf{q})}{1 + 3nu(\mathbf{q})/2\mu} \ . \tag{39}$$

Using (37) the interaction given by (15) and (17) can be written approximately as

$$\begin{aligned} U &= \frac{1}{2}\int d\mathbf{r}d\mathbf{r}' \cdot v_{\alpha\beta}(\mathbf{r}-\mathbf{r}')n_{0\alpha}(\mathbf{r})n_{0\beta}(\mathbf{r}') \cong \\ &\cong \frac{1}{2}\int d\mathbf{r}d\mathbf{r}' \cdot \tilde{v}_{\alpha\beta}(\mathbf{r}-\mathbf{r}')n_\alpha(\mathbf{r})n_\beta(\mathbf{r}') \quad , \end{aligned} \tag{40}$$

for short wavelengths (up to the self-interaction), and the Fermi liquid theory could now be started as before, in principle, with a f-function given by

$$f_{\alpha\beta}(\mathbf{k},\mathbf{k}') = -\tilde{v}_{\alpha\alpha}(\mathbf{k}-\mathbf{k}')\delta_{\alpha\beta} \ ; \tag{41}$$

the direct contributions are cancelled by the stabilizing background in this case, and the bare potential is replaced by the screened (or dressed) one; remarkably enough is the fact that the dressing factor $nv(\mathbf{q})/\mu$ is, practically, the same as the parameter $\overline{v}/(\hbar^2/ma^2)$ by means of which the validity of the perturbation theory is assessed by (4); this is why the potential dressing is irrelevant for weak short-range interaction; while for strong, long-range interaction the validity of the perturbation theory may remain, in principle, to be checked for the dressed interaction; for Coulomb interacting electrons it still does not hold. However, this is only part of the story;

because, the particle-hole excitations and the collective modes can obviously be obtained from the dynamics of $\delta n_{\mathbf{k}\alpha}$ in (26) and (27) with an f-function

$$f_{\alpha\beta}(\mathbf{k},\mathbf{k}') = v_{\alpha\beta}(\mathbf{q}) - \widetilde{v}_{\alpha\alpha}(\mathbf{k}-\mathbf{k}')\delta_{\alpha\beta} \quad , \quad \mathbf{q} = \mathbf{k}-\mathbf{k}' \to 0 \quad , \tag{42}$$

where the main part is played now by the direct bare interaction; indeed, the long-range collective modes are not screened by the background (but they do screen the individual motion of the particles). In other words, when the collective motion of the ensemble is pursued the first form of the interaction U in (40), with the bare potential, is valid, and not the second one, since the density variations involve now a local non-equilibrium, as for a collective motion of the ensemble as a whole. The zero-point contribution of these collective modes is to be added to the ground-state energy (and to the wavefunction), a competition appearing thereby between the long-range oscillations of the collective modes and the short-range contribution of the individual motion of the particles; this competition may usually be resolved, in principle, by minimizing the ground-state energy with respect to this splitting of the total number of the degrees of freedom.

For Coulomb interacting electrons $u = 0$ an f-function of the form (41) and (42) could be written as

$$f_{\alpha\beta}(\mathbf{k},\mathbf{k}') = -\frac{4\pi e^2}{|\mathbf{k}-\mathbf{k}'|^2 + q_{TF}^2}\delta_{\alpha\beta} \quad , \tag{43}$$

and

$$f_{\alpha\beta}(\mathbf{k},\mathbf{k}') = \frac{4\pi e^2}{q^2} - \frac{4\pi e^2}{|\mathbf{k}-\mathbf{k}'|^2 + q_{TF}^2}\delta_{\alpha\beta} \quad , \quad \mathbf{q} = \mathbf{k}-\mathbf{k}' \to 0 \quad , \tag{44}$$

where

$$q_{TF} = \left(4mk_F e^2/\pi\hbar^2\right)^{1/2} \tag{45}$$

as given by (38) (for a spherical Fermi surface) is the Thomas-Fermi wavevector (as a matter of fact, equation (36) is equivalent with the Poisson equation and with the linearized form of theThomas-Fermi equation). The screened Coulomb interaction is $v(\mathbf{r}) = e^2\exp(-q_{TF}r)/r$, and one can see that the long-range part of the Coulomb interaction is cut off, while the short-range part is practically unaffected, extending over a distance $a_{TF} \sim 1/q_{TF} \sim (a_H a)^{1/2}$, where $a_H = \hbar^2/me^2 = 0.53$Å is the Bohr radius (short-range oscillations of wavelength $\sim a/2$, called Friedel's oscillations, may appear for short-range corrections to the screened potential). As one can see, for instance from (38), the Coulomb potential is reduced in the long-range limit to $\tilde{v} \sim \mu/n \sim \mu a^3$, and the validity condition (4) for the perturbation theory is actually not satisfied, as expected; the formal perturbation parameter would be $q_{TF}^2/k_F^2 \sim a/a_H$, and, as known, one finds again that, formally, the electrons behave the freer the denser they are. One can say that the Coulomb interacting electrons are indeed strongly coupled. The random phase approximation does, in fact, restore the short-range variations of a uniform density, indicating the actual equilibrium picture of the electrons surrounded by a cloud of positive background, which screens out the bare Coulomb interaction; as a matter of fact the dielectric function is given by $\varepsilon = v(\mathbf{q})/\tilde{v}(\mathbf{q}) = 1 + (3n/2\mu)v(\mathbf{q}) = 1 + q_{TF}^2/q^2$ (and a dynamic dielectric function can be derived, depending on the excitation frequency too, by using a similar reasoning; incidentally, one may remark that for a classical plasma the factor 3/2 does not appear anymore in the screening formulas, while μ is replaced by the temperature T, leading thus to the Debye-Huckel formula). What is indeed remarkable is the magnitude of the Thomas-Fermi wavelength $a_{TF} \sim (a_H a)^{1/2}$, which is shorter than the inter-particle separation (and, of course, longer than the Bohr radius). As a general remark, one can see that dressing up the interaction by static random phase approximation is practically not of much avail in treating the interaction effects; a dynamic random phase approximation would, however, do better.

The dynamics of the Coulomb interacting electrons has however an essential element, namely the plasmons. Suppose that the particles are slightly displaced by $\mathbf{u}(\mathbf{r})$, such that $\mathbf{r}_i \to \mathbf{r}_i + \mathbf{u}(\mathbf{r}_i)$; the change in density can be written as

$$\delta n = -n div \mathbf{u} \quad , \tag{46}$$

where n is the electrons concentration. A decoupling can be carried out, by assuming that δn contains the long-range degrees of freedom, while the remaining density $n(\mathbf{r})$ corresponds to the short-range degres of freedom. Indeed, the interaction (15) can now be written as

$$U = \frac{1}{2}\int d\mathbf{r}d\mathbf{r}' \cdot v(\mathbf{r}-\mathbf{r}')\delta n(\mathbf{r})\delta n(\mathbf{r}') + \frac{1}{2}\int d\mathbf{r}d\mathbf{r}' \cdot v(\mathbf{r}-\mathbf{r}')n(\mathbf{r})n(\mathbf{r}') \quad , \tag{47}$$

(self-interaction omitted for the moment), and with the Fourier transform

$$\mathbf{u}(\mathbf{r}) = \frac{1}{V}\sum_{\mathbf{q}} \mathbf{u}_{\mathbf{q}} e^{i\mathbf{q}\mathbf{r}} \quad , \quad \delta n_{\mathbf{q}} = -in\mathbf{q}\mathbf{u}_{\mathbf{q}} \quad , \tag{48}$$

the first term in the interaction above becomes

$$U_{lr} = \frac{1}{2V}\sum_{\mathbf{q}} v(\mathbf{q})\delta n_{\mathbf{q}}\delta n_{-\mathbf{q}} = \frac{1}{2V}\sum_{\mathbf{q}} v(\mathbf{q})n^2q^2u_{\mathbf{q}}u_{-\mathbf{q}} \quad , \tag{49}$$

where only the longitudinal components of the displacement field $\mathbf{u}$ have been retained (and the $\mathbf{q}=0$-component is skipped from the summation). A similar decomposition can be performed for the kinetic part of the hamiltonian, which becomes

$$K + \frac{1}{2V}\sum_{\mathbf{q}} mn\, \dot{u}_{\mathbf{q}}\dot{u}_{-\mathbf{q}} \quad , \tag{50}$$

where the coupling between particles and the field $\mathbf{u}$ is neglected for the moment. Now, it is easy to see that the long-range part associated with the displacement field $\mathbf{u}$ gives the plasma oscillations,

$$\omega_p = \left(4\pi ne^2/m\right)^{1/2} \quad ; \tag{51}$$

the dispersion relation (51) extends approximately up to some wavevector $\sim \hbar\omega_p/\hbar v_F \sim (a/a_H)^{1/2}k_F$, at most, where the plasmons start to decay on electron-hole excitations (or create electron-hole excitations); to this extent one may neglect the electron-plasmon coupling. The energy of the zero-point plasmon oscillations is given by

$$E_{lr} = \sum_{\mathbf{q}} \hbar\omega_p/2 - \frac{Ne^2}{\pi}q_c = \frac{V}{12\pi^2}\hbar\omega_p q_c^3 - \frac{Ne^2}{\pi}q_c \quad , \tag{52}$$

where the self-interaction energy in (15) has been reintroduced; the cut-off wavevector q_c is therefore given by

$$q_c \sim 1/a^{3/4}a_H^{1/4} \quad , \tag{53}$$

and, quite remarkably, it is shorter than q_{TF}, though longer than k_F in the low-density limit:

$$q_c \sim q_{TF}(a_H/a)^{1/4} \sim k_F(a/a_H)^{1/4} \quad ; \tag{54}$$

actually, for metallic densities $q_c \sim k_F$. Therefore, the electron-plasmon decoupling is realized, the electron-electron (very short-range) interaction is largely ineffective, the ground-state energy is lowered by

$$E_{lr}(q_c) = -\frac{2}{3}\frac{Ne^2}{\pi}q_c = -\frac{2}{3}\frac{Ne^2}{\pi}(m\omega_p/\hbar)^{1/2} \quad , \tag{55}$$

and the ensemble may be termed a genuine electron liquid; the remaining short-range electron interaction leads to a weakly-coupled Fermi liquid, whose f-function is $f_{\alpha\beta}(\mathbf{k},\mathbf{k}') = -4\pi e^2/\left|\mathbf{k}-\mathbf{k}'\right|^2 \delta_{\alpha\beta}$ for a "hard" cut-off $|\mathbf{k}-\mathbf{k}'| > q_c$, which, however, has small effects; the corresponding effective mass of the electrons is practically left unchanged, their life-time effects are small, and a certain, small uncertainty in the ground-state energy is brought, corersponding to the Hartree-Fock contribution of this interaction; the spin susceptibility is likewise slightly affected. For very large inter-particle separation the Coulomb potential may realize its minimum value for periodic structures, and the liquid may become a solid, which is called a Wigner solid. A similar theory can formally be attempted for weak, short-range interactions, but it turns out immediately to be inconsistent, as expected. Therefore, the long-range electron degrees of freedom are taken in the oscillating plasmons, precisely those which screen out the bare Coulomb interaction (and which are in fact associated with the oscillations of the macroscopic electric field of polarization; according to (46) this field is proportional to the displacement $\mathbf{u}$); while the remaining short-range degrees of freedom are left for the motion of the individual electrons (associated with the microscopic, rapidly oscillating fields); the latter implies such strong excitation processes that they are much unfavoured, and this remaining part of the motion, though formally may be described perturbationally, involves practically small corrections to the free electron picture.