Differential operators on orbits of coherent states

S. Berceanu, A. Gheorghe

National Institute for Physics and Nuclear Engineering
Department of Theoretical Physics
PO BOX MG-6, Bucharest-Magurele, Romania
E-mail: Berceanu@theor1.theory.nipne.ro; Cezar@theor1.theory.nipne.ro

Abstract

We emphasize some properties of coherent state groups, i.e. groups whose quotient with the stationary groups, are manifolds which admit a holomorphic embedding in a projective Hilbert space. We determine the differential action of the generators of the representation of coherent state groups on the symmetric Fock space attached to the dual of the Hilbert space of the representation. This permits a realization of coherent state Lie algebras by first-order differential operators with holomorphic polynomial coefficients on Kähler coherent state orbits.

1 Introduction

The differential action of the generators of the groups on coherent state manifolds which have the structure of hermitian symmetric spaces can be written down as a sum of two terms, one a polynomial P, and the second one a sum of partial derivatives times some polynomials Q-s, the degree of polynomials being less than 3 [2, 3]. It is interesting to investigate the same problem as in [2, 3] on flag manifolds [11]. Some results are available [13], but they are not easily handled.

Our investigations on the differential action of the generators of hermitian groups on hermitian symmetric spaces have been extended to semisimple Lie groups acting on coherent state manifolds which admit a Kähler structure, and explicit formulas for the polynomials P and Q-s have been given [6]. Explicit formulas for the simplest example of a compact nonsymmetric coherent state manifold, $SU(3)/S(U(1)\times U(1)\times U(1))$, where the degree of the polynomial is already 3, have been also obtained [6]. Here we discuss in the context of the so called coherent state (shortly, CS)-groups [14, 15, 16, 21] the space of functions on which these differential operators act.

We emphasize some properties of CS-groups, i.e. groups whose quotient with the stationary groups are manifolds which admit a holomorphic embedding in a projective Hilbert space. This class of groups contains all compact groups, all simple hermitian groups, certain solvable groups and also mixed groups as the semidirect product of the Heisenberg group and the symplectic group [21]. We determine the differential action of the generators of the representation of the CS-group on the symmetric Fock space attached to the dual of the Hilbert space of the representation.

The coherent states are a useful tool of investigation of quantum and classical systems [25]. It was shown in [2, 3] that a linear Hamiltonian in the generators of the groups implies equivalent quantum and classical evolution. It was proved that for Hermitian symmetric spaces the evolution equation generated by Hamiltonians which are linear in the generators of the group is a matrix Riccati equation. It is interesting to see how it looks like the corresponding equation of motion generated by linear Hamiltonians for CS-manifolds. So, the present paper gives the technical tools for this calculation. The degrees of the polynomials P and Q-s for flag manifolds are grater than 2. We have underlined the realization of coherent state algebras by differential operators [9], giving explicit formulas for the semisimple case.

Another field of possible applications is the determination of the Berry phase [28] on CS-manifolds. In [2, 3] there were presented explicit expressions for the Berry phase for the complex Grassmann manifold. These results were used further [8] for explicit calculation of the symplectic area of geodesic triangles on the complex Grassmann manifold and its noncompact dual. Also we have considered explicit boson expansions for collective models on Kähler CS-orbits in [7].

The general framework in which this paper must be considered is the deep relationship between coherent states and geometry [4].

The paper is laid out as follows. §2 contains the definition of CS-orbits, in the context of Lisiecki [14, 15, 16] and Neeb [21]. The geometry of coherent state manifolds for compact groups was previously considered in [1]. §3 deals with the so called Perelomov's CS-vectors. The coherent vectors are defined taking into account that CS-representations are realized by highest weight representations and the manifold of coherent states is a reductive homogeneous space. In §4 we construct the space of functions on which the differential operators will act. In §5 we study the representations of Lie algebras of CS-groups by differential operators.

We use for the scalar product the convention: $(\lambda x, y) = \bar{\lambda}(x, y)$, $x, y \in \mathcal{H}, \lambda \in \mathbb{C}$.

2 CS-representations

Let us consider the triplet $(G, T, \mathcal{H})$, where T is a continuous, unitary representation of the Lie group G on the separable complex Hilbert space $\mathcal{H}$. Let us denote by $\mathcal{H}^\infty$ the dense subspace of $\mathcal{H}$ consisting of those vectors v for which the orbit map $G \rightarrow \mathcal{H}, g \mapsto T(g).v$ is smooth. Let us pick up $e_0 \in \mathcal{H}^\infty$ and let the notation: $e_{g,0} := T(g).e_0, g \in G$. We have an action $G \times \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty$, $g.e_0 := e_{g,0}$. When there is no possibility of confusion, we write just e_g for $e_{g,0}$. Let us denote by $[\ ] : \mathcal{H}^* := \mathcal{H} \setminus \{0\} \rightarrow \mathbb{P}(\mathcal{H}) = \mathcal{H}^*/\sim$ the projection with respect to the equivalence relation $[\lambda x] \sim [x]$, $\lambda \in \mathbb{C}^*$, $x \in \mathcal{H}^*$. So, $[.] : \mathcal{H}^* \rightarrow \mathbb{P}(\mathcal{H})$, $[v] = \mathbb{C}v$. The action $G \times \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty$ extends to the action $G \times \mathbb{P}(\mathcal{H}^\infty) \rightarrow \mathbb{P}(\mathcal{H}^\infty), g.[v] := [g.v]$.

Let us now denote by H the isotropy group $H := G_{[e_0]} := \{g \in G | g.e_0 \in \mathbb{C}e_0\}$. We shall consider (generalized) coherent states on complex homogeneous manifolds $M \cong G/H$, imposing the restriction that M be a complex submanifold of $\mathbb{P}(\mathcal{H}^\infty)$.

Definition 1. a) The orbit M is called a CS-*orbit* if there exits a holomorphic embedding $\iota : M \hookrightarrow \mathbb{P}(\mathcal{H}^\infty)$. In such a case M is also called CS-*manifold*.

b) $(T, \mathcal{H})$ is called a CS-*representation* if there exists a cyclic vector $0 \neq e_0 \in \mathcal{H}^\infty$ such that M is a CS-orbit.

c) The groups G which admit CS-representations are called CS-*groups*, and their Lie algebras $\mathfrak{g}$ are called CS-*Lie algebras*.

The G-invariant complex structures on the homogeneous space $M = G/H$ can be introduced in an algebraic manner. For $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Lie group G, let us define the unbounded operator $dT(X)$ on $\mathcal{H}$ by $dT(X).v := d/dt|_{t=0} T(\exp tX).v$ whenever the limit on the right hand side exists. We obtain a representation of the Lie algebra $\mathfrak{g}$ on $\mathcal{H}^\infty$, *the derived representation*, and we denote $\boldsymbol{X}.v := dT(X).v$ for $X \in \mathfrak{g}, v \in \mathcal{H}^\infty$. Extending dT by complex linearity, we get a representation of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ on the complex vector space $\mathcal{H}^\infty$. Lemma XV.2.3 p. 651 in [21] and Prop. 4.1 in [16] determine when a smooth vector generates a complex orbit in $\mathbb{P}(\mathcal{H}^\infty)$. Let now denote by $B :=< \exp_{G_{\mathbb{C}}} \mathfrak{b} >$ the Lie group corresponding to the Lie algebra $\mathfrak{b}$, with $\mathfrak{b} := \overline{\mathfrak{b}(e_0)}$, where $\mathfrak{b}(v) := \{X \in \mathfrak{g}_{\mathbb{C}} : X.v \in \mathbb{C}v\} = (\mathfrak{g}_{\mathbb{C}})_{[v]}$. The group B will be supposed to be closed in the complexification $G_{\mathbb{C}}$ of G, and in fact this assumption is justified for CS-groups G (cf. Lemma XII.1.2. p. 495 in [21]). Then the complex structure on M is induced by an embedding in a complex manifold, $i_1 : M \cong G/H \hookrightarrow G_{\mathbb{C}}/B$. We consider such manifolds which admit a holomorphic embedding $i_2 : G_{\mathbb{C}}/B \hookrightarrow \mathbb{P}(\mathcal{H}^\infty)$. Then the embedding $\iota = i_1 \circ i_2$, $\iota : M \hookrightarrow \mathbb{P}(\mathcal{H}^\infty)$ is a holomorphic embedding, in the sense that the complex structure comes as in Theorem XV.1.1 and Proposition XV.1.2 p. 646 in [21].

3 CS-vectors

Now we construct what we call Perelomov's (generalized) coherent state vectors, or simply CS-vectors, based on the homogeneous manifold $M \cong G/H$. Usually [25], this construction is done for (semi)simple Lie groups G with $H := K$, where K is a maximally compact subgroup of G. Here we do this construction for the CS-groups G in the meaning of Definition 1.

We denote also by T the holomorphic extension of the representation T of G to the complexification $G_{\mathbb{C}}$ of G, whenever this holomorphic extension exists. In fact, it can be shown that in the situations under interest in this paper, this holomorphic extension exists [18, 20]. Then there exists the homomorphisms χ_0 (χ), $\chi_0 : H \rightarrow \mathbb{T}$, ($\chi : B \rightarrow \mathbb{C}^*$), such that $H = \{g \in G | e_g = \chi_0(g) e_0\}$ (respectively, $B = \{g \in G_{\mathbb{C}} | e_g = \chi(g) e_0\}$), where $\mathbb{T}$ denotes the torus $\mathbb{T} := \{z \in \mathbb{C} | |z| = 1\}$.

For the homogeneous space $M = G/H$ of cosets $\{gH\}$, let $\lambda : G \rightarrow G/H$ be the natural projection $g \mapsto gH$, and let $o := \lambda(\mathbf{1})$, where $\mathbf{1}$ is the unit element of G. Choosing a section $\sigma : G/H \rightarrow G$ such that $\sigma(o) = \mathbf{1}$, every element $g \in G$ can be written down as $g = \tilde{g}(g) h(g)$, where $\tilde{g}(g) \in G/H$ and $h(g) \in H$. Then we have $e_g = e^{i\alpha(h(g))} e_{\tilde{g}(g)}$, where $e^{i\alpha(h(g))} = \chi_0(h)$. Now we take into account that M also admits an embedding in $G_{\mathbb{C}}/B$. We choose a local system of coordinates parametrized by z_g (denoted also simply z, where there is no possibility of confusion) on $G_{\mathbb{C}}/B$. Choosing a section $G_{\mathbb{C}}/B \rightarrow G_{\mathbb{C}}$ such that any element $g \in G_{\mathbb{C}}$ can be written as $g = \tilde{g}_b b(g)$, where $\tilde{g}_b \in G_{\mathbb{C}}/B$, and $b(g) \in B$, we have $e_g = \Lambda(g) e_{z_g}$, where $\Lambda(g) = \chi(b(g)) = e^{i\alpha(h(g))}(e_{z_g}, e_{z_g})^{-\frac{1}{2}}$.

Let us denote by $\mathfrak{m}$ the vector space orthogonal to $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$, i.e. we have the vector space decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Even more, it can be shown that the vector space decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is Ad H-invariant. The homogeneous spaces $M \cong G/H$ with this decomposition are called *reductive spaces* (cf. [22]) and it can be proved that the CS-manifolds are reductive spaces. More exactly, using Lemma III.2.(iii) in [19] and Lemma XV.2.5 p. 652 in [21], it can be proved that:

Remark 1. *The homogeneous coherent state manifold $M \cong G/H$, for which the isotropy representation has discrete kernel, or for admissible Lie algebras and faithful CS-representations, is a reductive space.*

So, *the tangent space to M at o can be identified with* $\mathfrak{m}$. Now remember (cf. Proposition XV.2.4 p 651, Proposition XV.2.6 p. 652 in [21] and Theorem XV.2.10 p. 655 in [21], where the algebra $\mathfrak{g}$ is supposed to be admissible (cf. Definition VII.3.2 at p. 252)) that for CS-groups, the CS-representations are highest weight representations (cf. Definition X.2.9 p. 399 in [21]), and the vector e_0 is a primitive element of the generalized parabolic algebra $\mathfrak{b}$ (cf. Definition IX.1.1 p. 328 in [21]).

Let us denote $\boldsymbol{X} := dT(X), X \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$, where $\mathfrak{U}$ denotes the universal enveloping algebra. Let $\tilde{g}(g) = \exp X, \tilde{g}(g) \in G/H,\ X \in \mathfrak{m},\ e_{\tilde{g}(g)} = \exp(\boldsymbol{X})e_0, X \in \mathfrak{m}$. Let us remember again Theorem XV.1.1 p. 646 in [21]. Note that $T_o(G/H) \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{g}_{\mathbb{C}}/\bar{\mathfrak{b}} \cong (\mathfrak{b}+\bar{\mathfrak{b}})/\bar{\mathfrak{b}} \cong \mathfrak{b}/h_{\mathbb{C}}$, where we have a linear isomorphism $\alpha : \mathfrak{g}/\mathfrak{h} \cong \mathfrak{g}_{\mathbb{C}}/\bar{\mathfrak{b}}$, $\alpha(X+\mathfrak{h}) = X+\bar{\mathfrak{b}}$ ([19]). We can take instead of $\mathfrak{m} \subset \mathfrak{g}$ the subspace $\mathfrak{m}' \subset \mathfrak{g}_{\mathbb{C}}$ complementary to $\bar{\mathfrak{b}}$, or the subspace of $\mathfrak{b}$ complementary to $\mathfrak{h}_{\mathbb{C}}$. Then let $x(s)$, $s \in [0,1]$, be the one-parameter subgroup generated by $X \in \mathfrak{m}'$ and $x^*(s)$ his image in the reductive homogeneous space $M \cong G/H \hookrightarrow G_{\mathbb{C}}/B$, i.e. $x^*(s) = \exp(sX).o$. If we choose a local *canonical* system of coordinates $\{z_\alpha\}$ with respect to the basis $\{X_\alpha\}$ in $\mathfrak{m}$', then we can introduce the vectors

$$e_z = \exp(\sum_{X_\alpha \in \mathfrak{m}'} z_\alpha \boldsymbol{X}_\alpha).e_0 \in \mathcal{H}. \tag{3.1}$$

We get

$$e_{\sigma(z)} = T(\sigma(z)),\ z \in M, \tag{3.2}$$

and we prefer to choose local coordinates such that

$$e_{\sigma(z)} = N(z) e_{\bar{z}}, \quad N(z) = (e_{\bar{z}}, e_{\bar{z}})^{-1/2}. \tag{3.3}$$

Equations (3.1), (3.2), and (3.3) define locally the *coherent vector mapping*

$$\varphi : M \rightarrow \bar{\mathcal{H}},\ \varphi(z) = e_{\bar{z}}, \tag{3.4}$$

where $\bar{\mathcal{H}}$ denotes the Hilbert space conjugate to $\mathcal{H}$. We call the vectors $e_{\bar{z}} \in \bar{\mathcal{H}}$ indexed by the points $z \in M$ *Perelomov's coherent state vectors* [25]. Below we give more details about this construction.

Let V be a generalized highest weight module with highest weight λ and primitive element v_λ. Then V carries a non-degenerate contravariant hermitian form if and only if $V \cong L(\lambda, \mathfrak{b}) := M(\lambda, \mathfrak{b})/R$, where $M(\lambda, \mathfrak{b})$ is the Verma module and R is the radical of the contravariant hermitian form (cf. Definition IX.1.8 p. 333 in [21]). Now we suppose that V is a highest weight module with respect to a positive system of roots Δ^+. Let $\mathfrak{n}^{\pm} := \sum_{\alpha \in \Delta^+} \mathfrak{g}^{\pm\alpha}$ and $e_0 := v_\lambda$ a primitive element. Let us now take into account that $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-)\mathcal{U}(\mathfrak{b})$ and let us choose a canonical system of coordinates in the highest weight module with respect to a fixed base of $\mathfrak{g}$. One has locally finite representations by direct exponentiation of the module $L(\lambda, \mathfrak{b})$ (cf. Corollary XII.2.7 p. 523 to the globalization Theorem XII.2.6 p. 521 in [21]), and the *Perelomov's coherent state vectors can be obtained by just taking the exponential of images by the highest weight representation of elements of* $\mathcal{U}(\mathfrak{n}^-)$. We can apply a *Gauss type decomposition* as furnished by the Lemma XII.1.2. p. 495 in [21] and get the coherent vector mapping given by equations (3.2), (3.3), (3.4).

4 The symmetric Fock space $\mathcal{F}_{\mathcal{H}}$ as reproducing kernel Hilbert space

We have considered homogeneous CS-manifolds $M \cong G/H$ whose complex structure comes from the embedding $i_1 : M \hookrightarrow G_{\mathbb{C}}/B$. We have chosen a section $\sigma : G_{\mathbb{C}}/B \to G_{\mathbb{C}}$, and $G_{\mathbb{C}}$ can be regarded as a complex analytic principal bundle $B \xrightarrow{i} G_{\mathbb{C}} \xrightarrow{\lambda} G_{\mathbb{C}}/B$.

Let us introduce the function $f'_\psi : G_{\mathbb{C}} \to \mathbb{C}$, $f'_\psi(g) := (e_g, \psi), g \in G, \psi \in \mathcal{H}$. Then $f'_\psi(gb) = \chi(b)^{-1} f'_\psi(g), g \in G_{\mathbb{C}}, b \in B$, where χ is the continuous homomorphism of the isotropy subgroup B of $G_{\mathbb{C}}$ in $\mathbb{C}^*$. If the homomorphism χ is holomorphic, then *the coherent states realize the space of holomorphic global sections* $\Gamma^{\mathrm{hol}}(M, L_\chi) = H^0(M, L_\chi)$ *on the* $G_{\mathbb{C}}$*-homogeneous line bundle* L_χ *associated by means of the character* χ *to the principal* B*-bundle* (cf. [23]). Here the holomorphic line bundle is $L_\chi := M \times_\chi \mathbb{C}$, also denoted $L := M \times_B \mathbb{C}$ (cf. [12, 29]).

The local trivialization of the line bundle L_χ associates to every $\psi \in \mathcal{H}$ a holomorphic function f_ψ on a open set in $M \hookrightarrow G_{\mathbb{C}}/B$. Let the notation $G_S := G_{\mathbb{C}} \setminus S$, where S is the set $S := \{g \in G_{\mathbb{C}} | \alpha_g = 0\}$, and $\alpha_g := (e_g, e_0)$. G_S is a dense subset of $G_{\mathbb{C}}$. We introduce the function $f_\psi : G_S \mapsto \mathbb{C}$, $f_\psi(g) = \frac{f'_\psi(g)}{\alpha_g}, \psi \in \mathcal{H},\ g \in G_S$. The function $f_\psi(g)$ on G_S is actually a function of the projection $\lambda(g)$, holomorphic in $M_S := \lambda(G_S)$. We have introduced the function $f_\psi(g) = f_\psi(z_g) = \frac{(e_{\bar{z}_g}, \psi)}{(e_{\bar{z}_g}, e_0)}$ $((e_{z_g}, e_0) \neq 0)$ and also the coherent state map $\varphi : M \to \overline{\mathcal{H}}^{\infty}, \varphi(z) = e_{\bar{z}}, z \in \mathcal{V}_0$, where the canonical coordinates $z = (z_1, \ldots, z_n)$ constitutes a local chart on $\mathcal{V}_0 := M_S \to \mathbb{C}^n$, such that $0 = (0, \ldots, 0)$ corresponds to $\{B\}$. Note also that $\mathcal{V}_0 \equiv M \setminus \Sigma_0$, where $\Sigma_0 := \lambda(S)$ is the set of points of M for which the coherent state vectors are orthogonal to $e_0 \in \mathcal{H}$, called *polar divisor* of the point $z = 0$ (cf. [4]).

Supposing that *the line bundle* L_χ *is already very ample*, $\mathcal{F}_{\mathcal{H}}$ is defined as $\mathcal{F}_{\mathcal{H}} := \{f \in L^2(M, L) \cap \mathcal{O}(M, L) | (f, f)_{\mathcal{F}_{\mathcal{H}}} < \infty\}$ with respect to the scalar product

$$(f, g)_{\mathcal{F}_{\mathcal{H}}} = \int_M \bar{f}(z) g(z) d\nu_M(z, \bar{z}), \tag{4.1}$$

where $d\nu_M(z, \bar{z})$ is the invariant measure $\frac{d\mu_M(z,\bar{z})}{(e_{\bar{z}}, e_{\bar{z}})}$, and $d\mu_M(z, \bar{z})$ represents the Haar measure on M. It can be shown that the $\mathcal{F}_{\mathcal{H}} := L^{2,\text{hol}}(M, L_\chi)$ *is a closed subspace of* $L^2(M, L_\chi)$ *with continuous point evaluation* (cf. [24]).

Note that eq. (4.1) is nothing else than the Parseval (*overcompletness*) identity [10]:

$$(\psi_1, \psi_2) = \int_{M=G/K} (\psi_1, e_{\bar{z}})(e_{\bar{z}}, \psi_2) d\nu_M(z, \bar{z}), \ (\psi_1, \psi_2 \in \mathcal{H}). \tag{4.2}$$

Let us now introduce the map

$$\Phi : \mathcal{H}^\star \to \mathcal{F}_{\mathcal{H}}, \Phi(\psi) := f_\psi, f_\psi(z) = \Phi(\psi)(z) = (\varphi(z), \psi)_{\mathcal{H}} = (e_{\bar{z}}, \psi)_{\mathcal{H}}, \ z \in \mathcal{V}_0, \tag{4.3}$$

where we have identified the space $\overline{\mathcal{H}}$ with the dual space $\mathcal{H}^\star$ of $\mathcal{H}$.

In fact, our supposition that L_χ is already a very ample line bundle implies the validity of eq. (4.2) (cf. Theorem XII.5.6 p. 542 in [21], Remark VIII.5 in [17], and Theorem XII.5.14 p. 552 in [21]). Rosenberg and Vergne have shown that the projectively induced line bundle $\mathbb{L} \cong L_\chi$ is ample, i.e. there exists $n \in \mathbb{N}_0$ such that $L^{2,\text{hol}}(M, \mathbb{L}^n) \neq \{0\}$ (cf. Theorem 2.15 in [27]; see also §4 in [16]). If the line bundle is only a ample one, not every highest weight representation leads to square integrable representations, and the highest weight vector $e_0 := e_\lambda$ in the definition of coherent state vectors has to verify a condition which generalizes the Harish-Chandra condition in the semisimple case (cf. Theorem XII.5.14 p. 552 in [21] and Remark VIII.5 in [17]).

It can be seen that the group-theoretic relation (4.2) on homogeneous manifolds fits into Rawnsley's (global) realization [26] of Berezin's coherent states on quantizable Kähler manifolds [10]. We emphasize that, strictly speaking, *equation (4.2) should be considered with a partition of unity.*

Let us introduce the notation

$$K := \Phi \circ \varphi, \ K : M \to \mathcal{F}_{\mathcal{H}}, \quad K_w := f_{e_{\bar{w}}} \in \mathcal{F}_{\mathcal{H}}. \tag{4.4}$$

It can be defined a function, also denoted K, $K : M \times \overline{M} \to \mathbb{C}$, which on $\mathcal{V}_0 \times \overline{\mathcal{V}}_0$ reads

$$K(z, \overline{w}) := K_w(z) = (e_{\bar{z}}, e_{\bar{w}})_{\mathcal{H}}. \tag{4.5}$$

For fixed $z \in M$ the function $K(z, \overline{w})$ is defined for $w \notin \Sigma_z$ [5]. In the compact case $K(z, \overline{w}) = 0$ for $z \in M, \ w \in \Sigma_z$.

Taking into account (4.3) and supposing that eq. (4.2) is true, it follows that the function K (4.5) is a reproducing kernel. Using the terminology of ref. [21], we have:

Proposition 1. *Let $(T, \mathcal{H})$ be a CS-representation and let us consider the Perelomov's CS-vectors defined in (3.1)-(3.3). Suppose that the line bundle L is very ample. Then*

i) *The function $K : M \times \overline{M} \to \mathbb{C}$, $K(z, \overline{w})$ defined by equation (4.5), is a reproducing kernel.*

ii) *Let $\mathcal{F}_{\mathcal{H}}$ be the space $L^{2,\text{hol}}(M, L)$ endowed with the scalar product (4.1). Then $\mathcal{F}_{\mathcal{H}}$ is the reproducing kernel Hilbert space $\mathcal{H}_K \subset \mathbb{C}^M$ associated to the kernel K (4.5).*

iii) *The evaluation map Φ defined in eqs. (4.3) extends to an isometry*

$$(\psi_1, \psi_2)_{\mathcal{H}^\star} = (\Phi(\psi_1), \Phi(\psi_2))_{\mathcal{F}_{\mathcal{H}}} = (f_{\psi_1}, f_{\psi_2})_{\mathcal{F}_{\mathcal{H}}} = \int_M \overline{f}_{\psi_1}(z) f_{\psi_2}(z) d\nu_M(z), \tag{4.6}$$

and the overcompletness eq. (4.2) is verified.

5 Representations of CS-Lie algebras by differential operators

We remember the definitions of the functions f'_ψ and f_ψ, which allow to write down

$$f_\psi(z) = (e_{\bar{z}}, \psi) = \frac{(T(\bar{g})e_0, \psi)}{(T(\bar{g})e_0, e_0)}, \; z \in M, \; \psi \in \mathcal{H}. \tag{5.1}$$

So, we get

$$f_{T(\overline{g'}).\psi}(z) = \mu(g', z) f_\psi(\overline{g'}^{-1}.z), \tag{5.2}$$

where

$$\mu(g', z) = \frac{(T(\overline{g'}^{-1}\overline{g})e_0, e_0)}{(T(\overline{g})e_0, e_0)} = \frac{\Lambda(g'^{-1}g)}{\Lambda(g)}. \tag{5.3}$$

We remember that $T(g).e_0 = e^{i\alpha(g)} e_{\tilde{g}} = \Lambda(g) e_{z_g}$ where we have used the decompositions $g = \tilde{g}.h$, $(G = G/H.H)$; $g = z_g.b$ $(G_{\mathbb{C}} = G_{\mathbb{C}}/B.B)$. We have also the relation $\chi_0(h) = e^{i\alpha(h)}$, $h \in H$ and $\chi(b) = \Lambda(b)$, $b \in B$, where $\Lambda(g) = \frac{e^{i\alpha(g)}}{(e_{\bar{z}}, e_{\bar{z}})^{1/2}}$. We can also write down another expression for multiplicative factor μ appearing in eq. (5.2) using the CS-vectors

$$\mu(g', z) = \Lambda(\bar{g'})(e_{\bar{z}}, e_{\bar{z'}}) = e^{i\alpha(\bar{g'})} \frac{(e_{\bar{z}}, e_{\bar{z'}})}{(e_{\bar{z'}}, e_{\bar{z'}})^{1/2}}. \tag{5.4}$$

The following assertion is easy to be checked using successively eq. (5.3):

Remark 2. *Let us consider the relation (5.1). Then we have (5.2), where μ can be written down as in equations (5.3), (5.4). We have the relation $\mu(g, z) = J(g^{-1}, z)^{-1}$, i.e. the multiplier μ is the cocycle in the unitary representation $(T_K, \mathcal{H}_K)$ attached to the positive definite holomorphic kernel K defined by equation (4.5),*

$$(T_K(g).f)(x) := J(g^{-1}, x)^{-1}.f(g^{-1}.x), \tag{5.5}$$

and the cocycle verifies the relation

$$J(g_1 g_2, z) = J(g_1, g_2 z) J(g_2, z). \tag{5.6}$$

Note that *the prescription (5.5) defines a continuous action of* G *on* $\mathrm{Hol}(M,\mathbb{C})$ *with respect to the compact open topology on the space* $\mathrm{Hol}(M,\mathbb{C})$. *If* $K : M \times M \to \mathbb{C}$ *is a continuous positive definite kernel holomorphic in the first argument satisfying* $K(g.x,\overline{g.y}) = J(g,x)K(x,\overline{y})J(g,y)^*$, $g \in G$, $x, y \in M$, *then the action of* G *leaves the reproducing kernel Hilbert space* $\mathcal{H}_K \subseteq \mathrm{Hol}(M,\mathbb{C})$ *invariant and defines a continuous unitary representation* $(T_K,\mathcal{H}_K)$ *on this space* (cf. Prop. IV.1.9 p. 104 in Ref. [21]).

Let us consider the triplet $(G,T,\mathcal{H})$. Let $\mathcal{H}^0 := \mathcal{H}^\infty$, considered as a pre-Hilbert space, and let $B_0(\mathcal{H}^0) \subset \mathcal{L}(\mathcal{H})$ denote the set of linear operators $A : \mathcal{H}^0 \to \mathcal{H}^0$ which have a formal adjoint $A^\sharp : \mathcal{H}^0 \to \mathcal{H}^0$, i.e. $(x, Ay) = (A^\sharp x, y)$ for all $x, y \in \mathcal{H}^0$. Note that if $B_0(\mathcal{H}^0)$ is the set of unbounded operators on $\mathcal{H}$, then the domain $\mathcal{D}(A^*)$ contains $\mathcal{H}^0$ and $A^*\mathcal{H}^0 \subseteq \mathcal{H}^0$, and it makes sense to refer to the closure $\overline{A}$ of $A \in B_0(\mathcal{H}^0)$ (cf. [21] p. 29; here A^* is the adjoint of A).

Let $\mathfrak{g}$ be the Lie algebra of G and let us denote by $\mathcal{S} := \mathcal{U}(\mathfrak{g}_\mathbb{C})$ the semigroup associated with the universal enveloping algebra equipped with the antilinear involution extending the antiautomorphism $X \mapsto X^* := -\bar{X}$ of $\mathfrak{g}_\mathbb{C}$. *The derived representation* is defined as

$$dT : \mathcal{U}(\mathfrak{g}_\mathbb{C}) \to B_0(\mathcal{H}^0), \quad \text{with} \quad dT(X).v := \left.\frac{d}{dt}\right|_{t=0} T(\exp tX).v, X \in \mathfrak{g}. \tag{5.7}$$

Then dT is a *hermitian representation* of $\mathcal{S}$ on $\mathcal{H}^0$ (cf. Neeb [21], p. 30). Let us denote his image in $B_0(\mathcal{H}^0)$ with $\mathbf{A}_M := dT(\mathcal{S})$. If $\Phi : \mathcal{H}^\star \to \mathcal{F}_\mathcal{H}$ is the (Segal-Bargmann) isometry (4.3), we are interested in the study of the image of $\mathbf{A}_M$ via Φ as subset in the algebra of holomorphic, linear differential operators, $\Phi\mathbf{A}_M\Phi^{-1} := \mathbb{A}_M \subset \mathfrak{D}_M$. The new results for semisimple Lie groups G for $\mathbb{A}_M$, $M \approx G/H$, are contained in the main theorem in [6, 9].

The *sheaf* $\mathfrak{D}_M$ (or simply $\mathfrak{D}$) *of holomorphic, finite order, linear differential operators on* M is a subalgebra of homomorphism $\mathcal{H}om_\mathbb{C}(\mathcal{O}_M,\mathcal{O}_M)$ generated by the sheaf $\mathcal{O}_M$ of germs of holomorphic functions of M and the vector fields. We consider also *the subalgebra* $\mathfrak{A}_M$ of $\mathbb{A}_M$ *of differential operators with holomorphic polynomial coefficients.* Let $U := \mathcal{V}_0$ in M, endowed with the coordinates $(z_1, z_2, \cdots, z_n)$. We set $\partial_i := \frac{\partial}{\partial z_i}$ and $\partial^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$, $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$. The sections of $\mathfrak{D}_M$ on U are $A : f \mapsto \sum_\alpha a_\alpha \partial^\alpha f$, $a_\alpha \in \Gamma(U,\mathcal{O})$, the a_α-s being zero except a finite number.

For $k \in \mathbb{N}$, let us denote by $\mathfrak{D}_k$ the subsheaf of differential operators of degree $\leq k$ and by $\mathfrak{D}'_k$ the subsheaf of elements of $\mathfrak{D}_k$ without constant terms. $\mathfrak{D}_0$ is identified with $\mathcal{O}$ and $\mathfrak{D}'_1$ with the sheaf of vector fields. The filtration of $\mathfrak{D}_M$ induces a filtration on $\mathfrak{A}_M$.

Summarizing, we have the following three objects which correspond each to other:

$$\mathfrak{g} \ni X \mapsto \boldsymbol{X} \in \mathbf{A}_M \mapsto \mathbb{X} \in \mathbb{A}_M \subset \mathfrak{D}_M, \text{ differential operator on } \mathcal{F}_\mathcal{H}. \tag{5.8}$$

Now we can see that

Proposition 2. *If* Φ *is the isometry (4.3), then* $\Phi dT(\mathfrak{g}_\mathbb{C})\Phi^{-1} \subseteq \mathfrak{D}_1$.

Proof. Let us consider an element in $\mathfrak{g}_{\mathbb{C}}$ and his image in $\mathfrak{D}_M$, via the correspondence (5.8):

$$\mathfrak{g}_{\mathbb{C}} \ni G \mapsto \mathbb{G} \in \mathfrak{D}_M; \quad \mathbb{G}_z(f_\psi(z)) = \mathbb{G}_z(e_{\bar{z}}, \psi) = (e_{\bar{z}}, \boldsymbol{G}\psi),$$

$$\boldsymbol{G} = dT(G) = \frac{d}{dt}|_{t=0} T(\exp(tG)).$$

Remembering equation (5.2) and determining the derived representation, we get

$$\mathbb{G}_z(f_\psi(z)) = (P_G(z) + \sum Q^i_G(z)\frac{\partial}{\partial z_i}) f_\psi(z); \tag{5.9}$$

$$P_G(z) = \frac{d}{dt}|_{t=0}\mu(\exp(tG), z); \; Q^i_G(z) = \frac{d}{dt}|_{t=o}(\exp(-tG).z)_i. \qquad \square$$

Now we formulate the following assertion:

Remark 3. *If (G,T) is a CS-representation, then $\mathbb{A}_M$ is a subalgebra of holomorphic differential operators with polynomial coefficients, $\mathbb{A}_M \subset \mathfrak{A}_M \subset \mathfrak{D}_M$. More exactly, for $X \in \mathfrak{g}$, let us denote by $\boldsymbol{X} := dT(X) \in \mathbf{A}_M$, where the action is considered on the space of functions $\mathfrak{F}_{\mathcal{H}}$. Then, for CS-representations, $\mathbb{X} \in \mathfrak{A}_1 = \mathfrak{A}_0 \oplus \mathfrak{A}'_1$.*

Explicitly, if $\lambda \in \Delta$ is a root and G_λ is in a base of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$, then his image $\mathbb{G}_\lambda \in \mathfrak{D}_M$ acts as a first order differential operator on the symmetric Fock space $\mathfrak{F}_{\mathcal{H}}$

$$\mathbb{G}_\lambda = P_\lambda + \sum_{\beta \in \Delta_{\mathfrak{m}'}} Q_{\lambda,\beta}\partial_\beta, \; \lambda \in \Delta, \tag{5.10}$$

where P_λ and $Q_{\lambda,\beta}$ are polynomials in z, and $\mathfrak{m}'$ is the subset of $\mathfrak{g}_{\mathbb{C}}$ which appears in the definition (3.1) of the coherent state vectors.

Actually, we don't have a proof of this assertion for the general case of CS-groups. For the compact case, there exists the calculation of Dobaczewski [13], which in fact can be extended also to real semisimple Lie algebras. For compact hermitian symmetric spaces it was shown [2] that degrees of the polynomials P and Q-s are ≤ 2 and similarly for the non-compact hermitian symmetric case [3]. Neeb [21] gives a proof of this Remark for CS-representations for the (unimodular) Harish-Chandra type groups. Let us also remember that: *If G is an admissible Lie group such that the universal complexification $G \to G_{\mathbb{C}}$ is injective and $G_{\mathbb{C}}$ is simply connected, then G is of Harish-Chandra type* (cf. Proposition V.3 in [17]). The derived representation (5.7) is obtained differentiating eq. (5.5), and we get two terms, one in $\mathfrak{D}_0$ and the other one in $\mathfrak{D}'_1$. A proof that the two parts are in fact $\mathfrak{A}_0$ and respectively $\mathfrak{A}'_1$ is contained in Prop. XII.2.1 p. 515 in [21] for the groups of Harish-Chandra type in the particular situation where the space $\mathfrak{p}^+$ in Lemma VII.2.16 p. 241 in [21] is abelian. We have presented explicit formulas for semisimple Lie groups and also the simplest example where the maximum degree of the polynomials P and Q-s is 3 (cf. [6, 9]). $\square$

Acknowledgment S.B. is grateful to Karl-Hermann Neeb for many suggestions, criticism and the proof of Remark 1. S.B. thanks Martin Schlichenmaier for useful discussions and comments, and hospitality extended to him at the Mannheim University. This investigation was partially supported by the MEC project CERES 66/2001 and the DFG-Romanian Academy Project 436 RUM 113/15.

References

[1] S. Berceanu and C. A. Gheorghe, *J. Math. Phys.* **28**, 2899-2907 (1987)

[2] S. Berceanu and A. Gheorghe, *J. Math. Phys.* **33**, 998-1007 (1992)

[3] S. Berceanu and L. Boutet de Monvel, *J. Math. Phys.* **34**, 2353-2371 (1993)

[4] S. Berceanu, *J. Geom. Phys.* **21**, 149–168 (1997)

[5] S. Berceanu, *Rep. Math. Phys.* **40**, 159-168 (1997)

[6] S. Berceanu and A. Gheorghe, *Differential operators on Kähler orbits of coherent states*, XVIII Workshop on Geometric Methods in Physics, "Algebraic and geometric aspect of quantization", Białowieża, Poland, July 4-10 (1999)

[7] S. Berceanu and A. Gheorghe, *Rom. Jour. Phys.* **45**, 285-288 (2000)

[8] S. Berceanu, *Coherent states, phases and symplectic areas of geodesic triangles*, in "Coherent States, Quantization and Gravity", Edited by M. Schlichenmaier et all, Warsaw University Press (2001); also in Math. DG/9903190

[9] S. Berceanu and A. Gheorghe, *Linear Hamiltonians on homogeneous Kähler manifolds of coherent states*, The V international workshop on Differential Geometry and its Applications, Timisoara - Romania, September 18-22 (2001); —- *Realization of Lie algebras by first-order differential operators with holomorphic polynomial coefficients on Kähler coherent state orbits*, Sophus Lie seminary, Humboldt University, Berlin 6-8 December (2001)

[10] F. A. Berezin, *Commun. Math. Phys.* **40**, 153-174 (1975)

[11] M. Bordemann, M. Foger and H. Römer, *Commun. Math. Phys.* **102**, 605-647 (1986)

[12] R. Bott, *Ann. Math.* **66**, 203-247 (1957)

[13] A. Dobaczewski, I, *Nucl. Phys.* A **369** 213-236; II, 237-257 (1981); III, **380**, 1-28 (1982)

[14] W. Lisiecki, *Ann. Ins. Henri Poincarè*, **53**, 245-258 (1990)

[15] W. Lisiecki, *Bull. Amer. Math. Soc.* **25**, 37-43 (1991)

[16] W. Lisiecki, *Rep. Math. Phys.* **35**, 327-358 (1995)

[17] K.-H. Neeb, Realization of general unitary highest weight representations, Preprint, Technische Hochschule Darmstadt **1662**, (1994)

[18] K.-H. Neeb, *Math. Ann.* **301**, 155-181 (1995)

[19] K.-H. Neeb, *Forum Math* **7**, 349-384 (1995)

[20] K.-H. Neeb, *Pacific J. Math.* **174:2**, 230-261 (1996)

[21] K.-H. Neeb, Holomorphy and Convexity in Lie Theory, de Gruyter Expositions in Mathematics 28, Walter de Gruyter, Berlin-New York (2000)

[22] K. Nomizu, *Amer. J. Math.* **76**, 33-65 (1954)

[23] E. Onofri, *J. Math. Phys.* **16**, 1087-1089 (1975)

[24] Z. Pasternak-Winiarski and J. Wojcieszynski, *Demonstratio Mathematica* **XXX**, 199-214 (1997)

[25] A. M. Perelomov, Generalized Coherent States and their Applications, Springer, Berlin (1986)

[26] J. H. Rawnsley, *Quart. J. Math. Oxford* **28**, 403-415, (1977)

[27] J. Rosenberg and M. Vergne, *J. Funct. Anal.* **62**, 8-37 (1985)

[28] A. Shapere and F. Wilczek, eds, Geometrical Phases in Physics, World Scientific, Singapore (1989)

[29] J. A. Tirao and J. A. Wolf, *Indiana Univ. Math. J.* **20**, 15-31 (1970)