Quantum Phase Transitions: a Renormalization Group Approach

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November 26, 2002

Abstract

We applied the Renormalization group method at finite temperature in order to study the d-dimensional dilute Bose gas. The flow-equations and the free energy are obtained for the case of arbitrary dimension, d, and the case $d = 2$ have been analyzed in the limit of low and high temperatures. The critical temperature, the coherence length and the specific heat are obtained for this particular case using a regular solution for the coupling constant, which are obtained from the scaling equations. A non-universal behavior, consisting in a logarithmic temperature dependence in the critical region have been obtained for the specific heat.

1 Introduction

The observed Bose-Einstein condensation (BEC) of the alkali atoms [1], the discovery of the quasi-condensed state of polarized atomic hydrogen, [2], has renewed the interest in the physics of dilute Bose sistems. The Bose-Einstein condensation in a three-dimensional (3D) system is a well known phenomena. For this case, the role of the weak repulsive potential has been studied by different theoretical and numerical methodes. However, it is generally accepted that the occurence of the condensed phase at finite temperature depends on the dimensionality of the system, and for interacting uniform systems BEC occurs only for $d > 2$.

The absence of BEC below three dimensions does not necessarily imply the lack of a superfluid phase transition in $d = 2$, assuming that well defined conditions are satisfied by the system. In this paper we consider the case of a dilute two-dimensional Bose gas. Recent experimental results on adsorbed polarized hydrogen on ^4He surfaces, which is considered to condense at finite temperatures [3] motivate our work. Another possible observation of a quasi-2D condensation is in high temperature superconductors, where it is agreed that the 2D $Cu - O$ planes are responsible for the main physical properties of the system. A possible explanation of the this unusual physical behavior is based on the hypothesis that the normal phase contain pre-formed pairs, which obey Bose-Einstein condensation below a certain critical temperature [3].

It is well known that in the most general case of a system obeyng a parabolic law dispersion for the characteristic particle energy, Bose-Einstein condensation does not occure in two-dimensions at finite temperatures. If we consider a repulsive interaction between the component bosons, the BEC ohenomena is still absent, but we can find in this system a superfluid phase transition at very low temperatures. From the theoretical point of view, the superflui phase was investigated by Popov [4] using the t-matrix approximation. He proved that for a temperature T below a critical value T_c, a macroscopic number of bosons with small momenta ($k < k_0$) behaves like a superfluid. Using a characteristic parameter, $\gamma = na^2$, n being the system density and a the scattering length, he evaluate the value of the cut-off momenta and obtain a small, but non-zero, "condensation" temperature. The same problem was reconsidered by Fisher and Hohemberg [5] by the Renormalization Group (RG) method. Using this method, different from the many body treatment, they succeed to obtain a value of the critical temperature, following the same diluteness condition and by considering the system initially in a disordered phase ($T > T_c$). A different diluteness condition was used by Kolomeisky-Straley, [6] in their RG treatment of the Bose gas in arbitrary dimenssion. The usual condition of the density variable is replaced in their approach by an alternative condition imposed on the chemical potential, μ. In the dilute limit, this quantity has to obey the inequality $\mu \ll \hbar^2/ma^2$, m being the particle mass.

We intend to reconsider in this paper the RG method for the 2D Bose gas, using this last diluteness condition mentioned above, related to the chemical potential. Starting from the normal phase, and following the method proposed in [6] we shall reevaluate the results from [5] by solving more accurately the flow equations.

The paper is structured as follows. In Sec. II we present the model and the basic RG results at finite temperature for d dimensional systems. In Sec. III we solve the RG flow equations for the specific 2D case and we spell out our main results for the transition temperature, coherence length and specific heat. We shall discuss in Sec. IV the importance of the present results and their connections to other theoretical calculations existing in the literature.

2 Model and Scaling Equations

We consider the dilute Bose system in d dimensions at finite temperature T. Our approach start with the action:

$$S_{eff} = S_{eff}^{(2)} + S_{eff}^{(4)} \tag{1}$$

where

$$S_{eff}^{(2)} = \frac{1}{2}\sum_{k}\left[\frac{\hbar^2 k^2}{2m} - \mu - \frac{|\omega_n|}{\Gamma}\right]|\phi(k)|^2 \tag{2}$$

and

$$S_{eff}^{(4)} = \frac{u}{4}\sum_{k_1}\cdots\sum_{k_4}\phi(k_1)\cdots\phi(k_4)\delta\left(\sum_{i=1}^{4}k_i\right) \tag{3}$$

and the following notation have been used:

$$\sum_{k}\cdots \to k_B T\sum_{n}\int\frac{d^d\mathbf{k}}{(2\pi)^d}\cdots \tag{4}$$

where

$$\int\frac{d^d\mathbf{k}}{(2\pi)^d}\cdots \;=\; K_d\int dk\, k^{d-1}\cdots$$

and

$$K_d = \frac{\pi^{d/2}}{2^{d-1}\Gamma(d/2)}$$

In Eq. (2) μ is the chemical potential of the bosonic system, described by the scalar field $\phi(k)$, and Γ is a parameter which controls the influence of the quantum fluctuations. We set initially $\Gamma = 1$, and the classical limit can be recovered by taking $\Gamma = 0$. The Eq. (3) represents the interaction between fluctuations, u ($u > 0$) being the bare interaction, characterized by a specific range, a.

The bare propagator of the bosonic system has the form:

$$G(\mathbf{k}, \omega_n) = \left[\frac{\hbar^2 k^2}{2m} - \mu - \frac{|\omega_n|}{\Gamma}\right]^{-1} \tag{5}$$

and the renormalization transformations are carried out by integrating over a momentum shell and summing over all bosonic frequencies.

After integrating out the degrees of freedom in the momentum shell, we rescale the variables as:

$$k = \frac{k'}{b}\ ,\ \omega_n = \frac{\omega_n{}'}{b^z}\ ,\ T = \frac{T'}{b^z} \tag{6}$$

and the bosonic field as:

$$\phi'(k', \omega_n{}') = b^{-(d+z+2)/2}\phi\left(\frac{k'}{b}, \frac{\omega_n{}'}{b^z}\right) \tag{7}$$

where z is the dynamic critical exponent. The scaling equations can be obtained after some simple algebra (see for example Ref. [5, 6] for $T \neq 0$ and [7] for $T = 0$) as:

$$\frac{d\Gamma(l)}{dl} = -(2-z)\Gamma(l) \tag{8}$$

$$\frac{dT(l)}{dl} = zT(l) \tag{9}$$

$$\frac{d\mu(l)}{dl} = 2\mu(l) - K_d F_\mu[\mu(l), T(l), \Gamma(l)]\ u(l) \tag{10}$$

$$\begin{aligned}\frac{du(l)}{dl} &= [4-(d+z)]u(l) - \frac{1}{4}K_d\left\{8F_{p-p}[\mu(l), T(l), u(l), \Gamma(l)]\right.\\ &\quad \left.+2F_{p-a}[\mu(l), T(l), u(l), \Gamma(l)]\right\} u^2(l)\end{aligned} \tag{11}$$

where the renormalization parameter l is defined as $l = \ln b$. A scaling equation for the free energy has been obtained:

$$\frac{dF(l)}{dl} = (d+z)F(l) + K_d F_f[\mu(l), T(l), u(l), \Gamma(l)] \tag{12}$$

where the constants F_μ, F_{p-p}, F_{p-a} and F_f from Eqs. (10 - 12) are given by the expressions:

$$\begin{aligned} F_\mu &= F_\mu[\mu(l), T(l), \Gamma(l)] \\ &= \frac{\Lambda^d \Gamma(l)}{\exp\left[\frac{\Gamma(l)}{k_B T(l)}\left(\frac{\hbar^2\Lambda^2}{2m} - \mu(l)\right)\right] - 1} \end{aligned} \tag{13}$$

$$\begin{aligned} F_{p-a} &= F_{p-a}[\mu(l), T(l), \Gamma(l)] \\ &= \frac{\Lambda^d \Gamma^2(l)}{2\Gamma(l)\left(\frac{\hbar^2\Lambda^2}{2m} - \mu(l)\right)} \coth\left[\frac{\Gamma(l)}{2k_B T(l)}\left(\frac{\hbar^2\Lambda^2}{2m} - \mu(l)\right)\right] \end{aligned} \tag{14}$$

$$\begin{aligned} F_{p-p} &= F_{p-p}[\mu(l), T(l)] \\ &= \frac{1}{4k_B T(l)} \frac{\Lambda^d \Gamma^2(l)}{\sinh^2\left[\frac{\Gamma(l)}{2k_B T(l)}\left(\frac{\hbar^2\Lambda^2}{2m} - \mu(l)\right)\right]} \end{aligned} \tag{15}$$

$$\begin{aligned} F_f &= F_f[\mu(l), T(l), \Gamma(l)] \\ &= k_B T(l) \ln\left\{1 - \exp\left[-\frac{\Gamma(l)}{k_B T(l)}\left(\frac{\hbar^2\Lambda^2}{2m} - \mu(l)\right)\right]\right\} \\ &- k_B T(l) \ln\left[1 - \exp\left(-\frac{\Gamma(l)}{k_B T(l)}\right)\right] \end{aligned} \tag{16}$$

Here Λ is the characteristic momenta cut-off of the order of $1/a$.

A general analytical solution for the equations (8) - (11) is hard to obtain. However, for the interesting 2D case using the diluteness condition, the flow equations will be solved for the critical region.

3 Results for the 2D system

One of the most interesting cases is the two-dimensional dilute Bose gas. We are going to evaluate some of the properties of such a system, inside the critical regime, corresponding to the disordered phase. We will solve the flow equations separately, considering only the quantum case (taking a nonzero value of Γ). For the bosonic system we have to consider $z = 2$.

In this case the scaling equations have the form:

$$\frac{d\Gamma(l)}{dl} = 0 \tag{17}$$

$$\frac{dT(l)}{dl} = 2T(l) \tag{18}$$

$$\frac{du(l)}{dl} = -\frac{mK_2}{2\hbar^2}u^2(l) \tag{19}$$

$$\frac{d\mu(l)}{dl} = 2\mu(l) - \frac{\Lambda^2 K_2 u(l)}{\exp\left(\frac{\hbar^2\Lambda^2}{2mk_BT(l)}\right) - 1} \tag{20}$$

The first scaling equation (17), have the simple solution:

$$\Gamma(l) = \Gamma \tag{21}$$

which can be consider equal to one over the entire range of the renormalization parameter, l.

The second equation, corresponding to the temperature variable have also a simple solution,

$$T(l) = Te^{2l} \tag{22}$$

The next equation which will be consider is for the copling constant. A simple approximation that can be use in order to solve this equation is to consider that in the critical region the chamical potential can be neglected due to the proximity of the critical point, $|\mu(l)| \ll 1$. As we already mention, Γ will be take as $\Gamma = 1$ and l_0 has been calculated as:

$$l_0 = \frac{4\pi\hbar^2}{mu_0} \tag{23}$$

The general solution for the coupling constant will be:

$$u(l) = \frac{4\pi\hbar^2}{m}\frac{1}{l + l_0} \tag{24}$$

Let us focus now on the scaling equation corresponding to the chemical potential. This will be the equation used to define the diluteness condition. The solution of the equation (20) has the form:

$$\mu(l) = -\frac{4\Lambda^2\hbar^2}{2m}e^{2l}\int_0^l \frac{dl'}{l' + l_0}\frac{e^{-2l'}}{\exp\left(\frac{\hbar^2\Lambda^2}{2mk_BT}e^{-2l'}\right) - 1} \tag{25}$$

This expression can be rewritten as:

$$\begin{aligned}\mu(l) &= -\frac{4\Lambda^2\hbar^2}{2m}e^{2l}\frac{2mk_BT}{\hbar^2\Lambda^2}\left\{\frac{1}{2l_0}\ln\left[1 - \exp\left(-\frac{\hbar^2\Lambda^2}{2mk_BT}\right)\right]\right. \\ &\left.-\frac{1}{2l_0}\left(1 + \frac{l}{l_0}\right)^{-1}\ln\left[1 - \exp\left(-\frac{\hbar^2\Lambda^2}{2mk_BT}e^{-2l}\right)\right]\right\} - \frac{2mk_BT}{\hbar^2\Lambda^2}F(l)\end{aligned} \tag{26}$$

where the special function $F(l)$ is defined by:

$$F(l) = \int_0^{2l}\frac{dx}{(x + 2l_0)^2}\ln\left[1 - \exp\left(-\frac{\hbar^2\Lambda^2}{2mk_BT}e^{-x}\right)\right] \tag{27}$$

Following the method from Ref. [6] we define a diluteness condition as $\mu(l) \ll \frac{\hbar^2\Lambda^2}{2m}$. Then, the renormalization procedure will be stopped at l^* defined also by:

$$\mu(l^*) = -\alpha\frac{\hbar^2\Lambda^2}{2m} \tag{28}$$

with $\alpha \leq 1$. Following the same procedure we calculate

$$e^{-2l^*} \simeq \frac{4}{\alpha}\left[\frac{4}{\alpha} - \ln\frac{4}{\alpha}\right]\frac{2mk_BT}{\hbar^2\Lambda^2}\frac{1}{\ln\frac{\alpha}{4}\frac{\hbar^2\Lambda^2}{2mk_BT}} \tag{29}$$

and if we introduce the effective temperature $T_0 = \hbar^2\Lambda^2\alpha/8mk_B$ Eq. (29) will be written as:

$$e^{-2l^*} = \frac{C(\alpha)}{4}\frac{T}{T_0}\frac{1}{\ln\frac{T_0}{T}} \tag{30}$$

where

$$C(\alpha) \;=\; \frac{4}{\alpha}\,\left(\frac{4}{\alpha}\;-\;\ln\frac{4}{\alpha}\right)$$

The temperature dependence of the coherent length $\xi(T)$ will be calculate using

$$\xi^{-2}(T) \;=\; \frac{2m}{\hbar^2}|\mu(l^*)| \tag{31}$$

We get for coherence length

$$\xi(T) \sim \left|\frac{\ln\frac{T_0}{T}}{\frac{T}{T_0}}\right|^{1/2} \tag{32}$$

The general equation giving the bosonic density can be written as [8]

$$n \;=\; e^{-\alpha l^*}\int\frac{d^dk}{(2\pi)^d}\frac{1}{\exp\left[\frac{1}{k_BT(l^*)}\left(\frac{\hbar^2k^2}{2m}-\mu(l^*)\right)\right]-1} \tag{33}$$

In the critical region, for the 2D case this equation becomes:

$$n = \frac{2mk_BT}{4\pi\hbar^2}\;\ln\frac{1}{1-\exp\left(-\frac{\hbar^2\Lambda^2}{2mk_BT}e^{-2l^*}\right)} \tag{34}$$

which gives in the low temperature limit

$$n = \frac{2mk_BT}{4\pi\hbar^2}\;\ln\ln\left(\frac{T_0}{T}\right) \tag{35}$$

This equation can be inverted [5] and taking $\Lambda \sim 1/a$ (a is the range of the interaction u) we obtain the critical temperature of QC transition as:

$$T_{2D} \;=\; \frac{\hbar^2}{2m}\,\frac{4\pi n}{\ln\ln\frac{1}{na^2}} \tag{36}$$

a result which agrees with the t - matrix calculations [8, 9] and also with the RG calculations from Ref. [6].

In order to calculate the specific heat in the critical region, we have to solve first the scaling equation for the free energy (12). In order to do this, we put this equation in the form:

$$\frac{dF(l)}{dl} = 4F(l) + C_f(\mu(l), T(l)) \tag{37}$$

where

$$C_f = K_2\Lambda^2 k_B T(l)\ln\left\{1-\exp\left[-\frac{\Gamma}{k_B T(l)}\left(\frac{\hbar^2\Lambda^2}{2m}-\mu(l)\right)\right]\right\} \tag{38}$$

Using the substitution $T(x) = Te^{2x}$ we write the solution of Eq. (37) as:

$$F(l) = \int_0^l dx\ e^{-4x} C_f\left(Te^{2x}\right) \tag{39}$$

where we approximate C_f as:

$$C_f = K_2\Lambda^2 k_B T\ln\left[1-\exp\left(-\frac{\Gamma}{k_B T}-\mu(l)\right)\right] \tag{40}$$

The expression for $F(l^*)$ becomes:

$$F(l^*) = \frac{\Lambda^2}{2\pi}k_B T\int_0^{l^*} dx\ e^{-2x}\ln\left(1-e^{-\frac{D}{T}e^{-2x}}\right) \tag{41}$$

where we made the notation

$$D = \frac{\Gamma}{k_B}\left(\frac{\hbar^2\Lambda^2}{2m}-\mu\right)$$

In the low temperature limit, corresponding to the quantum regime, Eq. (41) can be approximated as:

$$\begin{aligned} F &= \frac{\Lambda^2}{4\pi}k_B T\ln\frac{D}{T}\left[1-\frac{\frac{T}{T_0}\frac{C(\alpha)}{4}}{\ln\frac{T_0}{T}}\right] \\ &+ \frac{\Lambda^2}{4\pi}(k_B T)^2\frac{\frac{T}{T_0}}{\ln\left|\frac{T}{T_0}\right|}\ \ln\left[\frac{\ln\left|\frac{T}{T_0}\right|}{\left|\frac{T}{T_0}\right|}\right] \\ &+ \frac{\Lambda^2}{4\pi}k_B T\left[\frac{\frac{T}{T_0}}{\ln\left|\frac{T}{T_0}\right|}-1\right] \end{aligned} \tag{42}$$

From this equation we calculate the specific heat $C_v(T) = -T\partial^2 T/\partial T^2$ and the dominant contribution in temperature has the form:

$$C_v(T) = C_0\frac{\left|\frac{T}{T_0}\right|}{\ln^3\left|\frac{T}{T_0}\right|} \tag{43}$$

where $C_0 = C_0(\Lambda)$ which shows that the result is Λ - dependent. As we can see, both coherence length and specific heat present a non - universal temperature dependence in the critical region, just above the critical point, this dependence involving logarithmic terms.

4 Discussions

We reconsidered in this paper the problem of the two-dimensional Bose gas weakly interacting via a repulsiv potential, using the Renormalization Group Method. Our paper is complementary to that of Kolomeyski and Straley [6] which studied this system at $T = 0$. The method, first applied by Fisher and Hohenberg [5], give the possibility to calculate a critical temperature T_c, which is similar to that obtained by Popov [4] using the t matrix method. We showed that the generalization of the method given in Ref. [5], at finite temperatures, gives correct results for $d = 2$, also in the limit of low temperatures, using a diluteness condition written in terms of chemical potential. We have presented the calculated bosonic density, coherence length and specific heat, which showed non-universal behavior.

The study of the free energy [10, 11] gives us the possibility to calculate the coherence length as function of temperature and the specific heat in the critical region. For the interesting case $d = 2$ we calculated a critical temperature T_{2D} which is the critical temperature of QC transition. The agreement with the t-matrix calculation [4] obtained also in Ref. [5] demonstrated that the Popov method of the calculation is equivalent to RG method. The existent experimental data [3, 12, 13, 14] showed the existence of the QC transition predicted in [4] and [5], but more accurate measurements have to be performed to confirm the critical behavior of the coherence length and specific heat. These kind of measurements on the cuprate superconductors are also promising near the quantum critical point.

The case of $d = 3$ Bose gas has been studied in [15, 16, 17, 18] using the RG method obtaining for the critical exponent $\nu = 0.67$, which is in agreement with the measured values in ^4He experiments. A similar result can be obtained also from our calculations, but as in [15, 16, 17, 18], only in the high temperature limit.

A possible application of our calculation is in connection with high temperature superconducting materials, where due to the possible strongly correlation effects, a bosonic description can be considered more appropriate for the normal phase. A standard Bardeen-Cooper-Schrieffer mechanism, used for explanation of the superconducting state in standard superconductors, seems to be questionable because it is essentially a weack-coupling theory, which is not accurate for the strongly correlated nature of the high temperature superconductors.

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