On the Parametrization of Complex Hadamard Matrices

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Abstract

Complex Hadamard matrices are important in the mathematical structure of the quantum information theory being an essential tool for the construction of bases of unitary operators used in the theory. In this paper we provide a procedure for the parametrization of the complex Hadamard matrices for an arbitrary integer n starting from our previous results on parametrization of unitary matrices. More precisely we obtain a set of $(n-2)^2$ equations whose solutions give all the complex Hadamard matrices of size n.

Recently the mathematical structure of the quantum information theory was better understood by establishing a one-to-one correspondence between quantum teleportation schemes, dense coding schemes, orthogonal bases of maximally entangled vectors, bases of unitary operators and unitary depolarizers by showing that given any object of any one of the above types one can construct any object of each of these types by using a precise procedure. See [13] for details. The construction procedure will be efficient only to the extent that the unitary bases can be generated and the construction of these bases makes explicit use of the complex Hadamard matrices and the Latin squares. The aim of this paper is to provide a procedure for the parametrisation of the complex Hadamard matrices for an arbitrary integer n. More precisely we will obtain a set of $(n-2)^2$ equations whose solutions will give all the complex Hadamard matrices of size n. Complex n-dimensional Hadamard matrices are unitary $n \times n$ matrices whose entries have modulus $1/\sqrt{n}$.

The term *Hadamard matrix* has its root in the Hadamard's paper [9] where he gave the solution to the question of the maximum possible absolute value of the determinant of a complex $n \times n$ matrix whose entries are bounded by some constant, which, without loss of generality, can be taken equal to one. Hadamard shown that the maximum is attained by unitary matrices whose entries have the same modulus and he asked the question if the maximum

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can also be attained by orthogonal matrices. These last matrices have come to be known as *Hadamard matrices* in its honor and have many applications in combinatorics, coding theory, orthogonal designs, quantum information theory, etc., and a good reference about the obtained results is [1].

The first complex Hadamard matrices were found by Sylvester [10]. He observed that if we denote by $a_i,\ i = 0, 1, \ldots, n-1$ the solutions of the equation $x^n - 1 = 0$ for a prime n then the Vandermonde matrix

$$\frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} \end{pmatrix}$$

is unitary and Hadamard. In the same paper Sylvester found a method to obtain a Hadamard matrix of size mn if one knows two Hadamard matrices of order m and respectively n by taking their Kronecker product. Soon after the publication of the paper by Hadamard the interest was mainly on the *real* Hadamard matrices such that the Sylvester contribution fell into oblivion and the *complex Hadamard matrices* have been again reinvented by Turyn [11] in a particular case, i.e. matrices whose entries are $\pm 1, \pm i$ where $i = \sqrt{-1}$

The parametrisation of complex Hadamard matrices is a special case of a more general problem: that of reconstructing the phases of a unitary matrix from the knowledge of the moduli of its entries, problem which was a fashionable one at the end of eighties of the last century in the high energy physics community [2]-[3], [4]-[5]. An existence theorem as well as an estimation for the number of solutions was obtained by us in [7]. The particle physicists abandoned the problem when they realised realized that for $n \geq 4$ there exists a continuum of solutions, i.e. solutions depending on arbitrary phases, result that was considered uninteresting from their point of view.

Complex n-dimensional Hadamard matrices being unitary matrices whose entries have modulus $1/\sqrt{n}$, the natural class of looking for complex Hadamard matrices is the unitary group $\mathbf{U}(n)$.

Because in any group the product of two arbitrary elements is again an element of the group there is a freedom in choosing the "building" blocks to be used in a definite application. For example the high energy physicists working on CP violation problem in the framework of the standard model realized that for the Cabibbo-Kobayashi-Maskawa unitary mass matrix there is a natural constraint, namely the mass matrix is invariant under a rephasing transformation, i.e. a transformation of the form

$$a_{ij} \rightarrow e^{i(\alpha_i + \beta_j)} a_{ij}, \quad (\alpha_i, \beta_j \quad \text{arbitrary modulo } 2\pi)$$

where $a_{ij},\ i, j = 1, \ldots, n$ are the entries of the matrix A_n. Similarly in the case of a complex Hadamard matrix the multiplication of a row and/or a column by an arbitrary phase factor does not change its properties and consequently we can remove the phases of a row and column taken arbitrarily.

Taking into account that property we can write

$$A_n = d_n \tilde{A}_n d_{n-1}$$

where $\tilde{A}_n$ is a matrix with all the elements of the first row and the first column positive numbers and $d_n = (e^{i\varphi_1}, \ldots, e^{i\varphi_n})$ and $d_{n-1} = (1, e^{i\varphi_{n+1}}, \ldots, e^{i\varphi_{2n-1}})$ are two diagonal phase matrices. In the following we will consider that $A_n \equiv \tilde{A}_n$, i.e. A_n will be a matrix with positive entries in the first row and the first column.

Secondly we can permute any rows and/or columns and get an equivalent unitary matrix. This procedure can be seen as a multiplication of A_n at left and/or right by an arbitrary finite number of permutation unitary matrices $P_{ij},\ i \neq j,\ i,j = 1, \ldots, n$, whose all diagonal entries but a_{ii} and a_{jj} are equal to unity, $a_{ii} = a_{jj} = 0,\ a_{ij} = a_{ji} = 1,\ i \neq j$ and all the other entries vanish. Both the diagonal phase and permutation matrices generate subgroups of the unitary $\mathbf{U}(n)$ group; so we may consider them as gauge subgroups, i.e. any element of $\mathbf{U}(n)$ is defined modulo the action of a finite number of the above transformation which has as consequence a simplication of the calculations.

Besides for Hadamard matrices we will not distinguish between A_n and its complex conjugated matrix $\bar{A}_n$, the complex conjugation being equivalent to the sign change of all phases $\varphi_i \to -\varphi_i$ entering the parametrisation. More generally we shall consider equivalent two matrices whose phases can be obtained each other by an arbitrary non-singular linear transformation with constant coefficients.

To be efficient we need a parametrisation of unitary matrices. In our previous works [6]-[7] we have shown that starting from a partitioning of the matrix A_n in blocks

$$A_n = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and using some results from the contractions theory we arrive at the following form for A_n when $n = 2\,m$

$$A_n = \frac{1}{\sqrt{2}} \begin{pmatrix} A & B \\ C & -C\,A^*\,B \end{pmatrix}$$

which is unitary by construction. A, B and C have order $m \times m$. In general the above matrix will not be Hadamard even when A, B and C are as the simplest example shows; this happens only when either $C = A$ or $B = A$. Since the second case is obtained by transposing the matrix of the first one, as long as B and C are arbitrary we will consider only the matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} A & B \\ A & -B \end{pmatrix} \tag{1}$$

which will be the elementary two-dimensional array that will be used in construction of more complicated arrays of Hadamard matrices.

In the following we suppose that A and B are complex Hadamard matrices of size m each one depending on $p \geq 0$ respectively $q \geq 0$ free phases, i.e. Eq.(1) is a complex Hadamard matrix of size $2\,m$. They are normalized such that $AA^* = BB^* = I_m$ where I_m denotes the unit matrix of size m. Now we make use of Hadamard's trick [9] to get a Hadamard matrix depending on $p+q+m-1$ arbitrary phases. Indeed we can multiply B at left by the diagonal matrix $d = (1, e^{i\,\varphi_1}, \ldots, e^{i\,\varphi_{m-1}})$ without modifying the Hadamard property. In this way Hadamard obtained a continuum of solutions for the case $n = 4$. We denote $B_1 = d \cdot B$ and then the matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} A & B_1 \\ A & -B_1 \end{pmatrix} \tag{2}$$

will be unitary Hadamard depending on $p+q+m-1$ parameters. From Eq.(1) we obtain in general two non-equivalent $2\,m \times 2\,m$ Hadamard matrices by taking $B = A$, and $B = A^*$; if B is not equivalent to A we obtain others two different matrices, one being Eq.(2) and the second one is given by $B_1 \to B_2 = d \cdot B^*$ where * denotes the adjoint. The above procedure can be iterated by taking the matrix Eq.(1) as a new A block obtaining a Hadamard matrix of the form

$$\frac{1}{2}\begin{pmatrix} A & B & C & D \\ A & -B & C & -D \\ A & B & -C & -D \\ A & -B & -C & D \end{pmatrix} \tag{3}$$

which is a $4\,m$-dimensional array similar to Williamson array, and so on. In contradistinction to the Williamson array the A, B, C, D blocks satisfy no supplementary conditions, excepting their unitarity, for obtaining Hadamard matrices, and the elementary array Eq.(1) is different from that appearing in the constructions of Williamson type that has the form

$$\frac{1}{\sqrt{2}}\begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$

Moreover the above array is not unitary when A and B are, this happens only when $A = BA^*B$. As an application of the formula (3) we consider the following case: $a_{11} = a_{12} = a_{21} = -a_{22} = b_{11} = b_{12} = c_{11} = c_{12} = d_{11} = d_{12} = 1/\sqrt{2}$ and $b_{21} = -b_{22} = e^{is}/\sqrt{2}$, $c_{21} = -c_{22} = e^{it}/\sqrt{2}$, $d_{21} = -d_{22} = e^{iu}/\sqrt{2}$ where the notation is self-explanatory, and we obtain an eight-dimensional Hadamard matrix depending on three arbitrary phases s, t, u.

When $A = B$ Eq.(1) can be written as

$$\frac{1}{\sqrt{2}}\begin{pmatrix} A & A \\ A & -A \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & \epsilon \end{pmatrix} \otimes A$$

where $\epsilon = -1$, i.e. the first factor is the Sylvester Vandermonde matrix of the second roots of unity, and $\otimes$ is the ordinary Kronecker product, $A \otimes B = [a_{ij}B]$; of course the first factor can be any complex Hadamard matrix of order m. Now we want to define a new product the aim being a more general construction of Hadamard matrices. Let M and N be two matrices of the same order m whose elements are matrices M_{ij} of order n and respectively N_{ij} of order p. The new product denoted by $\tilde{\otimes}$ is given as

$$Q = M \tilde{\otimes} N$$

which is a matrix of order mnp, where

$$Q_{ij} = \sum_{k=1}^{k=m} M_{ik} \otimes N_{kj}$$

We will use here the above formula only in the following case: $M = m_{ij}$ where m_{ij} are complex scalars, not matrices and N is an arbitrary diagonal matrix $N = (N_{11}, \cdots, N_{mm})$ where N_{ii} are matrices of order p obtaining

$$Q = \begin{pmatrix} m_{11}N_{11} & \cdot & \cdot & m_{1m}N_{mm} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ m_{1m}N_{11} & \cdot & \cdot & m_{mm}N_{mm} \end{pmatrix} \tag{4}$$

If the matrices M and N_{ii}, $i = 1, \ldots, m$ are Hadamard so will be the matrix (4) and this form is the most general Williamson-type array we have obtained. If in the above relation we take $m_{11} = m_{12} = m_{21} = -m_{22} = 1/\sqrt{2}$ and $N_{11} = A$ and $N_{22} = B$ then (4) reduces to Eq.(1). If now m_{ij} are the same as above and

$$N_{11} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -e^{is} & e^{is} \\ 1 & -1 & e^{is} & -e^{is} \end{pmatrix}$$

is the complex four-dimensional Hadamard matrix and

$$N_{22} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{it} & 0 & 0 \\ 0 & 0 & e^{iu} & 0 \\ 0 & 0 & 0 & e^{iv} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -e^{iy} & e^{iy} \\ 1 & -1 & e^{iy} & -e^{iy} \end{pmatrix}$$

we obtain an eight-dimensional matrix depending now on five arbitrary phases s, t, u, v, y instead of three as in the preceding example obtained by using the Williamson-type array (3).

The most convenient for our purposes is the parametrisation of unitary matrices under the form of a product of n diagonal matrices containing phases interlaced with $n-1$ orthogonal matrices each one generated by a real vector $v \in \mathbf{R}^n$. See [8] for details. We consider that this form will be more appropriate to design and implement software packages for solving the moduli equations for arbitrary n and allow us to find explicitly these equations.

In the following we use the standard form of Hadamard matrices, i.e. the entries of the first row and of the first column are positive and equal $1/\sqrt{n}$ and obtain that the first simplest entries of the unitary matrix have the form

$$a_{22} = -\frac{1}{(n-1)\sqrt{n}} - \frac{n-2}{n-1} \, cos\, b_1 \, e^{i\alpha_1}, \ldots$$

$$a_{k2} = -\frac{1}{(n-1)\sqrt{n}} + \sqrt{\frac{n-2}{n-1}} \left(\frac{cos\, b_1 \, e^{i\alpha_1}}{\sqrt{(n-1)(n-2)}} + \ldots + \frac{sin\, b_1 \ldots cos\, b_{k-2} \, e^{i\alpha_{k-2}}}{\sqrt{(n-k+2)(n-k+1)}} \right.$$

$$\left. - \sqrt{\frac{n-k}{n-k+1}} sin\, b_1 \ldots sin\, b_{k-2} \, cos\, b_{k-1} \, e^{i\alpha_{k-1}} \right), \qquad k = 3, \ldots, n-1$$

$$a_{23} = -\frac{1}{(n-1)\sqrt{n}} + \frac{\cos\, b_1 \, e^{i\alpha_1}}{n-1} + \sqrt{\frac{n-3}{n-1}} \, \sin\, b_1 \, \cos\, c_1 \, e^{i(\alpha_1+\beta_1)}, \ldots, \text{ etc.}$$

By Latin letters we denoted the angles and by Greek letters the phases which parametrize the given matrix.

The matrix elements get more complicated when going from the upper left corner to right bottom corner. The entries a_{22}, a_{32} and a_{23} lead, for example, to the following moduli equations

$$(n-2) \cos^2 b_1 + \frac{2}{\sqrt{n}} \cos\, b_1 \, \cos\, \alpha_1 - 1 = 0$$

$$\sin b_1 \left((n-3) \sin b_1 \, \cos^2 b_2 + 2\sqrt{\frac{n-3}{n-1}} \cos b_2 \left(\frac{\cos \alpha_2}{\sqrt{n}} - \cos b_1 \, \cos(\alpha_1 - \alpha_2) \right) \right.$$

$$\left. - \sin\, b_1 \right) = 0 \tag{5}$$

$$\sin b_1 \left((n-3) \sin b_1 \, \cos^2 c_1 + 2\sqrt{\frac{n-3}{n-1}} \cos c_1 \left(-\frac{\cos(\alpha_1 + \beta_1)}{\sqrt{n}} + \cos b_1 \, \cos \beta_1 \right) \right.$$

$$\left. - \sin\, b_1 \right) = 0$$

and so on. The number of equations (5) equals $(n-2)^2$.

It is easily seen that other equations contain as factors $sin\, b_2, \ldots, sin\, b_{n-2}$, $sin\, c_1, \ldots$, etc.. Thus a particular solution can be obtained when

$$sin\, b_1 = 0$$

which implies $b_1 = 0, \pi$, and from the first equation (5) we get

$$cos\, \alpha_1 = \pm \frac{(n-3)\sqrt{n}}{2}$$

It is easily seen that the above equation has solution only for $n = 2, 3, 4$; for $n \geq 5$ the factor $sin\, b_1$ will be omitted from Eqs.(5) because then $b_1 \neq 0, \pi$. When $n = 2$ we obtain $\alpha_1 = \pi/4$ so $a_{22} = -1/\sqrt{2}$. If $n = 3$, then $\alpha_1 = \pi/2$ and from the first Eq.5 one gets

$$a_{22} = -\frac{1}{2\sqrt{3}} - \frac{i}{2} = \frac{1}{\sqrt{3}} e^{\frac{4\pi i}{3}}, \text{ etc.}$$

The case $n = 4$ leads to $\alpha_1 = \pi$ which gives

$$a_{22} = -a_{23} = -a_{32} = \frac{1}{2} \quad \text{and} \quad a_{33} = -\frac{e^{i(\alpha_2+\beta_1)}}{2}$$

After the substitution $\alpha_2 + \beta_1 = t$ one finds the standard complex form of the 4×4 matrix found by Hadamard. To view what is the origin of the phase $\alpha_2 + \beta_1$ we have to look at the moduli equations. They have the form

$$2\, cos^2 b_1 + cos\, b_1\, cos\alpha_1 - 1 = 0$$

$$sin\, b_1 (cos\alpha_2 - 2\, cos b_1\, cos(\alpha_1 - \alpha_2)) = 0$$

$$sin\, b_1 (2\, cos b_1\, cos\beta_1 - cos(\alpha_1 + \beta_1)) = 0$$

$$cos\, 2b_1\, cos(\alpha_1 - \alpha_2)\, cos\, \beta_1 + cos\, b_1\, cos(\alpha_2 + \beta_1) + sin(\alpha_1 - \alpha_2)\, sin\beta_1 = 0$$

and we see that that the above system splits into two cases. In the first one when $sin\, b_1 = 0$ the rank of the system is two which explains the above dependence of a_{33} on two phases and in the second case when $sin\, b_1 \neq 0$ the rank is three and the dependence is only on one arbitrary phase. However in this case there is no final difference between the two cases. The solution of the above system is obtained directly but for $n \geq 5$ the problem is difficult and needs more powerful techniques. Particular solutions can be obtained rather easily, e.g. for $n = 6$ we get

$$\frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i & -i & i \\ 1 & i & -1 & e^{it} & -e^{it} & -i \\ 1 & -i & -e^{-it} & -1 & i & e^{-it} \\ 1 & -i & e^{-it} & i & -1 & -e^{-it} \\ 1 & i & -i & -e^{it} & e^{it} & -1 \end{pmatrix}$$

matrix that depends on an arbitrary phase. However the problem of solving the system of Eqs.(5) in full generality is a difficult problem and we will consider it elsewhere.

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