Some applications of Riemannian submersions in physics

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Abstract

We review some applications of Riemannian submersions in physics. We describe a compactification scheme for the Kaluza-Klein theory triggered by a scalar sector in the form of a non-linear sigma model. Another application refers to the Kaluza-Klein monopole which was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional Kaluza-Klein theory.

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1 Introduction

Many results on the Riemannian submersions are relevant in various areas of mathematical physics as Kaluza-Klein theories, Yang-Mills equations, strings, supergravity. Interest in higher dimensional theories has been ignited once again in recent years due largely to the discovery that the underlying symmetry of the fundamental interactions is geometrical, local and gauged.

A current trend in modern physics is the search for a theory which provides a unification of gravity with the other fundamental forces of nature. One of the early possibilities for such a unification was suggested by Kaluza [1] and later expanded upon by Klein [2]. It was shown within a five dimensional extension of Einstein's theory of general relativity how both gravity and electromagnetism could be treated on a similar footing. Both interactions were described as part of the five dimensional metric. The fifth coordinate was made invisible through a "cylindrical condition": it was assumed that in the fifth direction, the world curled up into a cylinder of very small radius (10^{-33} cm, Planck's length).

A natural generalization of the original Kaluza-Klein idea which incorporates non-Abelian gauge fields is to consider a higher than five dimensional theory in which the gauge fields become part of the metric in the same way as the electromagnetic field did in Kaluza's theory.

In the next Section we describe a compactification scheme for the Kaluza-Klein theory triggered by a scalar sector in the form of a non-linear sigma model.

The last Section is devoted to the Kaluza-Klein monopole in connection with the Hopf maps. In physics, the Hopf maps represent systems with non-trivial topological properties, e.g. the Z_2 kink or sine-Gordon soliton, $U(1)$ magnetic monopole or vortex in a superconducting sheet, $SU(2)$ instanton, etc. Other physical realizations of the Hopf maps are possible.

After a brief presentation of the formalism of the magnetic charges, the Dirac monopole is described in terms of Hopf maps. Finally the Kaluza-Klein monopole is constructed by embedding the Taub-NUT gravitational instanton into five-dimensional Kaluza-Klein theory. Let us note also that the same object has re-emerged in the study of monopole scattering. In the long-distance limit, neglecting radiation, the relative motion of two monopoles is described by the geodesics of the Taub-NUT space [3, 4].

2 Kaluza-Klein theories and Riemannian submersions

2.1 Kaluza-Klein ansatz

In the modern Kaluza-Klein theories one starts with the hypothesis that space-time has $(4 + m)$ dimensions. The extra m spatial dimensions are static and curled up into a compact manifold of unobservable small size, typically of the order of the Planck length. It is not assumed that space-time is of the form $M^4 \times M^m$ where M^4 is the Minkowski space and M^m is a compact space. Rather, such spaces should correspond to ground state solutions. Physical fields are then introduced as fluctuations around these ground state solutions.

In a large class of physically interesting models in $(4 + m)$ dimensions, the compactification of the extra m dimensions are produced spontaneously by the non-trivial vacuum configuration of an antisymmetric tensor field. It is almost common to these models to get a huge cosmological constant for the space-time if the extra dimensions are Planck-sized.

Another mechanism for space-time compactification was proposed by Omero and Percacci [5] and Gell-Mann and Zwiebach [6]. Their mechanism uses a scalar sector in the form of a non-linear sigma model to trigger the compactification. The compactified space becomes isomorphic to the manifold in which the scalar fields take values and the four dimensional space has no cosmological term at the classical level. However, all symmetries in the extra dimensions are broken by the scalars so one would have to resort to solitons or bound states for the massless gauge bosons or put them in by hand.

In a few recent papers [7, 8, 9, 10] it was investigated the possibility to extend the model, assuming that the internal m-dimensional space can be larger than the manifold in which the fields take values. The general solution of the model can be expressed in terms of harmonic maps [11, 12] satisfying Einstein equations. It was shown that a very general class of solutions is given by Riemannian submersions from the extra dimensional space onto the space in which the scalar fields take values. Investigating the isometries of the metric of the extra m-dimensional space we found that the gauge fields associated with the vertical Killing vectors are massless.

2.2 Generalized non-linear sigma model in curved space

The model we shall discuss consists of Einstein gravity in $(4+m)$ dimensions coupled to a nonlinear sigma model [5, 6] :

$$S = \frac{1}{2}\int_{M^{4+m}} \left(-\frac{R}{2} + \frac{g^{ij}}{\lambda^2} h_{ab}(\Phi) \frac{\partial \Phi^a}{\partial z^i} \frac{\partial \Phi^b}{\partial z^j} \right) dv \tag{1}$$

where dv is the volume element of the oriented (pseudo-)Riemannian manifold M^{4+m} with the metric g_{ij} and scalar curvature R. We parametrize $(4+m)$ dimensional space-time M^{4+m} by local coordinates $z^i = (x^\mu, y^p)$ with the indices taking the values $i, j = 1, 2, ..., 4+m$; $\mu = 0, 1, 2, 3$; $p = 5, 6, ..., 4+m$. The scalar field $\Phi^a(z)$ are thought of as coordinates of a n-dimensional compact space B with metric h_{ab}. Latin letter from the beginning of the alphabet $(a, b, c, ...)$ will take the values $5, 6, ..., 4+n$ and λ^2 is a constant giving the strength of the self-coupling of the scalar fields.

The action (1) contains two parts. The first term is the usual gravitational action without a cosmological term. The second term is related to the energy of the map Φ between the manifold M^{4+m} and B:

$$E(\Phi) = \frac{1}{2}\int_{M^{4+m}} e(\Phi) dv \tag{2}$$

where the energy density is

$$e(\Phi) = \frac{1}{2} Tr(\Phi^* h) = \frac{1}{2} g^{ij} h_{ab} \frac{\partial \Phi^a}{\partial z^i} \frac{\partial \Phi^b}{\partial z^j} \tag{3}$$

and $\Phi^* h$ is the pull back of the metric h by the map Φ.

Varying the action with respect to g^{ij} we obtain the Einstein equation

$$S_{ij} - \frac{1}{2} R g_{ij} = 2T_{ij} \tag{4}$$

with the energy-momentum tensor

$$T = \frac{1}{\lambda^2} S_\Phi = \frac{1}{\lambda^2}(\Phi^* h - e(\Phi) g)\,. \tag{5}$$

We remark that the energy-momentum tensor (5) is (up to the coupling constant λ^2) the stress-energy tensor of the map $\Phi : (M^{4+m}, g) \to (B, h)$ [11, 12]:

$$(S_\Phi)_{ij} = \frac{1}{2} g^{kl} h_{ab} \frac{\partial \Phi^a}{\partial z^k} \frac{\partial \Phi^b}{\partial z^l} g_{ij} - h_{ab} \frac{\partial \Phi^a}{\partial z^i} \frac{\partial \Phi^b}{\partial z^j}\,. \tag{6}$$

From equations (4) and (5) we can extract the scalar curvature

$$R = \frac{2}{\lambda^2} h_{ab} \frac{\partial \Phi^a}{\partial z^i} \frac{\partial \Phi^b}{\partial z^j} g^{ij} \tag{7}$$

such that the Einstein equations become

$$S_{ij} = \frac{2}{\lambda^2} h_{ab} \frac{\partial \Phi^a}{\partial z^i} \frac{\partial \Phi^b}{\partial z^j} \,. \tag{8}$$

The map Φ is called harmonic if it is an extremal of the energy integral $E(\Phi)$ (2). The corresponding Euler-Lagrange equations are the equations of the harmonic maps [11, 12]:

$$\tau(\Phi)^c = g^{ij} (\nabla(d\Phi))^c_{ij} = 0 \tag{9}$$

where $\tau(\Phi)$ is the tension field of the map Φ.

The stress-energy tensor S_Φ has divergence:

$$div S_\Phi = - < \tau(\Phi), d\Phi > \tag{10}$$

and consequently if Φ is harmonic, then S_Φ is conservative (i.e. $div S_\Phi = 0$).

On the other hand S_Φ is obviously conservative, from equation (4), since the Einstein field tensor is divergence free

$$(S_{ij} - \frac{1}{2} R g_{ij})_{;i} = 0 \tag{11}$$

as a consequence of Bianchi's second identity.

We shall use this fact in conjunction with equation (10) to come to the conclusion that Φ is harmonic if the map is a differentiable submersion almost everywhere [7, 8, 9, 10].

With this observation the study of the coupled equations (8) and (9) is reduced to the search of submersions $\Phi : M^{4+m} \to B$ satisfying the Einstein equation (8), while equation (9) is automatically verified. Probably there are solutions of the system of equations (8) and (9) which are not submersions, but the rank of them is not maximal. The class of solutions given by Riemannian submersions is quite general enough and presents a large physical interest.

In order to recover the solutions from references [5, 6], let us consider a submersion $\Phi : M^{4+m} \to B$. For any $p \in M^{4+m}$ there is a local coordinate system (z^i) and a local coordinate system around $\Phi(p) \in B$ such that the submersion Φ is given by the equations

$$\Phi^a(z) = y^a \, , a = 5, 6, ..., 4 + n \, . \tag{12}$$

Assuming that Φ does not depend on the coordinates $x^\mu \, , \mu = 0, 1, 2, 3$ and choosing $m = n$ (the dimension of the extra space and the dimension of the manifold B are the same) we get precisely the solutions from references [5, 6].

Using the parametrization (12) we can deduce from (8) some properties of the Ricci tensor S_{ij}. In the above local charts we have

$$S_{ij} = \frac{2}{\lambda^2} h_{ab} \delta_{ai} \delta_{bj} \, , a, b = 5, 6, ..., 4 + n \tag{13}$$

and the other components of S_{ij} vanish.

This means that for the background metric $\bar{g}_{\mu\nu}$ we can take any solution of the Einstein equations in vacuum without a cosmological term

$$S_{\mu\nu} = 0 \, , \mu, \nu = 0, 1, 2, 3 \, . \tag{14}$$

If $m > n$, equation (13) also implies the vanishing of some components of the Ricci tensor in the extra space. Therefore the theory admits a $M^4 \times M^m$ background where the four dimensional space-time can be taken flat (Minkowski) and has no cosmological term at the classical level. The extra space M^m roll up to form the manifold B which we choose to be compact.

2.3 Gauge fields and Killing vectors

As it was observed in [5, 6] the model admits a "background" solution $M^{4+m} = M^4 \times M^m$ where M^4 is a flat (Minkowski) space and the sigma field Φ represents an identity map from the extra space M^m to the space B of dimensions $m = n$. In general the extra space M^m rolls up to form the manifold B which we choose to be compact. In what follows we shall assume that the submersion $\Phi : M^{4+m} \to B$ is independent on the space-time coordinates $x^\mu(\mu = 0, 1, 2, 3)$. Moreover we shall suppose that we know a "background" metric $\bar{g}^{ij}(i, j = 1, 2, ...4 + m)$ solution of equation (2). As usual in the standard Kaluza-Klein theory, the spin-1 gauge bosons arise from the expansion of the metric tensor g^{ij} around the "background" metric $\bar{g}^{ij}$ [13]:

$$g^{rs}(z) = \bar{g}^{rs}(z) \quad + \quad A^{\mu\alpha}(x)A^{\beta}_{\mu}(x)X^{r}_{\alpha}(y)X^{s}_{\beta}(y) + ... \qquad r, s = 5, 6, ..., 4 + m \tag{15}$$

where X_α is a set of Killing vectors on the manifold M^m and the gauge fields $A^\alpha_\mu(x)$ appear in conjunction with them. Including the fluctuations (15) into the action (1), we shall get an additional term representing the interaction between the gauge fields A_μ and the scalar fields Φ^a :

$$\frac{1}{2\lambda^2} \cdot \int_{M^4} A^{\mu\alpha}(x)A^{\beta}_{\mu}(x)\ dv_1 \cdot \int_{M^m} h_{ab}X^{r}_{\alpha}(y)X^{s}_{\beta}(y)\frac{\partial\Phi^a}{\partial y^r}\frac{\partial\Phi^b}{\partial y^s}\ dv_2 \tag{16}$$

where dv_1 and dv_2 are the volume elements for the manifolds M^4 and M^m respectively. In general such a term is not zero, which means that the gauge fields acquire masses. This is the main drawback of the previous solutions [5, 6] since there are no massless gauge bosons in the theory.

In what follows we shall investigate the possibility to generate massless gauge fields. For this purpose we shall use an extra dimensional space larger than the space in which the scalar fields take values.

Indeed from equation (16) the necessary condition in order that a Killing vector X_α of M^m yields a massless gauge boson is [7]

$$X_\alpha \Phi^a = 0 \qquad for \qquad a = 5, 6, ..., 4 + m. \tag{17}$$

Writing a vector field X_α of M^m in a local chart

$$X_\alpha = \sum_{5}^{4+m} X^p_\alpha \frac{\partial}{\partial y^p} \tag{18}$$

we get that a vector field vanishes on the submersion map, equation (17), if it is vertical

$$X_\alpha = \sum_{r=4+n}^{4+m} X^r_\alpha \frac{\partial}{\partial y^r}, \tag{19}$$

that is X_α is tangent to the fibres of the submersion.

We recall that for each $q \in B$, the fibre $\Phi^{-1}(q)$ is a submanifold of M^m of dimension $m - n$. As a final remark, we mention that in references [5, 6] $m = n$, the fibres are discrete sets and there are no vertical Killing vectors.

Some explicit constructions of the submersion $\Phi : M^m \to B$ with vertical Killing vectors in order to get massless gauge bosons are presented in [8, 9].

3 Kaluza-Klein monopole

3.1 The duality of electricity and magnetism

In what follows we shall review briefly the formalism of the magnetic charges which was first formulated by Dirac [14] over 70 years ago. Their existence has been under active experimental investigation ever since.

We start from classical electromagnetism in Minkowski space which is described by Maxwell's equations in terms of the electromagnetic field tensor $F^{\mu\nu}, (\mu, \nu = 0, 1, 2, 3)$:

$$\begin{aligned} \partial_\nu F^{\mu\nu} &= -j^\mu\,, &(20)\\ \partial_\nu\, {}^*F^{\mu\nu} &= 0 &(21)\end{aligned}$$

where

$$j^\mu = (\rho, \vec{j}) \quad , \quad F^{0i} = -E^i \quad , \quad F^{ij} = -\epsilon_{ijk} B^k \quad , \quad i, j, k = 1, 2, 3\,, \qquad (22)$$

$\vec{E}$ and $\vec{B}$ being the electric and magnetic fields respectively. A point in the Minkowski space is $x^\mu = (t, \vec{x})$ and the dual field tensor ${}^*F^{\mu\nu}$ is

$${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}\,. \qquad (23)$$

In vacuum, where the electric current j^μ vanishes, the Maxwell equations are symmetric under the *duality transformation*

$$F^{\mu\nu} \to {}^*F^{\mu\nu} \;, \quad {}^*F^{\mu\nu} \to -F^{\mu\nu} \qquad (24)$$

which corresponds to the interchange of electricity and magnetism

$$\vec{E} \to \vec{B}\;,\; \vec{B} \to -\vec{E}\,. \qquad (25)$$

This symmetry is broken by the presence of the electric current j^μ in equation (20). It is possible to restore the symmetry introducing a magnetic current $k^\mu = (\sigma, \vec{k})$ in the right hand side of equation (21) giving the modified Maxwell's equations

$$\begin{aligned} \partial_\nu F^{\mu\nu} &= -j^\mu\,, &(26)\\ \partial_\nu\, {}^*F^{\mu\nu} &= -k^\mu\,. &(27)\end{aligned}$$

They will be symmetric under the duality transformation (24) supplemented by

$$j^\mu \to k^\mu \ , \ k^\mu \to -j^\mu \, . \tag{28}$$

Let us examine into details the simplest possible situation: a particle of mass m and electric charge q moves in the field of a magnetic monopole of strength g fixed at the origin

$$\vec{B} = \frac{g}{r^3}\vec{r}. \tag{29}$$

Let us remark that equation (21) is automatically satisfied and the magnetic current, k_μ, breaks the dual symmetry. Dirac [14] was able to circumvent this difficulty, showing that a dually symmetric electromagnetism could be quantized provided that the condition

$$\frac{qg}{\hbar c} = \frac{n}{2} \tag{30}$$

was satisfied. Here n is an integer and we shall henceforth adhere to the convention of units $\hbar = c = 1$ where $\hbar$ is the Planck constant and c is the velocity of light. This is the famous Dirac quantization condition which implies *charge quantization.*

Let us observe that in the presence of a monopole, the vector potential cannot exist everywhere because $^*F^{\mu\nu}$ satisfies equation (27) rather than equation (21). The best we can do is to define an $\vec{A}$ such that $\vec{B}$ is given by $\vec{\nabla} \times \vec{A}$ everywhere except on a line joining the origin to infinity. To make things more specific let us consider the magnetic field due to an infinitely long and thin solenoid placed along the negative z axis with its positive pole g at the origin. The line occupied by the solenoid is called the *Dirac string.* The vector potential $\vec{A}$ of the solenoid can be written, in polar coordinates, as

$$A_r = A_\theta = 0 \ \ , \ \ A_\phi = g\frac{1-\cos\theta}{r\sin\theta} \tag{31}$$

which is singular on the negative z axis. It is obvious that, by a suitable choice of coordinates, the Dirac string may be chosen along any direction and in fact we can choose a continuous curve from the origin to infinity. Choosing the Dirac string along the positive z axis we have

$$A_r = A_\theta = 0 \ \ , \ \ A_\phi = -g\frac{1+\cos\theta}{r\sin\theta} \, . \tag{32}$$

The Dirac string is a considerable embarrassment in the monopole theory. It is quite disturbing to find that the vector potential $\vec{A}$ that describe a Dirac monopole has a string singularity even though it can argue that the string is undetectable.

Wu and Yang [15] have recast the theory of the Dirac monopole into a form which avoids the use of a singular vector potential. The sphere surrounding the monopole is divided into two overlapping regions R_a and R_b. R_a excludes the negative z axis and in this region $\vec{A}$ is defined as in (31). The second region R_b excludes the positive z axis and the vector potential $\vec{A}$ is defined as in (32). It is quite obvious that $\vec{A}^a$ and $\vec{A}^b$, the vector potentials in the regions R_a and R_b respectively, are both finite in their own domain. In the region of overlap they differ by a gauge transformation:

$$A^b_\mu = A^a_\mu - \frac{i}{q} S \frac{\partial S^{-1}}{\partial x^\mu}\,, \tag{33}$$

with $S = e^{2igq\phi}$. If S is not single-valued, then the change in the phase of the wave function of a particle with the charge q, as the particle is transported around the equator, is ill defined. So we must demand

$$qg = \frac{n}{2} \tag{34}$$

which is precisely the Dirac quantization condition (30).

3.2 The Dirac monopole and the Hopf map

From the above presentation of the physics of the Dirac monopole it is quite transparent that the presence of point magnetic monopoles necessitates a fibre bundle formulation of electrodynamics [16, 17, 18]. The Hopf fibering of S^3 over a base space S^2 with fibre S^1 yields the Wu-Yang potentials which describe the Dirac monopole.

For this purpose we start with the Hopf map, which maps S^3 onto S^2. S^3 may be parameterized by x_1, x_2, x_3 and x_4, coordinates in R^4, obeying

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\,. \tag{35}$$

On the other hand S^2 may be parameterized by ξ_1, ξ_2 and ξ_3, coordinates in R^3, obeying

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = 1\,. \tag{36}$$

The Hopf map is given by

$$\xi_1 = 2(x_1x_3 + x_2x_4) \ \ , \ \ \xi_2 = 2(x_2x_3 - x_1x_4) \ \ , \ \ \xi_3 = x_1^2 + x_2^2 - x_3^2 - x_4^2 \,. \quad (37)$$

To see more neatly the relationship among Wu-Yang potentials we proceed to a parameterization of the sphere S^3 by the Euler angles $\psi\,,\theta\,,\phi$:

$$\begin{aligned} x_1 &= \cos\left(\frac{\phi+\psi}{2}\right)\cos\frac{\theta}{2} \quad , \quad x_2 = \sin\left(\frac{\phi+\psi}{2}\right)\cos\frac{\theta}{2} \ , \\ x_3 &= \cos\left(\frac{\psi-\phi}{2}\right)\sin\frac{\theta}{2} \quad , \quad x_4 = \sin\left(\frac{\psi-\phi}{2}\right)\sin\frac{\theta}{2} \end{aligned} \quad (38)$$

with $0 \leq \theta \leq \pi$ and $-\pi \leq (\psi+\phi)/2 \leq \pi$.

The Hopf map (37) then gives

$$\xi_1 = \cos\phi\sin\theta \ \ , \ \ \xi_2 = \sin\phi\sin\theta \ \ , \ \ \xi_3 = \cos\theta\,. \quad (39)$$

The angles θ, ϕ may be identified with the polar angles on the sphere S^2 and ψ is the angle on the S^1 fibre.

The magnetic field of a monopole (29) is described by a 2−form σ_2 [18]

$$B = g\sigma_2 = g\sin\theta d\theta\wedge d\phi\,. \quad (40)$$

The 2−form σ_2 is closed but not exact on S^2 and we have shown that there is no vector potential $\vec{A}$ on S^2 such that $B = dA$ or in components $\vec{B} = \vec{\bigtriangledown}\times\vec{A}$. On the other hand, σ_2 as a 2−form on S^3 is exact since all closed forms on S^3 are exact taking into account that the second cohomology group of S^3 is trivial, $H^2(S^3) = 0$.

Therefore on S^3, B is exact and there exists a 1−form A such that

$$B = dA. \quad (41)$$

We can easily construct the following as a solution (not unique)

$$A = 2g(x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4) = -g(d\psi + \cos\theta d\phi) \quad (42)$$

verifying equation (41) with B given by (40)

Now taking the sections τ_a and τ_b of equations $\psi = -\phi$ and $\psi = \phi$, respectively and considering the pullback of A under such sections, we get [19, 16, 17]:

$$\tau_a : \psi = -\phi \quad : \quad A^a_\phi = g(1-\cos\theta)\,, \quad A^a_\theta = 0\,, \quad (43)$$

$$\tau_b : \psi = \phi \quad : \quad A^b_\phi = -g(1+\cos\theta)\,, \quad A^b_\theta = 0\,, \quad (44)$$

which together with $A^a_r = A^b_r = 0$ allow to recognize the Wu-Yang potentials (31) and (32).

3.3 Kaluza-Klein monopole and Taub-NUT gravitational instanton

In what follows we shall briefly present the Kaluza-Klein monopole [20, 21] which was obtained by embedding the Taub-NUT gravitational [22] instanton into five-dimensional Kaluza-Klein theory.

For this purpose it is necessary to construct soliton solutions in Kaluza-Klein theories. By solitons we mean non-singular solutions of the classical field equations which represent spatially localized lumps that are topologically stable. The construction of a soliton in Kaluza-Klein theories is analogous to the magnetic monopole of 't Hooft and Polyakov [19] that occurs in non-Abelian gauge theories.

The Kaluza-Klein monopole in a five-dimensional theory can be viewed as a principal fiber bundle with M^4 as the base manifold and $U(1)$ as the structure group. The vacuum is the trivial bundle $M^4 \times S^1$ but of course there exist topologically inequivalent bundles. At spatial infinity a solution will describe a S^1 bundle over S^2 (the boundary of the 3$-$dimensional space) and there exist an infinite collection of such bundles, each characterized by an integer which can be identified with the magnetic charge of the soliton.

The simplest and basic soliton is the magnetic monopole [20, 21]. It is a generalization of the self-dual Taub-NUT solution [23] and is described by the following metric:

$$\begin{aligned} ds^2 = & - dt^2 \\ & + V(r)[dx^4 + \vec{A}(\vec{r})d\vec{r}]^2 + V^{-1}(r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2) \end{aligned} \quad (45)$$

where $\vec{r}$ denotes a three-vector $\vec{r} = (r, \theta, \phi)$, the gauge field $\vec{A}$ is that of a monopole (29) and function $V(r)$ is $V(r) = \frac{r}{g+r}$.

The trivial term $-dt^2$ corresponding to time was added, converting the four-dimensional Newman-Unti-Tamburino line element into a static solution of the five-dimensional vacuum Einstein's equations. There would appear to be a coordinate singularity when $r = 0$, where the Killing vector field has a fixed point. This is the so called NUT singularity and is absent if x^4 is periodic with period $4\pi g$ [23]. From equation (29) it is clear that the gauge field A_μ is of a monopole and has a Dirac string singularity running from $r = 0$ to ∞. As usual this singularity is an artifact if and only if the period of x^4 is equal to $4\pi g$.

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