Influence of third order dispersion on the bound state of two solitons of the NLS equation

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Abstract

Using Karpman-Soloviev perturbation procedure the influence of the third order dispersion on the bound state of two solitons of the NLS equation is investigated. The problem has two small parameters (supposed to be of the same order): the small overlap of the two well separated solitons, and the amplitude of the third order dispersion. If the velocities of the two solitons are the same, a bound state is formed, with an oscillating expression for the distance between solitons. When the third order dispersion is introduced a slow monotonous increasing function of time is superposed over this oscillatory behaviour.

1 Introduction

The problem of a train of N initially equal and equidistant solitons of the nonlinear Schrödinger (NLS) equation has attracted a great interest in the last decade, mainly due to its possible relevance to the physics of pulse propagation in optical fibers [1]-[3]. The N-soliton solutions of a completely integrable system is easily obtained by the inverse scattering transform method, and can be found in any textbook on solitons. But the method fails to give an answer when the solitons parameters are practically equal, a situation encountered in a train of N-solitons. Actually in such a train the solitons are initially quite well separated and a perturbation approach of the problem is well justified. The unperturbed state corresponds to N independent solitons, while the perturbation is given by the overlap of two neighbour solitons. Several perturbation approaches are available [4]-[8], but in what concerns us in the present paper we mention and shall use the "quasiparticle approach", developed many years ago by Karpman and Solov'ev for two-soliton system [4], based on the perturbation theory for the one soliton solution [8], [9]. The problem of $N \geq 3$ interacting solitons was discussed by many authors [10]-[18], and it was discovered that the soliton positions are obeying the equations of the complex Toda chain with N nodes, which is a completely integrable system. This explains the stability of such systems, observed in numerical calculations.

Although the theoretical discussion in [4]-[18] is quite general, including any type of perturbations, not only the perturbation generated by the overlap of the solitons, explicit results concerning a specific type of perturbation are only briefly mentioned. It is the aim of the present paper to discuss the effect of a third order dispersive term, $\epsilon \frac{\partial^3 u}{\partial x^3}$, on the bound state of two initially identical and well separated solitons. In the next section, using the first order perturbation theory, the evolution equations for the one-soliton parameters are written down, and the main results of Karpman and Solov'ev [4] are reviewed. We shall concentrate our attention on the time dependence of the distance $r(t)$ between the two solitons. If $\epsilon = 0$ this has an oscillatory behaviour of period T [4]. In section three the effect of the third order dispersive term on $r(t)$ is determined in first order in ϵ. Actually we succeeded to calculate an interpolating function $\bar{r}(t_n)$, which coincides with $r(t)$ at discrete times $t_n = nT$. A polynomial increase of $\bar{r}(t_n)$ with n is found. Few conclusions and remarks are given in the last section.

2 Basic equations

Consider the perturbed NLS equation

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = i\epsilon R(u). \qquad (1)$$

When $\epsilon = 0$ it has the one-soliton solution

$$u_S(x,t) = 2\nu \frac{exp(i\phi(x,t))}{\cosh z(x,t)} \qquad (2)$$

where

$$\begin{aligned} z(x,t) &= 2\nu(x - \xi(t)), \qquad & \phi(x,t) &= 2\mu(x - \xi(t)) + \delta(t) \\ \xi(t) &= 2\mu t + \xi_0, \qquad & \delta(t) &= 2(\mu^2 + \nu^2)t + \delta_0. \end{aligned} \qquad (3)$$

Here μ, ν are the real and imaginary part of the eigenvalue ζ of the associated spectral problem to the NLS equation, $\zeta = \mu + i\nu$, and ξ_0, δ_0 are the initial position and phase respectively.

When $\epsilon \neq 0$ the first order perturbation theory gives [4], [8], [9]

$$\begin{aligned} \frac{d\mu}{dt} &= \epsilon M(u), \qquad & \frac{d\nu}{dt} &= \epsilon N(u) \\ \frac{d\xi}{dt} &= 2\mu + \epsilon C(u), \qquad & \frac{d\delta}{dt} &= 2(\mu^2 + \nu^2) + 2\mu\epsilon C(u) + \epsilon D(u) \end{aligned} \qquad (4)$$

where

$$\begin{aligned} M(u) &= \frac{1}{2} Im \int_{-\infty}^{+\infty} \frac{\tanh z}{\cosh z} R(u) e^{-i\phi} dz \\ N(u) &= \frac{1}{2} Re \int_{-\infty}^{+\infty} \frac{1}{\cosh z} R(u) e^{-i\phi} dz \\ C(u) &= \frac{1}{4\nu^2} Re \int_{-\infty}^{+\infty} \frac{z}{\cosh z} R(u) e^{-i\phi} dz \\ D(u) &= \frac{1}{2\nu} Im \int_{-\infty}^{+\infty} \frac{1 - z\tanh z}{\cosh z} R(u) e^{-i\phi} dz \end{aligned} \qquad (5)$$

Let us consider we have a superposition of two practically identical and well separated solitons, $u = u_1 + u_2$, where each u_1, u_2 is of the form (3) and for convenience we assume $\xi_1 > \xi_2$. With the notations

$$r = \xi_1 - \xi_2, \qquad \psi = \delta_2 - \delta_1, \qquad p = \nu_2 - \nu_1$$
$$\nu = \frac{\nu_1 + \nu_2}{2}, \qquad \mu = \frac{\mu_1 + \mu_2}{2}, \qquad q = \mu_2 - \mu_1 \tag{6}$$

the working approximations are

$$\frac{|p|}{\nu} \ll 1, \quad \frac{|q|}{\mu} \ll 1, \quad \nu r \gg 1, \quad |p| r \ll 1. \tag{7}$$

The first three inequalities reflect the fact that the two solitons have practically the same parameters and are well separated, while the fourth one is a convenient approximation which will simplify the results.

The perturbation due to the overlap of the two solitons is given by

$$\epsilon R(u_{n_1}) = i(u_{n_1}^2 u_{n_2}^* + 2|u_{n_1}|^2 u_{n_2}) \qquad (n_1 \neq n_2 = 1 \ or \ 2)$$

where u_{n_2} has to be evaluated at the position of n_1-soliton. Using the notations (6) one obtains

$$\epsilon R(u_1) = 16\nu^3[3i\cos(2\mu r + \psi) - \sin(2\mu r + \psi)]\frac{e^{i\phi}}{\cosh^2 z} e^{-z} e^{-2\nu r}$$
$$\epsilon R(u_2) = 16\nu^3[3i\cos(2\mu r + \psi) + \sin(2\mu r + \psi)]\frac{e^{i\phi}}{\cosh^2 z} e^{z} e^{-2\nu r} \tag{8}$$

and the small parameter characterizing the perturbation is given by $e^{-2\nu r}$. Taking into account only this type of perturbation the calculation of the integrals appearing in (5) is straightforward, and the evolution equations for the parameters of the two solitons write [4] (n=1 and 2)

$$\frac{d\mu_n}{dt} = (-)^n 16\nu^3 e^{-2\nu r}\cos(2\mu r + \psi)$$
$$\frac{d\nu_n}{dt} = (-)^n 16\nu^3 e^{-2\nu r}\sin(2\mu r + \psi) \tag{9}$$
$$\frac{d\xi_n}{dt} = 2\mu_n + 4\nu e^{-2\nu r}\sin(2\mu r + \psi)$$
$$\frac{d\delta_n}{dt} = 2(\mu_n^2 + \nu_n^2) + 8\mu\nu e^{-2\nu r}\sin(2\mu r + \psi) + 24\nu^2 e^{-2\nu r}\cos(2\mu r + \psi)$$

It is immediately seen that μ and ν are constants of motion, while p and q are satisfying the equations

$$\frac{dp}{dt} = 32\nu^3 e^{-2\nu e} \sin(2\mu r + \psi), \qquad \frac{dq}{dt} = 32\nu^3 e^{-2\nu e} \cos(2\mu r + \psi) \tag{10}$$

It is convenient to introduce the complex quantity $Y = q + ip$, and from (10) it satisfies the equation

$$\frac{dY}{dt} = 32\nu^3 e^{-2\nu r} e^{i(2\mu r + \psi)}. \tag{11}$$

Also the relative distance between the solitons $r(t)$ and phase difference $\psi(t)$ are satisfying the evolution equations

$$\frac{dr}{dt} = -2q \tag{12}$$

$$\frac{d\psi}{dt} = 4(\nu p + \mu q). \tag{13}$$

By a straightforward calculation one can prove that the quantity

$$\Lambda^2 = Y^2 - 16\nu^2 e^{-2\nu r} e^{i(2\mu r + \psi)} \tag{14}$$

is also a constant of motion [4]. Then the equation (11) writes

$$\frac{dY}{dt} - 2\nu(Y^2 - \Lambda^2) = 0, \tag{15}$$

whose solution is

$$Y = -\Lambda \tanh[2\Lambda\nu t - (\alpha_1 + i\alpha_2] \tag{16}$$

where α_1, α_2 are real integration constants. Writing $Y = n_0 + im_0$ the system of two solitons is completely characterized by the set of constants μ, ν, n_0 and m_0.

In the following we shall consider the case $m_0 = 0$ and $n_0 \neq 0$. Then it is possible to consider also $\alpha_2 = 0$ and Y becomes $(\alpha_1 \rightarrow \alpha)$

$$Y = -in_0 \tanh(2in_0\nu t - \alpha) \tag{17}$$

Separating the real and the imaginary part we get

$$p_0(t) = n_0 \frac{\sinh 2\alpha}{\cos 4n_0\nu t + \cosh 2\alpha}, \qquad q_0(t) = n_0 \frac{\sin 4n_0\nu t}{\cos 4n_0\nu t + \cosh 2\alpha} \tag{18}$$

Then the equation for $r(t)$ can be integrated with the result [4]

$$r(t) = \frac{1}{2\nu} ln[\frac{8\nu^2}{n_0^2}(\cosh 2\alpha + \cos 4n_0\nu t)] \tag{19}$$

which is a periodic function of t with the period

$$T = \frac{\pi}{2\nu|n_0|}. \tag{20}$$

For $t = 0$ we have $q(t = 0) = 0$ and the two solutions have the same velocity. Now we can verify the approximations (7). From $\nu r_0 \ll 1$ and $\frac{|p_0|}{\nu} \ll 1$ we get

$$\frac{4\nu \cosh\alpha}{|n_0|} \gg 1 \quad and \quad \frac{|n_0|}{\nu}\tanh\alpha \ll 1$$

both being satisfied if $|n_0| \ll \nu$.

3 Third order dispersion effect

Beside the perturbation generated by the overlap of the two well separated solitons we shall consider an additional perturbation in the form of a third order dispersive term, $i\epsilon\frac{\partial^3 u}{\partial x^3}$ and one assume the small parameters ϵ and $e^{-2\nu r}$ to be of the same order of magnitude. This perturbation acts on each soliton separately, and its explicit expression is given by

$$\epsilon R(u) = \epsilon 8[i\mu(-\mu^2 + 3\nu^2 - \frac{6\nu^2}{\cosh^2 z}) + \nu(3\mu^2 - \nu^2 + \frac{6\nu^2}{\cosh^2 z})\tanh z]u \tag{21}$$

with $u(z)$ given in (2). The integrals in (5) are easily performed and instead of (9) the following evolution equations for the soliton parameters are obtained $(n = 1, 2)$

$$\begin{aligned}
\frac{d\mu_n}{dt} &= (-)^n 16\nu^3 e^{-2\nu r}\cos(2\mu r + \psi) \\
\frac{d\nu_n}{dt} &= (-)^n 16\nu^3 e^{-2\nu r}\sin(2\mu r + \psi) \\
\frac{d\xi_n}{dt} &= 2\mu_n + 4\nu e^{-2\nu r}\sin(2\mu r + \psi) + 4\epsilon(3\mu_n^2 + \nu_n^2) \\
\frac{d\delta_n}{dt} &= 2(\mu_n^2 + \nu_n^2) + 8\mu\nu e^{-2\nu r}\sin(2\mu r + \psi) + 24\nu^2 e^{-2\nu r}\cos(2\mu r + \psi) + \\
&\quad 16\epsilon\mu_n(\mu_n^2 - \nu_n^2)
\end{aligned} \tag{22}$$

It easily seen that μ, ν remain constants of motion and the evolution equations (10), (11) for p, q *and* Y are unchanged, while the evolution equations for $r(t)$ and ψ become

$$\frac{dr}{dt} = -2q - 8\epsilon(3\mu q + \nu p) \tag{23}$$

$$\frac{d\psi}{dt} = 4(\mu q + \nu p) + 16\epsilon[(3\mu^2 - \nu^2)q - 2\mu\nu p + \frac{1}{4}q(q^2 - p^2)]$$

It is convenient to use the same quantity Λ^2 defined in (14), but now it will be no more a constant of motion and will satisfy the evolution equation

$$\frac{d\Lambda^2}{dt^2} = -8\epsilon \left(3\mu Y^* - i\nu Y + \frac{i}{4\nu}q(q^2 - p^2)\right) \frac{dY}{dt} \tag{24}$$

We shall assume that any quantity Q can be written as $Q = Q_0 + \epsilon Q_1$, where Q_0 is the value of Q when $\epsilon = 0$, and Q_1 is the first order contribution due to the third order dispersive perturbation. For instance $\Lambda = \Lambda_0 + \epsilon\Lambda_1$, with $\Lambda_0 = in_0$ and from (24) we get

$$\frac{d\Lambda_1}{dt} = \frac{4i}{n_0} \left(3\mu Y_0^* - i\nu Y_0 + \frac{i}{4\nu}q_0(q_0^2 - p_0^2)\right) \frac{dY_0}{dt} \tag{25}$$

where $Y_0 = q_0 + ip_0$ and q_0, p_0 are given by (18). With the initial condition $\Lambda_1(t = 0) = 0$ this equation can in principle be integrated as containing only known functions. Also for Y we can write $Y = Y_0 + \epsilon Y_1$, where Y_1 is satisfying the equation

$$\frac{dY_1}{dt} = 4\nu(Y_0 Y_1 - \Lambda_0\Lambda_1), \qquad Y_1(t = 0) = 0. \tag{26}$$

Separating the real and imaginary part we obtain the expressions for q_1 and p_1 respectively. Writing $r(t) = r_0(t) + \epsilon r_1(t)$, $\quad r_1(t = 0) = 0$, the time dependence of $r_1(t)$ is determined by

$$\frac{dr_1}{dt} = -2[q_1 + 4(3\mu q_0 + \nu p_0)]. \tag{27}$$

Although it contains only elementary functions, the integration is tediously and the final result would be very complicated and irrelevant for a simple interpretation. Therefore instead to find the exact expression of $r_1(t)$ we shall determine its variation over a period T

$$\Delta r_1(t) = r_1(t+T) - r_1(t) \tag{28}$$

If we know it at the discrete times $t_0 = 0, t_1 = T, ...t_n = nT...$ an interpolation function $\bar{r_1}(t_n)$ is easily found

$$\bar{r_1}(t_n) = \sum_{j=0}^{n-1} \Delta r_1(jT). \tag{29}$$

This function will coincide with the exact $r_1(t)$ for the discrete times t_n and will have quite a simple monotonous variation with n. It is easily seen that

$$\frac{d\Delta r_1}{dt} = -2\Delta q_1 \tag{30}$$

where $\Delta q_1 = q_1(t+T) - q_1(t)$ is the real part of $\Delta Y_1 = Y_1(t+T) - Y_1(t)$. For $\Delta Y_1(t)$ we get

$$\frac{d\Delta Y_1}{dt} = 4\nu(Y_0 \Delta Y_1 - in_0 \Delta\Lambda_1) \tag{31}$$

and from (25) $\Delta\Lambda_1$ is a constant. In this way our problem is considerably reduced. We have to determine $\Delta\Lambda_1$ integrating (25) over a period T and then find $\Delta Y_1, \Delta r_1$ and finally $\bar{r_1}(t_n)$ from (31), (30) and (29) respectively. It is convenient to introduce the integration variable $x = 4\nu n_0 t - \pi$. Then $\int_0^T ...dt \to \int_{-\pi}^{\pi} ...dx$ and we can fully exploit the symmetry properties of the integrand. We remain with the integral

$$\Delta\Lambda_1 = 8n_0 \int_0^{\pi} \frac{\sinh 2\alpha}{(\cosh 2\alpha - \cos x)^3} \left[(3\mu + i\nu)(1 - \cosh 2\alpha \cos x) - \right.$$
$$\left. (3\mu - i\nu)\sin^2 x + i\frac{n_0^2}{4\nu}\sin^2 x \frac{\sinh^2 2\alpha - \sin^2 x}{(\cosh 2\alpha - \cos x)^2} \right] dx \tag{32}$$

Straightforward calculations give

$$\begin{aligned} \Delta\Lambda_1 &= -A + iB \\ A &= 24\pi\mu\nu \frac{1}{\sinh^2}\left(\frac{n_0}{\nu}\right) \\ B &= \frac{\pi}{2}\nu^2 \frac{1 + 6\tanh^2 2\alpha}{\sinh^2 2\alpha}\left(\frac{n_0}{\nu}\right)^3 \end{aligned} \tag{33}$$

As $\nu \ll |n_0|$ we see that B is negligible compared with A and $\Delta\Lambda_1$ is practically a real quantity. Now the equation (31) written in x-variable is

$$\begin{aligned} \frac{dY_1}{dx} - if(x)\Delta Y_1 &= -i\Delta\Lambda_1 \\ f(x) &= \frac{\sinh 2\alpha - i\sin x}{\cosh 2\alpha + \cos x} \end{aligned} \qquad (34)$$

whose solution is

$$\Delta Y_1 = -i\Delta\Lambda_1 exp\left(i\int_0^x f(x_1)dx_1\right)\int_0^x dx_1 e^{-i\int_0^{x_1} f(x_2)dx_2}.$$

We find

$$\int_0^x f(x_1)dx_1 = 2arctg(\tanh\alpha\tan\frac{x}{2}) + iln\frac{\cosh 2\alpha + \cos x}{\cosh 2\alpha + 1}$$

and assuming $\tanh\alpha \simeq 1$ the first term is practically equal with x, an approximation we shall use in the following. Finally we get

$$\Delta Y_1 = -\frac{i}{2}\Delta\Lambda_1\frac{2i\cosh 2\alpha(1-e^{ix}) + xe^{ix} + \sin x}{\cosh 2\alpha + \cos x} \qquad (35)$$

and neglecting B with respect to A in (33) we obtain

$$-2\Delta q_1 = A\frac{2\cosh 2\alpha(1-\cos x) + x\sin x}{\cosh 2\alpha + \cos x}. \qquad (36)$$

Then $\Delta r_1(t)$ is obtained directly integrating the previous relation with respect to t

$$\Delta r_1(x) = A\int_0^x \frac{2\cosh 2\alpha(1-\cos x) + x\sin x}{\cosh 2\alpha + \cos x}dx.$$

We shall evaluate it for $t = mT$, i.e. $x_m = 2\pi m$. The integral $\int_0^{2\pi m} ...dx$ is written as a sum of m integrals $\sum_{j=0}^{m-1}\int_0^{2\pi(j+1)} ...dx$, and each such integrals is reduced to $\int_{-\pi}^{\pi} ...dx$ with the variable change $x \to x - \pi(2j+1)$. Straightforward calculations give

$$\Delta r_1(x_m) = \frac{12\pi^2}{\sinh^2 2\alpha}\frac{\mu}{\nu}\left(2\frac{\cosh 2\alpha}{\sinh 2\alpha}(\cosh 2\alpha + 1) -\right.$$

$$\ln(\cosh 2\alpha + 1) - (1 + ln2 + 2\alpha))\, m. \quad (37)$$

As we have supposed $\tan\alpha \simeq 1$ this expression can be simplified, and the interpolation function $\bar{r_1}(x_n)$ is obtained by performing a summation over m. The final result is

$$\bar{r_1}(n) = \frac{6\pi^2}{\sinh^2 2\alpha}\frac{\mu}{\nu}[2\cosh 2\alpha + 1 - ln2(\cosh 2\alpha + 1) - 2\alpha]n(n-1). \quad (38)$$

4 Conclusions

The main result of this paper consists in the evaluation of the effects of a third order dispersive perturbation on the bound state of two practically identical and well separated solitons of the NLS equation. The calculations are done in a first order perturbation theory and we concentrated on the time evolution of the distance $r(t)$ between the two solitons, which was written as $r_0(t) + \epsilon r_1(t)$. Here $\epsilon r_1(t)$ is the first order contribution of the third order dispersion, superposed on the oscillating result $r_0(t)$, valid in the absence of this supplementary perturbation. We succeeded to evaluate exactly an interpolation function $\bar{r_1}(t_n)$ coinciding with $r_1(t)$ for $t = nT$. This is an increasing function of n, given by (38), and depends only on the mean value parameters μ, ν of the soliton system, and the parameter α which characterizes the unperturbed solution. The main conclusion which can be drawn is that for $t \simeq O(\frac{1}{\sqrt{\epsilon}})$, $\epsilon\bar{r_1}(t) \simeq O(1)$ and the previous perturbational approach will be no more valid. This result disagrees with some recent results for a perturbed train of N solitons, based on a theory of a perturbed Toda chain [14], [16]. Further investigations are necessary to see the origin of this disagreement.

References

[1] A. Hasegawa, Y. Kodama, *"Solitons in Optical Communications"* (Oxford Univ. Press, Oxford, 1995)

[2] G.P. Agrawal, *"Nonlinear Fiber Optics"* (Academic, San Diego, 1995)

[3] C. Desem, P.L. Chu, in *"Optical Solitons. Theory and Experiments"*, eds. J.R. Taylor, p. 127 (Cambridge Univ. Press, Cambridge, 1992)

[4] V.I. Karpman, V.V. Solov'ev, Physica D **3**, 487 (1981)

[5] K.A. Gorshkov, L.A. Ostrovski, Physica D **3**, 428 (1981)

[6] T. Okamawari, A. Hasegawa, Y. Kodama, Phys. Rev. A **51**, 3203 (1995)

[7] D. Anderson, M. Lisak, Optics Lett. **11**, 174 (1986)

[8] Yu.S. Kivshar, B.A. Malomed, Rev. Mod. Phys. **61**, 763 (1989)

[9] V.I. Karpman, Physica Scripta **20**, 462 (1974)

[10] M. Romagnoli, S. Wabnitz, P. Franco, M. Midrio, F. Fontana, G. Bordogna, Optics Comm. **108**, 65 (1994)

[11] J.M. Arnold, IEE Proc. **140**, 359 (1993);
Opt. Lett. **21**, 31 (1996)

[12] J.M. Arnold, J. Opt. Soc. Am. A **15**, 1450 (1988)

[13] V.S. Gerdjikov, D.J. Kaup, I.M. Uzunov, E.G. Evstatiev Phys. Rev. Lett. **77**, 3943 (1996);
I.M. Uzunov, V.S. Gerdjikov, M. Gölles, F. Ledered, Opt. Comm. **125**, 237 (1996)

[14] V.S. Gerdjikov, I.M. Uzunov, E.G. Evstatiev, G.L. Diankov, Phys. Rev. E **55**, 6039 (1997)

[15] V.S. Gerdjikov, E.G. Evstatiev, D.J. Kaup, G.L. Diankov, I.M. Uzunov, Phys. Lett. A **241**, 323 (1998);
in *"Adv. Photonics with Second-Order Nonlinear Processes"*, eds. A.D. Boardman et al. (Kluwer Acad. Press, Netherlands, 1999), p. 219

[16] V.S. Gerdjikov, I.M. Uzunov, in *"Geometry, Integrability and Quantization"*, eds. I.M. Mladenov, G.L. Naber (Coral Press, Sofia, 2000), p. 79

[17] D.C. Calvo, T.R. Akylos, Physica D **101**, 270 (1997)

[18] V.S. Shechesovich, Phys. Rev. E **65**, 46614 (2002)