C. MICROSCOPIC NUCLEAR MODELS

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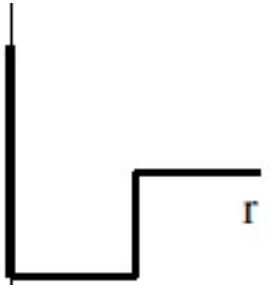
C.01. Nuclear mean field:

In nuclei the selfconsistent potential “seen” by one nucleon is created by all other nucleons

Various shapes of the mean field

Finite square well

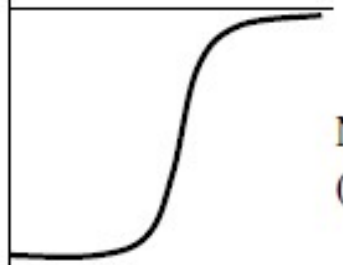


Infinite energy
Sharp edge

Harmonic oscillator potential

Infinite energy
Edge not sharp enough

Realistic Wood-Saxon potential



Nuclear potential
(in-between)

$$V(r) = \frac{-V_o}{1+\exp[(r-R)/a_{skin}]}$$

$V_0 \sim 50\ MeV$

$R = 1.25A^{1/3}\ fm$

$a_{skin} \sim 0.52\ fm$

Spherical harmonic oscillator

is a good approximation of the mean field describing low lying single particle states

Potential is given by:

$$V(r) = \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2) = \frac{1}{2} m\omega^2 r^2 = \frac{\hbar\omega}{2} q^2$$

$$q \equiv \frac{r}{r_0}$$

where the ho length parameter is: $r_0 = \sqrt{\frac{\hbar}{m\omega}}$

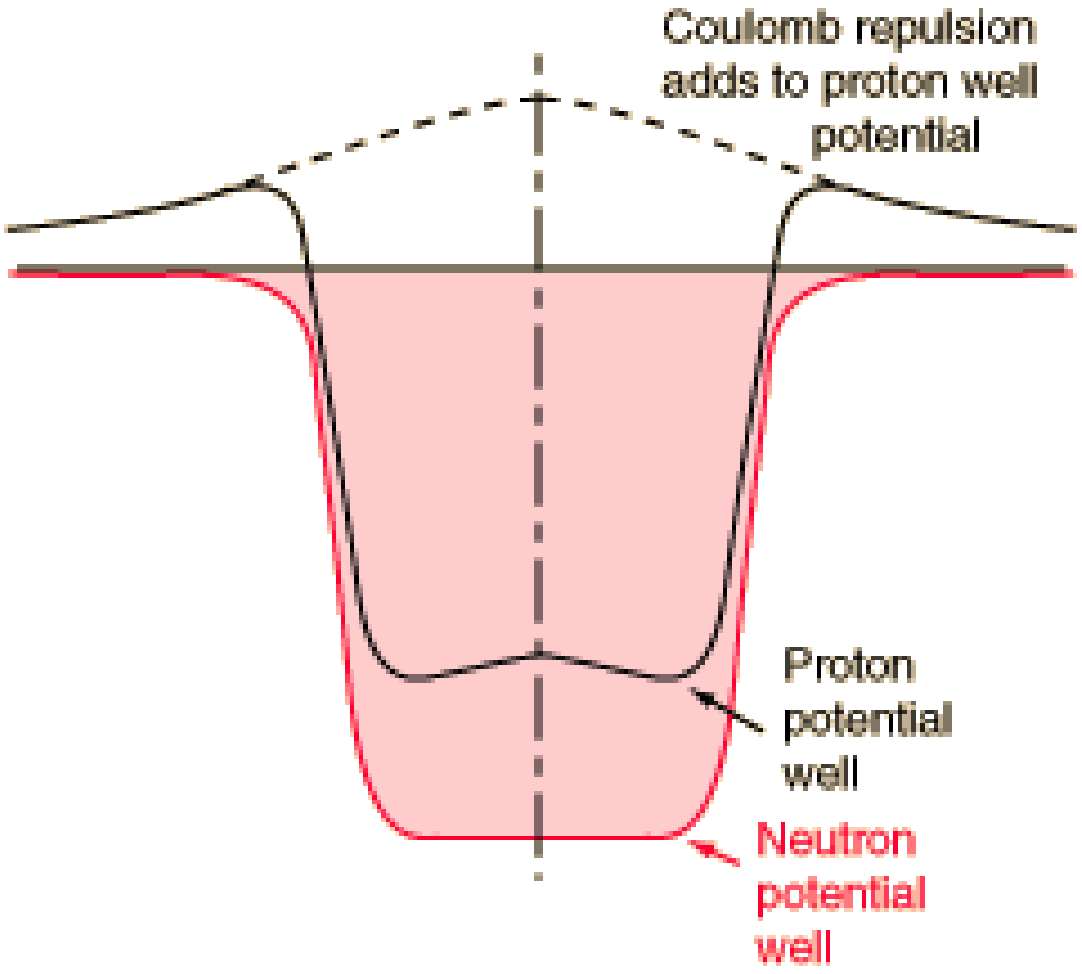
The energy spectrum is the sum of three one-dimensional eigenvalues:

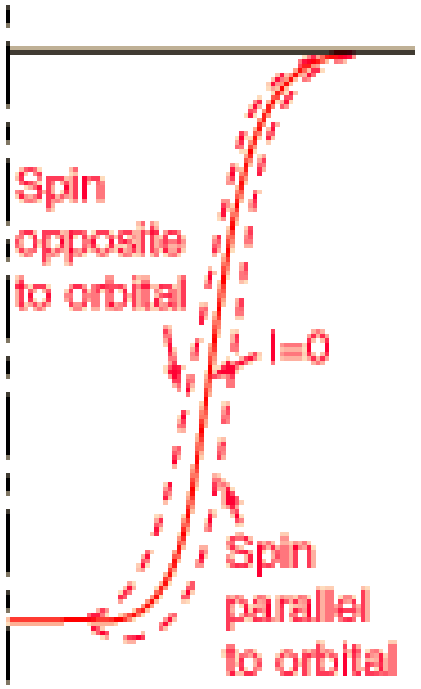
$$E = \hbar\omega \left(n_x + n_y + n_z + \frac{3}{2} \right) = \hbar\omega \left(N + \frac{3}{2} \right)$$

where $N=2n+l=0,1,2,3,\ldots$ is called principal quantum number,
$n=0,1,2,\ldots N/2$ radial quantum number and
$l=0,1,2,\ldots,N$ angular momentum

Realistic mean field potential

for protons and neutrons is given by a dependence between the square well and ho.
Woods-Saxon dependence is the most used potential.





Spin-orbit interaction

is very strong in nuclei
and it is peaked on the nuclear surface
(is proportional to the derivative of the central potential)

$$V_{so}(r) = \kappa \frac{dV_c(r)}{dr} (\mathbf{l}.\mathbf{s})$$

The expectation value of the scalar product <(**l**.**s**)> can be derived by using the expression:

$$\mathbf{j} = \mathbf{l} + \mathbf{s}$$

$$\mathbf{j}^2 = \mathbf{l}.\mathbf{l} + \mathbf{s}.\mathbf{s} + 2(\mathbf{l}.\mathbf{s})$$

$$\langle (\mathbf{l}.\mathbf{s}) \rangle = \frac{\hbar^2}{2} [\, j(j+1) - l(l+1) - s(s+1)]$$

$$= \frac{\hbar^2}{2} \left[j(j+1) - l(l+1) - \frac{3}{4} \right]$$

Spin-orbit splitting

Total spin has two values :

$$j = l + \frac{1}{2}$$

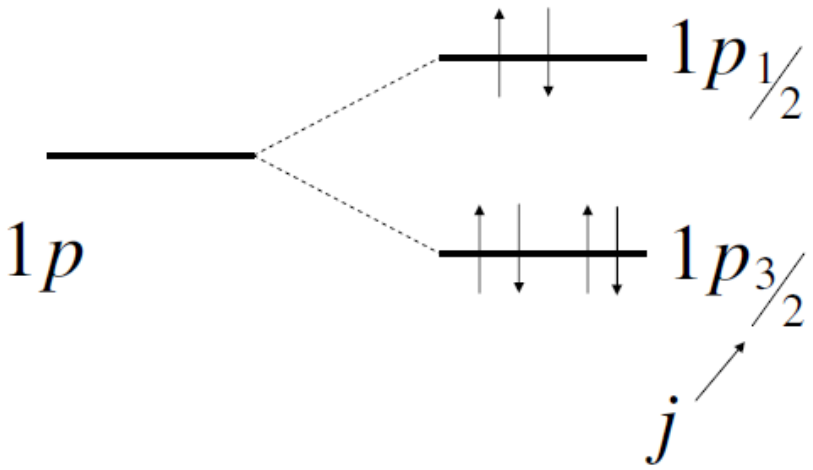
$$\langle \mathbf{l}.\mathbf{s} \rangle = \frac{\hbar^2}{2}\left[\left(l+\frac{1}{2}\right)\left(l+\frac{3}{2}\right) - l(l+1) - \frac{3}{4}\right] = \frac{\hbar^2}{2} l$$

$$j = l - \frac{1}{2}$$

$$\langle \mathbf{l}.\mathbf{s} \rangle = \frac{\hbar^2}{2}\left[\left(l-\frac{1}{2}\right)\left(l+\frac{1}{2}\right) - l(l+1) - \frac{3}{4}\right] = -\frac{\hbar^2}{2}(l+1)$$

Thus, the spin-orbit scalar product becomes:

$l=1$



Eigenfunctions in the nuclear mean field with a spin-orbit coupling

The single particle wave function is called spin-orbit harmonics and it can be written by using the ket notation, or the coordinate form:

$$\left| jm \right\rangle = \sum_{m_l + m_s = m} \left\langle lm_l ; \frac{1}{2} m_s \mid jm \right\rangle \left| lm_l \right\rangle \left| \frac{1}{2} m_s \right\rangle \Leftrightarrow$$

$$\psi_{jm}(\mathbf{r}, \mathbf{s}) \equiv \left\langle \mathbf{r}, \mathbf{s} \mid jm \right\rangle = R_l(r) \sum_{m_l + m_2 = m} \left\langle lm_l ; \frac{1}{2} m_s \mid jm \right\rangle Y_{lm_l}(\vartheta, \varphi) \chi_{\frac{1}{2} m_s}(\mathbf{s})$$

where $R_l(r)$ is the radial wave function,
which can be expanded in terms of eigenfunctions
of the spherical ho potential $R_{nl}(r)$.
Y_l is the angular harmonics and $\chi_{1/2}$ the spin function.

Single particle eigenfunction

$$\psi_{\varepsilon ljm}(\mathbf{r}, \mathbf{s})$$

is characterized by:

ε: energy, or alternatively:

n: number of nodes of the radial wave function, which can be expanded in terms of ho eigenfunctions. Thus, n coincides with the radial quantum number of the largest ho component

l: angular momentum

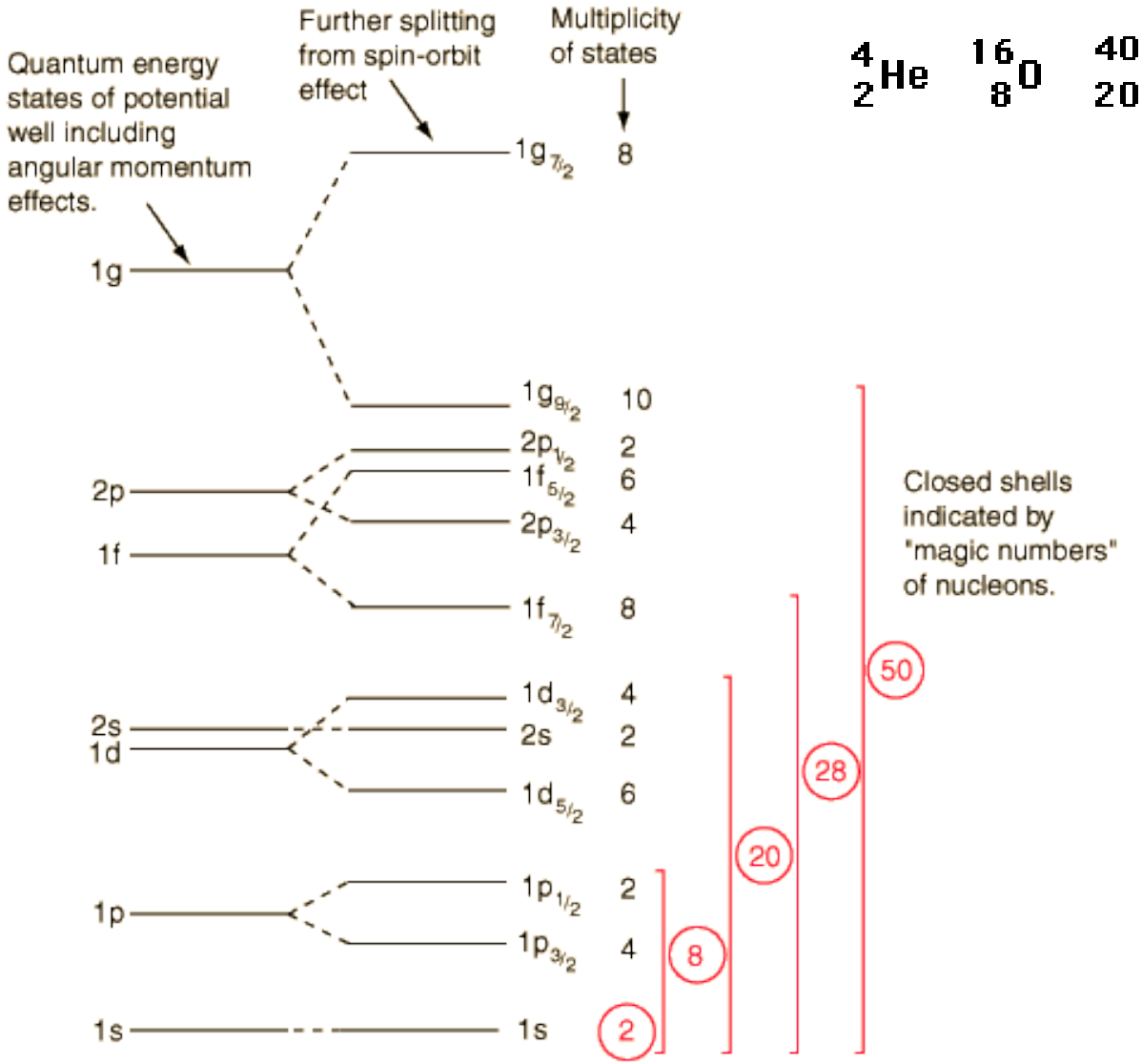
j: total spin

m: spin projection

Notice that the parity is given by the $(-)^l$ rule

C.02. Shell model

Magic numbers separating the shells appear due to the spin-orbit splitting of eigenvalues



$^{4}_{2}He$ $^{16}_{8}O$ $^{40}_{20}Ca$ $^{48}_{20}Ca$ $^{208}_{82}Pb$

Spectroscopic notation of of levels:

nl_j

n: radial q.numb.

j: total spin

l=0 → *s*

l=1 → *p*

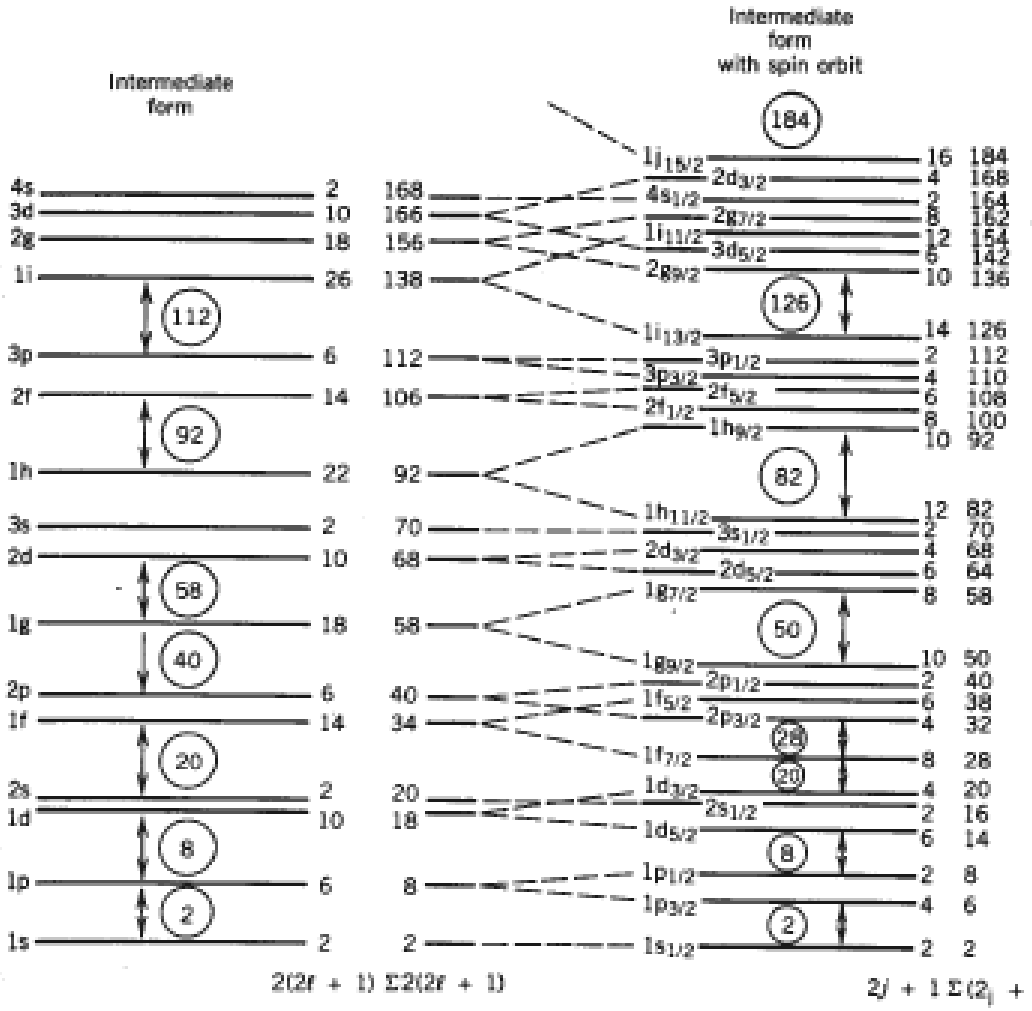
l=2 → *d*

l=3 → *f*

l=4 → *g*

l=5 → *h*

Spherical shell model scheme

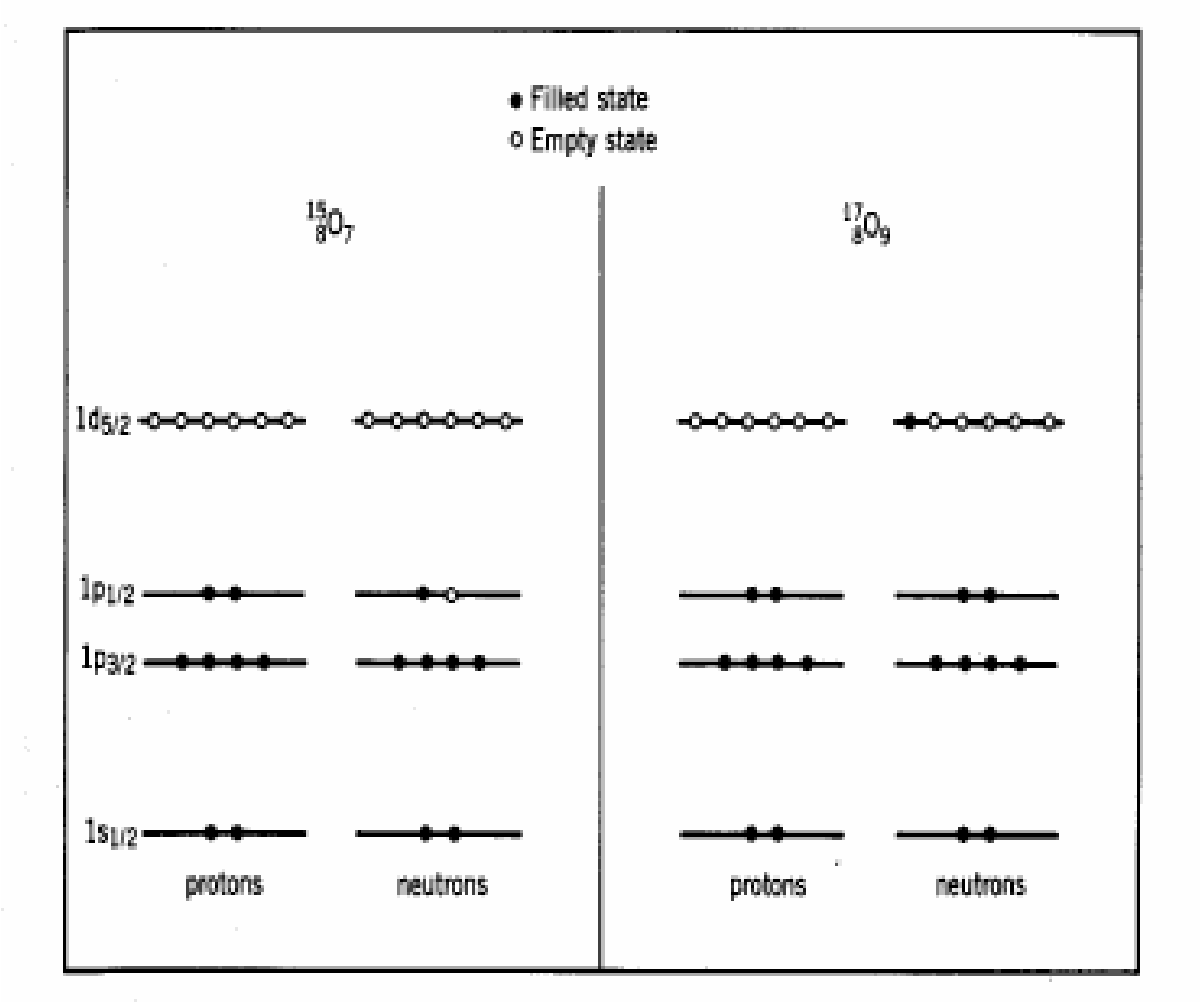


Energy level scheme

- s, p, d, f ….. $l = 0, 1, 2.....$ (as in ATOM)

The last nucleon of an odd-even (even-odd) nucleus

determines the nuclear properties
(spin, quadrupole and magnetic moments)



Shell Model

- Z or N = MAGIC ± 1
- Exclusion Principle: the extra nucleon (or 'hole') determines *I*.

	$^{15}_{8}O$	$^{16}_{8}O$	$^{17}_{8}O$
Z	8	8	8
N	7	8	9
Level	One n in 1p(1/2)	1p (1/2) full	One n in 1d(5/2)
I	1/2	0	5/2

"Doubly Magic"

C.03. Many body dynamics

Identical particles

The probability of a system with several particles remains unchanged with respect to the permutation of particle coordinates:

$$\left|\Psi(...,r_i,...,r_j,...)\right|^2 = \left|\Psi(...,r_j,...,r_i,...)\right|^2$$

Therefore the wave function can be symmetric or antisymmetric:

$$\Psi(...,r_i,...,r_j,...) = \pm\Psi(...,r_j,...,r_i,...)$$

Connection between symmetry and statistics

The symmetric wave function describes a system of bosons with integer spins, like for instance alpha particles (2 protons+2 neutrons). The antisymmetric wave function describes a systems of fermions with half-integer spins, like for instance protons, neutrons or electrons.

Independent particle motion

The wave function of independent particles is a symmetrised or antisymmetrised normalized product of single particle wave functions. For instance a system of N independent fermions is described by a normalized determinant (called Slater determinant):

$$\Psi_N(r_1,...,r_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(r_1)\psi_1(r_2).....\psi_1(r_N) \\ ..... \\ \psi_N(r_1)\psi_N(r_2)...\psi_N(r_N) \end{vmatrix}$$

Lines correspond to states.
Rows correspond to coordinates.

Pauli principle

Two fermions cannot occupy the same state
(if two lines are equal then the determinant vanishes).

Second quantization

is a **representation** describing a system with a variable number of particles.
We will give basic details for fermions.
A normalized determinant describing a system of N fermions
can be written as a product of N creation operators
acting in the space of occupation numbers, called **“Fock space”**.

$$\left|\Psi_N\right\rangle = \hat{\mathbf{a}}_N^+ \hat{\mathbf{a}}_{N-1}^+ \ldots \hat{\mathbf{a}}_1^+ \left|0\right\rangle$$

where |0> is called vacuum state with 0 particles and
$\mathbf{a}_k^+$ is the creation operator on the k-th state.
The annihilation and creation operators obey the following anticommutator:

$$\hat{\mathbf{a}}_i \hat{\mathbf{a}}_j^+ + \hat{\mathbf{a}}_j^+ \hat{\mathbf{a}}_i = \delta_{ij}$$

The expansion of a determinant of the order N in terms of the elements
of the first line multiplied by determinants of the N-1 order is written as follows:

$$\left|\Psi_N\right\rangle \propto \hat{\mathbf{a}}_N^+ \left|\Psi_{N-1}\right\rangle$$

One body operator

like for instance number of particle operator
or electric multipole transition operators
has the following representation:

$$\hat{\mathbf{T}} = \sum_{ij} \hat{\mathbf{a}}_i^+ \left\langle i \left| \hat{\mathbf{T}} \right| j \right\rangle \hat{\mathbf{a}}_j$$

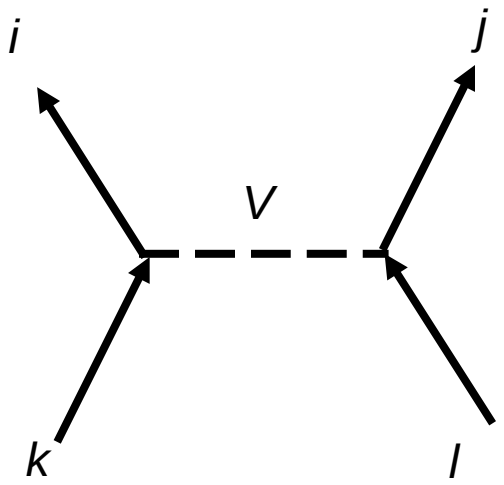
i.e. is the sum of transitions annihilating the state *j*
and creating the state *i*

Two body operator

like for instance potential energy operator
has the following representation:

$$\hat{\mathbf{V}} = \sum_{ijkl} \hat{\mathbf{a}}_i^+ \hat{\mathbf{a}}_j^+ \left\langle ij \left| \hat{\mathbf{V}} \right| kl \right\rangle \hat{\mathbf{a}}_l \hat{\mathbf{a}}_k$$

i.e. is the sum of transitions annihilating the states *kl*
and creating the states *ij*. It is represented
by the right diagram from right to the left.



C.04. Hartree-Fock (HF) mean field

In the lowest approximation each nucleon moves independently
in the mean field created by the other N-1 nucleons.

The mean field Hartree-Fock equations are derived
by using the variational principle:

$$\delta\left\langle \Psi \left| \hat{\mathbf{H}} \right| \Psi \right\rangle = 0$$

where the Hamiltonian is given by
the sum between kinetic and potential components:

$$\hat{\mathbf{H}} = \sum_{ij} \hat{\mathbf{a}}_i^+ \left\langle i \left| \hat{\mathbf{T}} \right| j \right\rangle \hat{\mathbf{a}}_j + \sum_{ijkl} \left\langle ij \left| \hat{\mathbf{V}} \right| kl \right\rangle \hat{\mathbf{a}}_i^+ \hat{\mathbf{a}}_j^+ \hat{\mathbf{a}}_l \hat{\mathbf{a}}_k$$

One obtains the following system of equations for single particle states:

$$\hat{\mathbf{H}}_{MF}\left|k\right\rangle = \varepsilon_k \left|k\right\rangle$$

with the selfconsistent nuclear mean field (MF):

$$\left\langle k \left| \hat{\mathbf{H}}_{MF} \right| k \right\rangle \equiv \left\langle k \left| \hat{\mathbf{T}} \right| k \right\rangle + \sum_n \left(\left\langle kn \left| \hat{\mathbf{V}} \right| kn \right\rangle - \left\langle kn \left| \hat{\mathbf{V}} \right| nk \right\rangle \right)$$

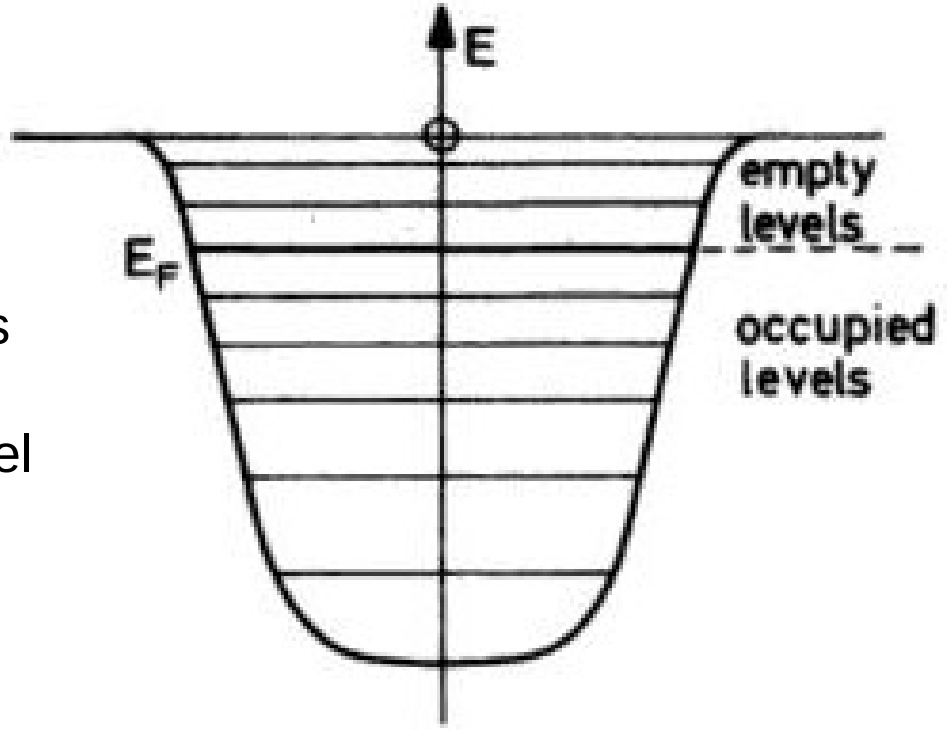
direct (Hartree) term exchange (Fock) term

HF equations are solved by iterations:
The eigenstates |k> are used to compute the HF Hamiltonian, which is used to obtain new eigenstates, until the convergency is achieved.

In the spherical nuclear mean field

each spherical level is filled by 2j+1 nucleons
with different projections

Fermi level
corresponds
to the last
occupied level

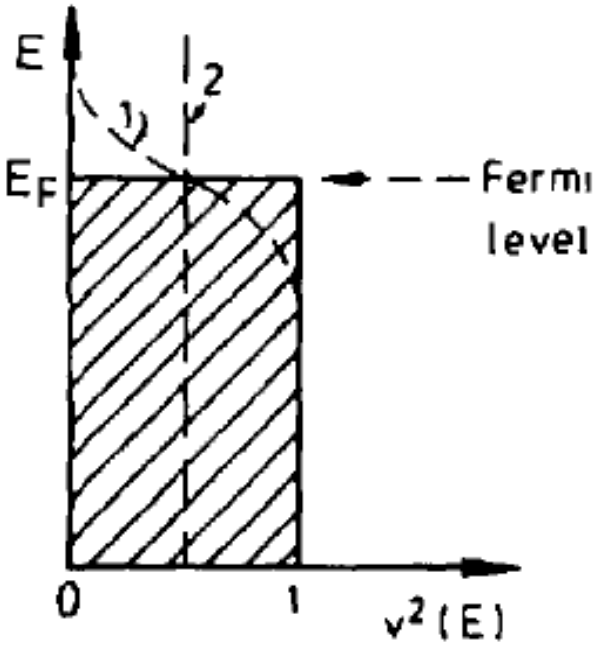


Ground state is a Slater determinant
of HF eigenstates
obeying the Pauli exclusion principle

C.05. Thomas-Fermi model

describes a gas if independent fermions within the semiclassical limit.
The space phase is defined by **k** (or **p**) and **r** coordinates.
The "elementary quantum cel" of the phase space is:
$\Delta\mathbf{k}\Delta\mathbf{r} \rightarrow (2\pi)^3$ or $\Delta\mathbf{p}\Delta\mathbf{r} \rightarrow (2\pi\hbar)^3$
The density of states at temperature T=0 is:

$$n = \frac{N}{V} = 2\frac{4\pi p_F^3}{3}\frac{V}{(2\pi\hbar)^3}\frac{1}{V} = \frac{p_F^3}{3\pi^2\hbar^3}$$



where we considered 2 particles with opposite spins on each state and p_F is the maximal (Fermi) momentum corresponding to the Fermi level (right figure).
Thus, Fermi momentum becomes:

$$p_F = \hbar\left(3\pi^2\frac{N}{V}\right)^{1/3}$$

where:

$$V = 4\pi r_0^3 A$$

For N=Z=A/2 one obtains: $$p_{Fp} = p_{Fn} = \frac{\hbar}{r_0}\left(\frac{9\pi}{8}\right)^{1/3}$$

By using the standard values
r_0=1.2, ħc=197.3 MeV.fm, $m_N c^2$=938.9 MeV,
one obtains the Fermi kinetic energy

$$E_{Fp} = \frac{p_{Fp}^2}{2m_N} = E_{Fn} = \frac{p_{Fn}^2}{2m_N} \approx 33 MeV$$

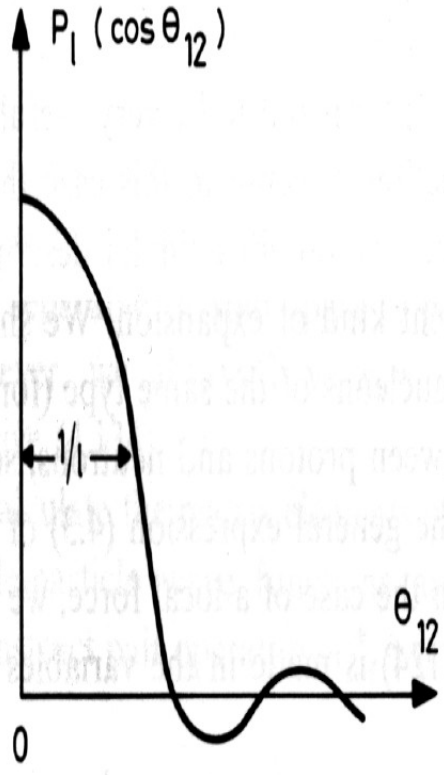
This energy corresponds to the kinetic energy of the highest occupied orbit (smallest binding energy).
The average energy / nucleon is:

$$\langle E \rangle_p = \frac{\int_0^{p_F} E_{kin} d\mathbf{p}}{\int_0^{p_F} d\mathbf{p}} = \frac{3}{5}\frac{p_{Fp}^2}{2m_N} = \frac{3}{5} E_{Fp} \approx 20 MeV$$

C.06. Residual interaction

acts among nucleons in the mean field.
The total Hamiltonian is the sum between the mean field and residual terms:

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}_{MF} + \hat{\mathbf{V}}_{res}$$



Multipole expansion of the residual interaction is given by

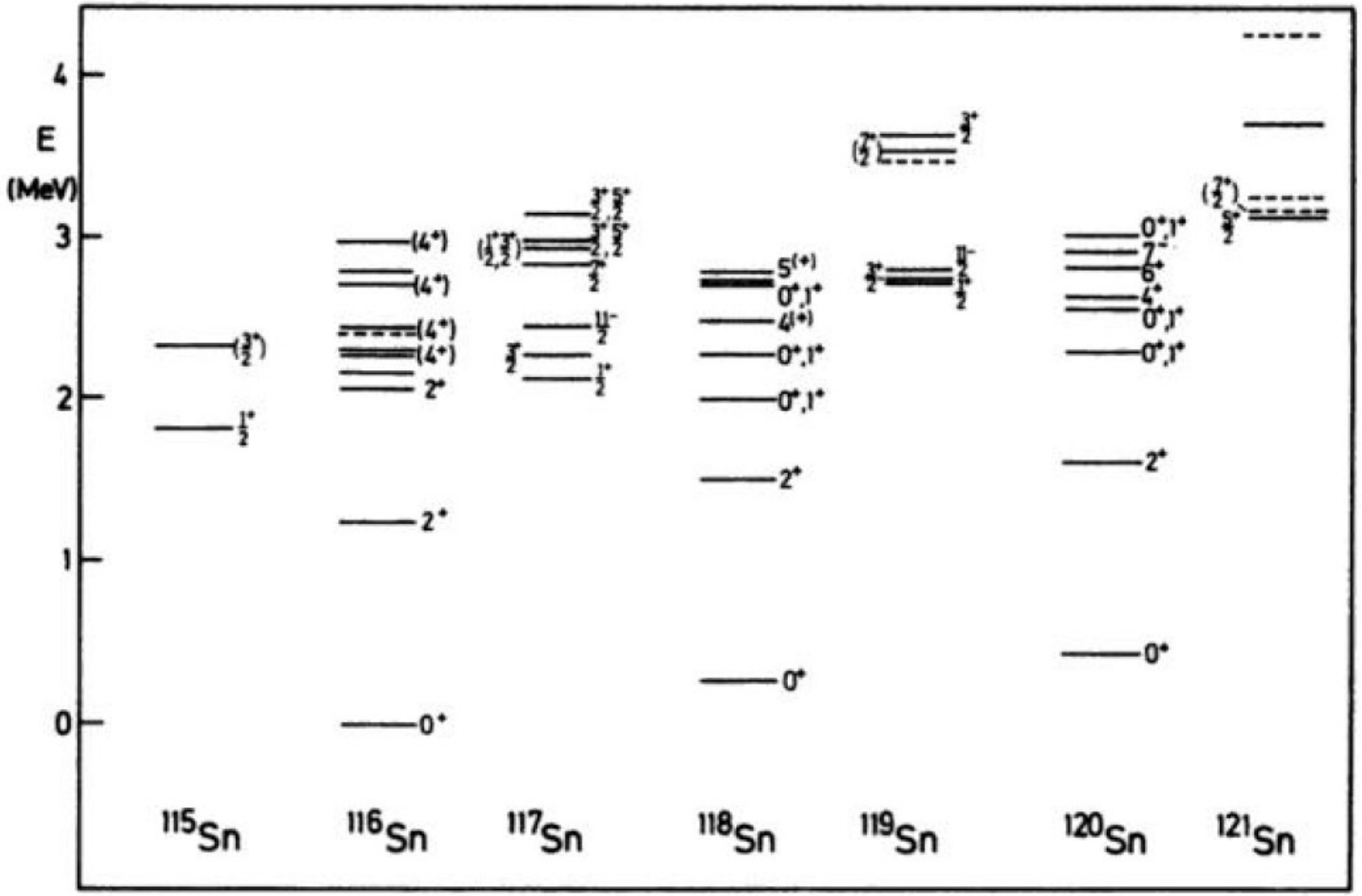
$$\hat{\mathbf{V}}_{res}(\mathbf{r_1},\mathbf{r_2}) = \sum_l V_l(r_1,r_2) P_l(\cos\vartheta_{12})$$

where P_l is the Legendre polinomial

The most important terms of the residual interaction are:
l=0 : monopole pairing interaction
l=2 : quadrupole-quadrupole interaction

C.07. Pairing interaction

Low-lying states in even-even nuclei are separated
by an energy gap from the ground state, like in Sn isotopes below



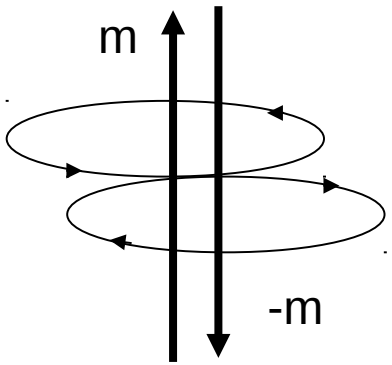
Monopole pairing coupling explains this gap

Pairing wave function is given
by the coupling of two single particle states
to the total angular momentum 0:

$$\Psi_j(\mathbf{r_1},\mathbf{r_2}) = \left[\psi_j(\mathbf{r_1}) \otimes \psi_j(\mathbf{r_2})\right]_0 = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} (-)^{j-m} \psi_{jm}(\mathbf{r_1}) \psi_{j-m}(\mathbf{r_2})$$

where $\psi_{nljm}(\mathbf{r_k})$ is the wave function of the k-th nucleon in the spin-orbit coupling.

The pictorial representation
of the pairing coupling is given
by two nucleons rotating
in opposite directions



Residual monopole particle-particle (p-p) interaction describes pairing correlations.

Monopole pairing Hamiltonian for protons/neutrons is:

$$\hat{\mathbf{H}}_P = \sum_j (\varepsilon_j - \lambda)\hat{\mathbf{N}}_j - G\sum_{jk} \hat{\mathbf{P}}_j^+ \hat{\mathbf{P}}_k$$

Here ε_j are single particle energies provided by the mean field and λ is the Lagrange multiplier taking care on the conservation of the number of particles in nucleus. G is the pairing strength.

We used a short hand notation: $(nlj) \rightarrow j$)

where:

$$\hat{\mathbf{N}}_j = \sum_m \hat{\mathbf{a}}_{jm}^+ \hat{\mathbf{a}}_{jm}$$

$$\hat{\mathbf{P}}_j^+ = \sum_m \hat{\mathbf{a}}_{jm}^+ \hat{\mathbf{a}}_{j-m}^+ (-)^{j-m}$$

The quasiparticle is a superposition of a creation and annihilation operators (Bogoliubov-Valatin transformation)

$$\hat{\boldsymbol{\alpha}}^{+}_{jm} = \mathrm{u}_j \hat{\mathbf{a}}^{+}_{jm} + \mathrm{v}_j \, \hat{\mathbf{a}}_{j-m} (-)^{j-m}$$

It approximately diagonalizes the pairing Hamiltonian:

$$\hat{\mathbf{H}}_P \rightarrow \sum_j E_j \hat{\boldsymbol{\alpha}}^{+}_j \hat{\boldsymbol{\alpha}}_j$$

$$E_j = \sqrt{(\varepsilon_j - \lambda)^2 + \Delta^2}$$

Here E_j is the quasiparticle energy.
Close to the Fermi level, where $\varepsilon_j \approx \lambda$, one has $E_j \approx \Delta$ and therefore 2Δ is the energy necessary to break a pair of particles in even-even nuclei.

Ground state (BCS vacuum) is defined by:

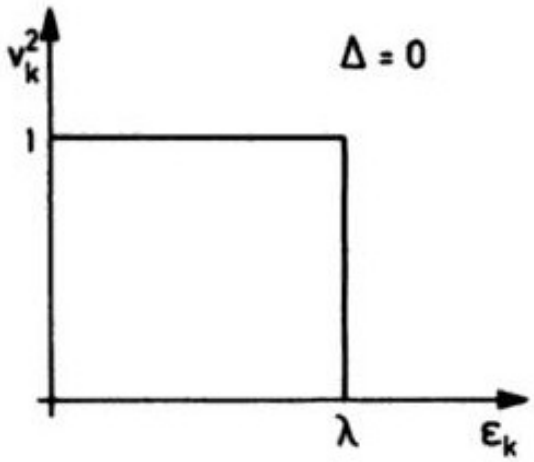
$$\hat{\boldsymbol{\alpha}}_k \left| BCS \right\rangle = 0$$

Gap parameter describes superfluid nuclei between closed shells

$$\Delta = G\sum_j \left\langle BCS|\hat{\mathbf{P}}_j^+|BCS\right\rangle$$

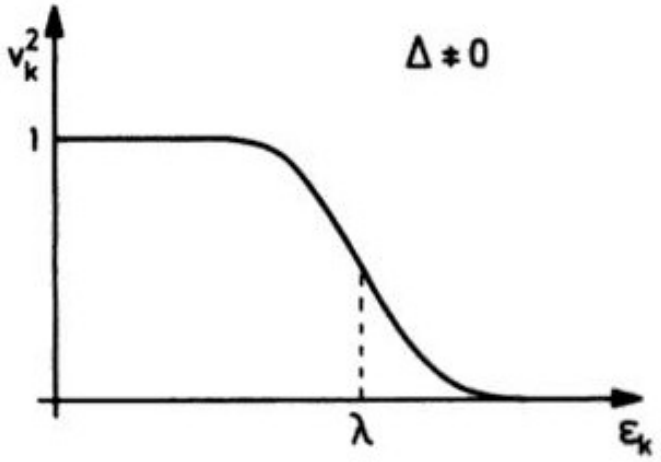
Distribution of occupation probabilities

Normal system (closed shell)



Fermi level

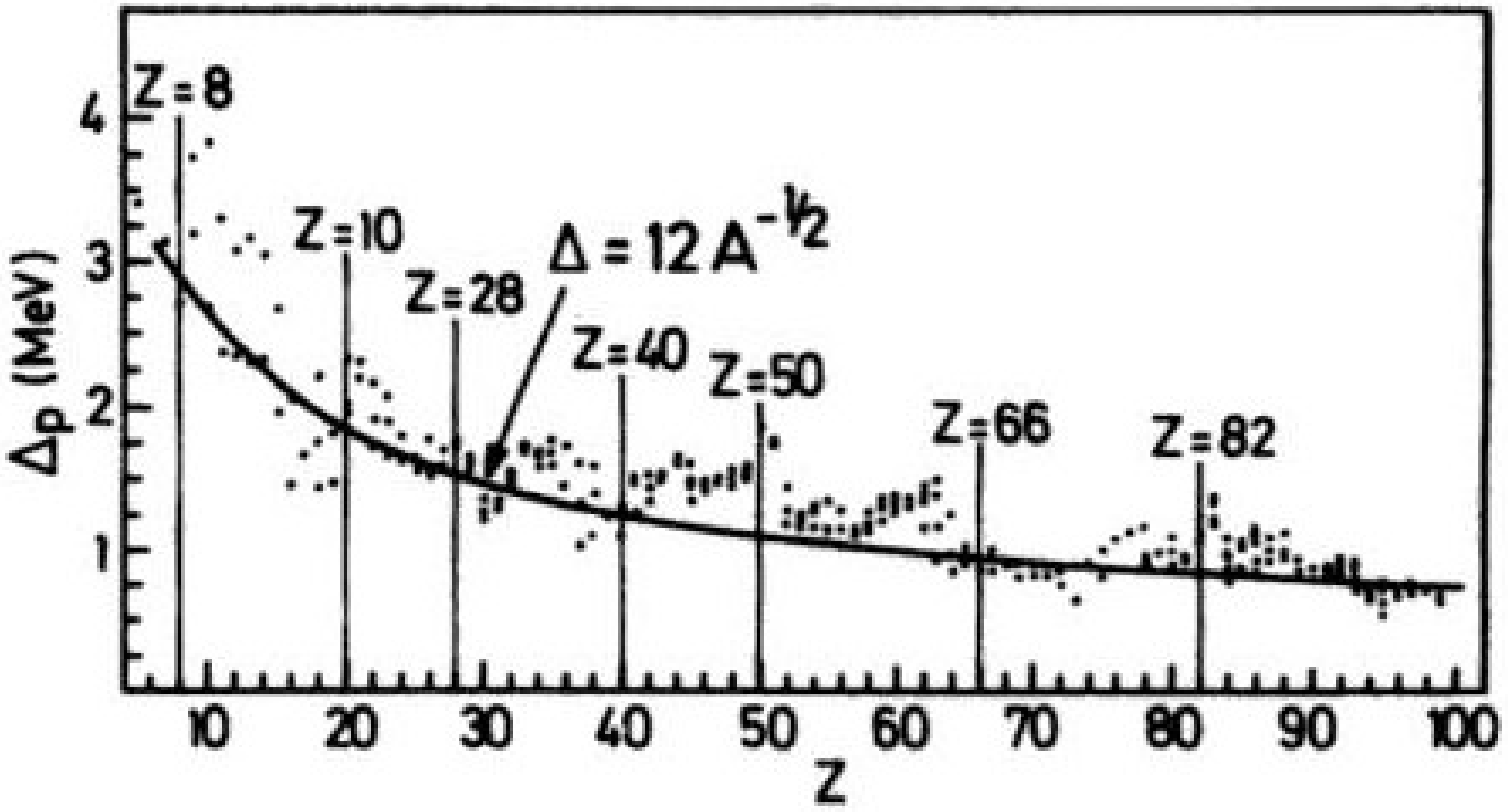
Superfluid system



$$\hat{\boldsymbol{\alpha}}_{jm}^+ \approx \hat{\mathbf{a}}_{jm}^+;\ \varepsilon_j > \lambda$$

$$\approx (-)^{j-m}\hat{\mathbf{a}}_{j-m};\ \varepsilon_j < \lambda$$

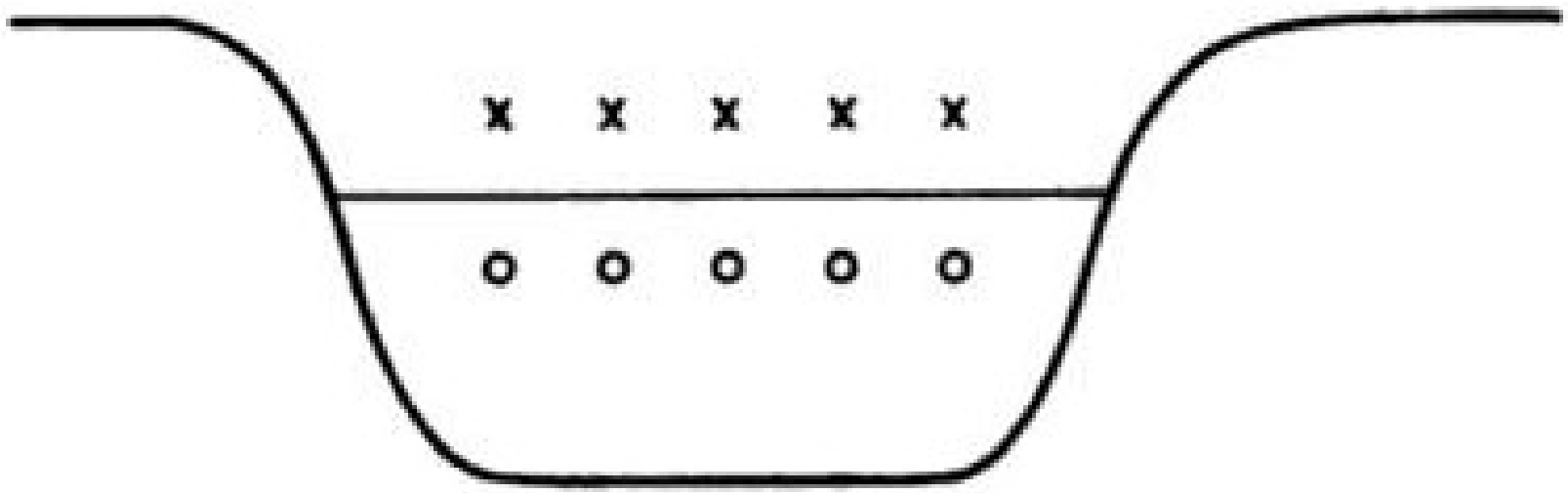
Proton pairing gap versus Z



C.08. Collective excitations

The superposition of pairs between particle (croses) and hole (open circles) states
describes collective excitations in nuclei

Elementary particle-hole (p-h) excitation: $\hat{\mathbf{a}}_p^+ \hat{\mathbf{a}}_h \left| \Psi_{gs} \right\rangle$



Types of p-h collective excitations:

1) low-lying surface vibrations

2) giant resonance (motion of protons against neutrons)

Residual Hamiltonian describing these excitations is given by the multipole-multipole interaction $Q_\lambda Q_\lambda$.

Quadrupole term with $\lambda=2$ describes low-lying states.

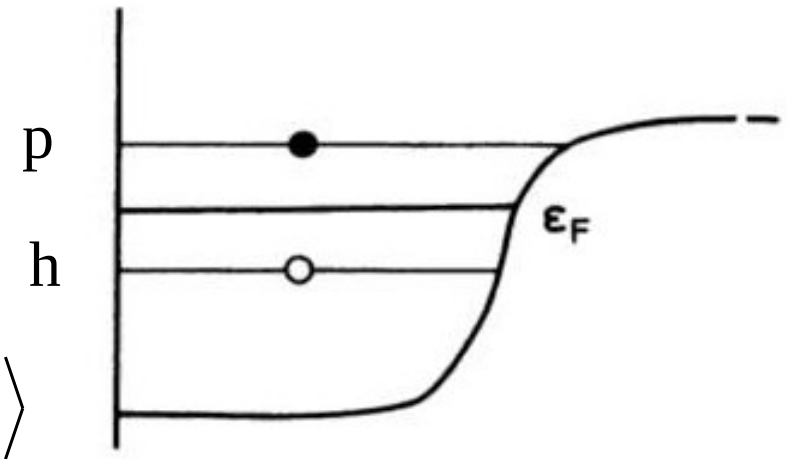
Dipole term with $\lambda=1$ describes giant dipole resonance.

$$\hat{\mathbf{H}}_Q = \sum_j \varepsilon_j \hat{\mathbf{N}}_j - F(\hat{\mathbf{Q}}_\lambda \otimes \hat{\mathbf{Q}}_\lambda)_0$$

where :

$$\hat{\mathbf{N}}_j = \sum_m \hat{\mathbf{a}}^+_{jm} \hat{\mathbf{a}}_{jm}$$

Wave function is a
collective p-h excitation,
i.e. a superposition of
various elementary p-h terms
(Tamm-Dancoff transformation)



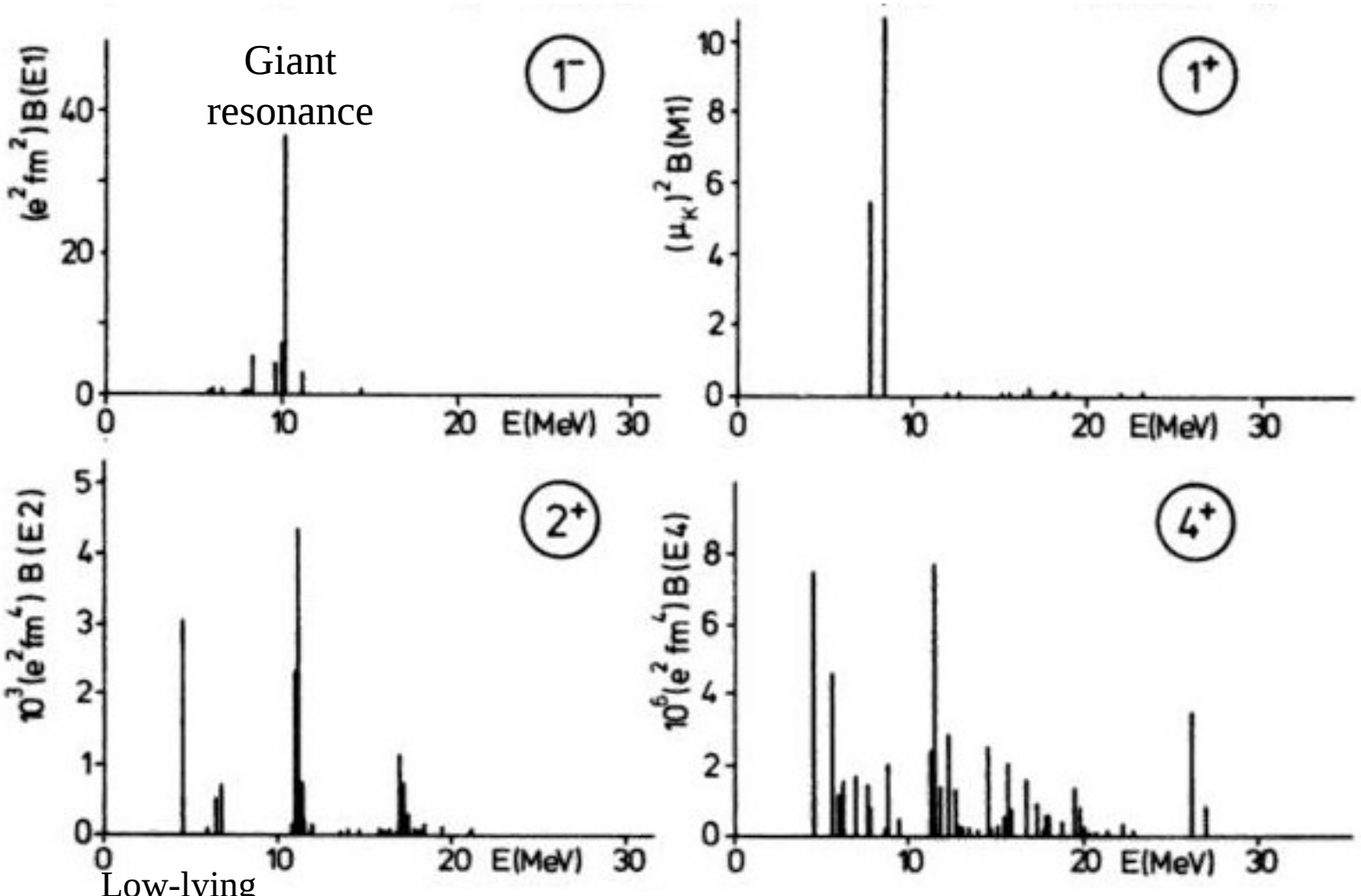
$$\hat{\Gamma}^{+}_{\lambda\mu}(n) = \sum_{ph} X^{\lambda}_{ph}(n)(\hat{\mathbf{a}}^{+}_{p} \otimes \hat{\mathbf{a}}_{h})_{\lambda\mu} \left| \Psi_{gs} \right\rangle$$

This wave function
diagonalizes the Hamiltonian

$$\hat{\mathbf{H}}_{Q} \rightarrow \sum_{n} \omega_{n} \hat{\mathbf{\Gamma}}^{+}_{\lambda\mu}(n) \hat{\mathbf{\Gamma}}_{\lambda\mu}(n)$$

where *n* denotes the collective
excitation (eigenmode)

Distribution of collective excitations for various multipolarities versus energy



C.08. Collective model

Shell Model assumes independent behaviour of nucleons

Liquid-Drop Model assumes the opposite

Collective Model takes features from both.

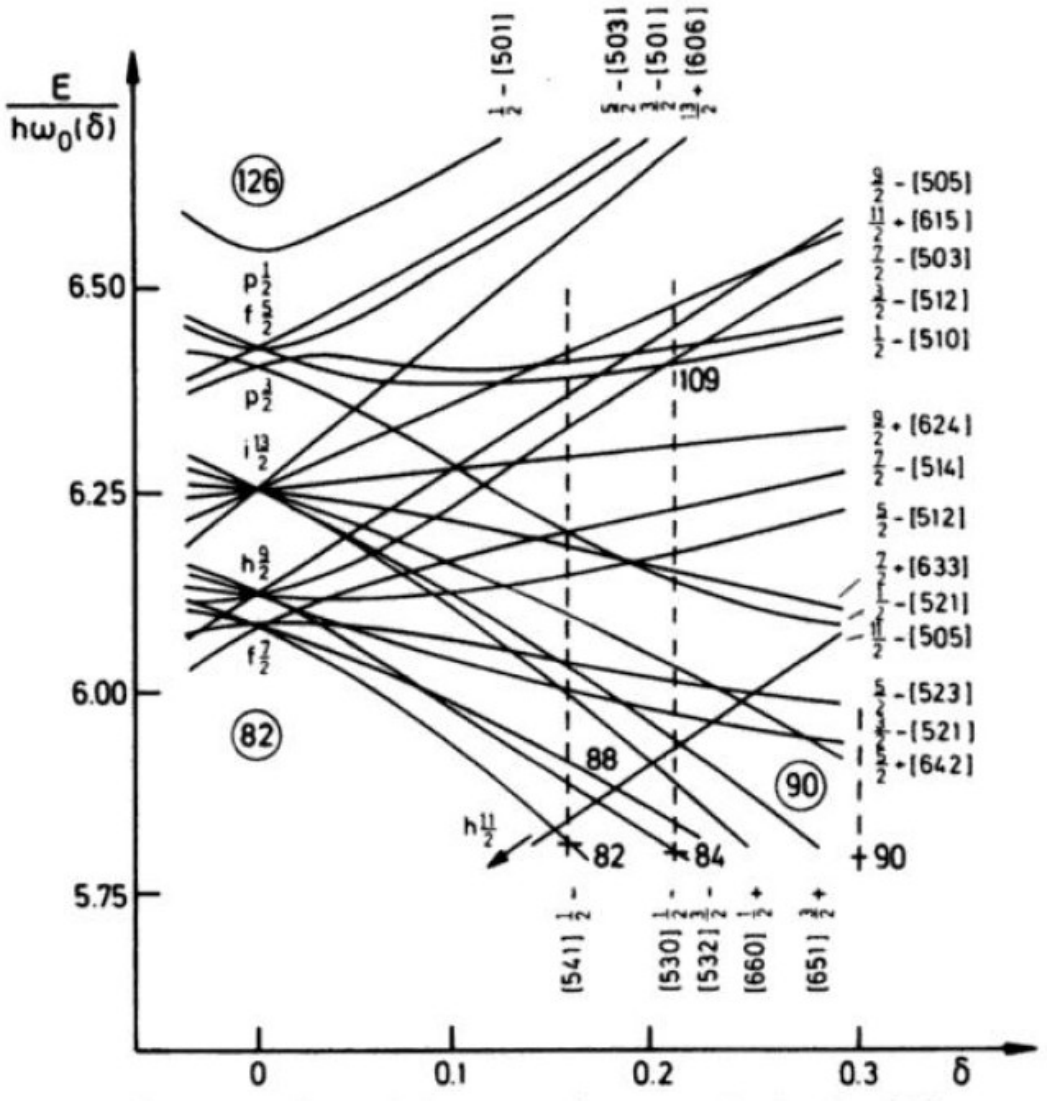
Nucleons in unfilled subshells move independently in a net potential created by the filled 'core' nucleons (as in Shell Model)

The potential is allowed to deform, as in a liquid. (The Shell Model assumes a static, spherically symmetric potential).

"Valence nucleons"

Nilsson mean field for single particle states in the deformed intrinsic system

Single particle energy versus deformation



Deformed Hamiltonian in the intrinsic system, connected to the nuclear symmetry axis, is given by the m=0 component of the quadrupole operator:

$$\hat{\mathbf{H}}_{def} = \hat{\mathbf{H}}_{sph} - \beta_2 m_N \omega^2 r^2 Y_{20}$$

$$\delta \propto \beta_2$$

Each deformed level can accommodate two nucleons with opposite spin projections +m, -m

Effect of the deformation:

In axially symmetric nuclei spherical levels are splitted into (2j+1)/2 deformed levels, as for instance B(E2:2->0) values from various 2^+ levels (right figures)

The lowest level in the deformed nucleus ^{240}Pu has E_2=43 keV and the strongest transition.

(c) corresponds to a spherical approach,
(d) corresponds to a deformed approach.