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On equations of motion on complex grassmann manifold

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ABSTRACT: We investigate the equations of motion on the "classical" phase space which corresponds to quantum state space in the case of the complex Grassmann manifold appearing in the Hartree - Fock problem. First and second degree polynomial Hamiltonians in bifermion operators are considered. The "classical" motion corresponding to linear Hamiltonians is described by a Matrix Ricatti equation.

In the case of second degree Hamiltonians in bifermion operators, the calculation is based on a formula which expresses the mean value of a product of two bifermion operators in terms of mean values of bifermion operators.

1. INTRODUCTION

Lately a considerable effort was devoted to accomodate the geometrical ideas and methods in physics (Abraham and Marsden 1978, Guillemin and Sternberg 1984). Particular attention has been paid to different physical models which have Grassmannian geometrical structure. The Grassmann manifolds appear in field theoretical models (see for example, Berezin 1978), in instanton theory (Perelomov 1982), also in Kadomtsev-Petviashvili hierarchy (Sato 1981, Segal and Wilson 1985, Presseley and Segal 1986), as well as in supersymmetry theories and topological quantum field theories (Witten 1987).

In other physical situations, more general objects, which generalize the Grassmann manifold are considered. For example, in reference Berceanu and Gheorghe 1987, using group-theoretical methods, perfect Morse functions (Morse 1934, Berceanu 1984) on compact manifold of coherent states (Perelomov 1972) admiting a Kählerian C-space structure (Wang 1954) have been constructed, considering linear Hamiltonians in the generators of the group of symmetry. In reference Berceanu and Gheorghe 1989, the authors have restricted to the complex Grassmann manifold which appears in the Hartree-Fock problem, and they have shown that linear Hamiltonians in bifermion operators lead to energy functions which satisfy the Morse-Bott inequalities as equalities. The problem of isolated critical points in the Hartree-Fock case has also been considered by Rosensteel and Ihring 1980.

In the present paper we restrict on the complex Grassmann manifold, focusing our interest to the motion problem on this manifold. We consider polynomial hermitian Hamiltonians in bifermion operators, which lead to real valued "classical" energy functions (Berezin 1978). Without entering into the subtle interplay between

quantum and classical concepts, we only emphasize that the equations of motion presented here represent a dequantization which leads from time-evolution in state space to trajectories on phase space. We hope that this study will contribute to the concrete description of this problem in the physical situation of time-dependent Hartree-Fock case.

The paper is organized as follows. In §2 the main definitions and notations are introduced. The equations of motion are presented in a form that can be interpreted geometrically as "classical" motion described with symplectic methods (Guillemin and Sternberg 1984, Kramer and Saraceno 1981). We choose a complex parametrization for the wave function, currently appearing in the coherent state approach (Perelomov 1982, 1983). The Slater determinant manifold is parametrized as Grassmann manifold (Berceanu and Gheorghe 1989). Details are also given in Appendix 1.

The case of linear Hamiltonians in bifermion operators is considered in §3. The equations of motion are a first-order system of differential equations, the right hand side being a second degree polynomial. These equations can be put in the form of Matrix Ricatti equations (Reid 1972). The Matrix Ricatti equations appear in different branches of Applied Mathematics, as Linear Systems theory, transmission line phenomena, the theory of stochastic processes, optimal control and filtering (Hermann 1973,1974, Hermann and Martin 1982, 1983). The phase portrait of Matrix Ricatti equation in the language of Differentiable Dynamical Systems (Smale 1967) is intensively studied (Schneider 1973, Hermann and Martin 1982,1983, Shayman 1986).

In §4 the problem of integration of differential equations of §3 is reduced by standard methods (Hartman 1964) to a linear algebra problem. In a given chart, to the Matrix Ricatti equation is associated a first order autonomous linear system of differential equations (Levin 1959). The solution of the Matrix Ricatti equation, in a given chart, is written explicitely for the case of distinct eigenvalues of a matrix which characterize the energy function associated to the linear Hamiltonian in bifermion operators. In the variables related to the initial variables through a generalized linear transformation, the trajectories are curves on a torus. The globalization problem of the solution (Schneider 1973) is discussed, noting that the Matrix Ricatti equation is a flow on the Grassmann manifold (Hermann and Martin 1982,1983, Shayman 1986).

In §5 we present the equations of motion on complex Grassmann manifold in the case of some particular second degree Hamiltonians in bifermion operators. The calculation is based on a formula which expresses the mean value of a product of bifermion operators in terms of mean values of bifermion operators. Some details on calculations are added in other two Appendices.

2. DEFINITIONS AND NOTATIONS

In this section we collect the main definitions and notation we use along this paper.

2.1. *Equations of motion*

A complex Hilbert space K with scalar product $<\cdot|\cdot>$ is considered.

We assume that the wave function is parametrized by the complex variables Z_i, $i = 1,\dots,N$

$$|\psi> = |\psi(Z_1,\dots,Z_N)> .$$

To get the equations of motion, we can invoke the time-dependent variational principle (Kramer and Saraceno 1981, Yamamura and Kuriyama 1987), starting with the Lagrangian

$$L(Z,\overline{Z}) = \frac{i}{2} <\psi|\psi>^{-1} \sum_{j=1}^{N} (\dot{Z}_j <\psi|\partial\psi/\partial Z_j> - \dot{\overline{Z}}_j <\frac{\partial\psi}{\partial \overline{Z}_j}|\psi>) -$$

$$- <\psi|\psi>^{-1} <\psi|H|\psi> ,$$

where H denotes the Hamiltonian and $\overline{Z}$ is the complex conjugate of Z.

The equations of motion can be read as

$$i\begin{pmatrix} 0 & N \\ -\overline{N} & 0 \end{pmatrix} \begin{pmatrix} \dot{Z} \\ \dot{\overline{Z}} \end{pmatrix} = -\begin{pmatrix} \frac{\partial}{\partial Z} \\ \frac{\partial}{\partial \overline{Z}} \end{pmatrix} H , \qquad (2.1)$$

where

$$N = (N_{i\bar{j}})_{1\le i,j\le N} = (\frac{\partial^2}{\partial Z_i \partial \overline{Z}_j} \ln<\psi|\psi>) , \qquad (2.2)$$

$$H = <\psi|\psi>^{-1} <\psi|H|\psi> . \qquad (2.3)$$

2.2. *The complex Grassmann manifold*

Following the quantum mechanics notations, we consider here the fermion algebra $A(n)$. This is an associative algebra with unit element e over the complex field $\mathbb{C}$, having as generators the creation and annihilation operators a_p^+, a_p, $p=1,\dots,n$, satisfying the usual anticommutation relations

$$\{a_p^+, a_q\} = e\delta_{pq} , \quad \{a_p^+,a_q^+\} = \{a_p,a_q\} = 0 , \quad p,q = 1,\dots,n. \qquad (2.4)$$

The algebra $A(n)$ is realized as a subalgebra of the algebra of linear operators that act on K. The usual decomposition of K is

$$K = \bigoplus_{r=0}^{n} K_r ,$$

where K_r is the subspace whose basis is formed by the vectors

$$a_{p_1}^+ a_{p_2}^+ \dots a_{p_r}^+|0>, \quad 1\le p_1<p_2< \dots <p_r\le n, \quad 0 \le r \le n ,$$

and $|0\rangle$ for $r = 0$. Here $a_p|0\rangle = 0$, $1 \leq p \leq n$ and $\langle 0|0\rangle = 1$.

Let C_{pq} denote the bifermion operators

$$C_{pq} = a_p^+ a_q \quad , \quad p,q = 1,\dots,n. \tag{2.5}$$

From the anticommutation relations (2.4), the following commutation relations are derived

$$[C_{pq}, C_{rs}] = \delta_{qr}C_{ps} - \delta_{ps}C_{rq} \quad , \quad 1 \leq p,q,r,s \leq n \quad , \tag{2.6}$$

which are specific to the algebra $u(n)$ of the unitary group $U(n)$.

Since the space K is dimensionally finite, one may obtain a representation $\rho : GL(n, \mathbb{C}) \to A(n)$, and a representation $\rho' : gl(n, \mathbb{C}) \to A(n)$

$$\rho(e^{X'}) = \exp X \quad , \quad \rho'(X') = X \quad ,$$
$$X' = \sum_{p,q=1}^{n} x_{pq}E_{pq} \; , \quad X = \sum_{p,q=1}^{n} x_{pq}C_{pq} \quad , \quad \bar{x}_{pq} = -x_{qp} \quad , \tag{2.7}$$

and

$$(E_{pq})_{ij} = \delta_{ip}\delta_{jq} \quad , \quad 1 \leq i,j \leq n \quad . \tag{2.8}$$

Now we fix a subspace K_A, $A \leq n$, and let us consider the vectors

$$|\psi\rangle = a^+_{\sigma(1)}a^+_{\sigma(2)} \cdots a^+_{\sigma(A)}|0\rangle \in K_A, \quad \sigma \in S(A,n) \quad .$$

Here $S(A,n)$ denotes the set of C^n_A Schubert symbols, defined to be the permutations $\sigma : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ with the property that the restriction of σ^{-1} to the subsets $\{1,2,\dots,A\}$ and $\{A+1,\dots,n\}$ are increasing.

Equations (2.4) lead to the relations

$$C_{ij}|\psi_o\rangle = |\psi_o\rangle\delta_{ij} \quad ,$$
$$C_{im}|\psi_o\rangle = 0 \quad , \quad 1 \leq i,j \leq A \quad ,$$
$$C_{mi}|\psi_o\rangle \neq 0 \quad , \quad C_{mm'}|\psi_o\rangle = 0 \quad , \quad A+1 \leq m,m' \leq n \quad , \tag{2.9}$$

where $|\psi_o\rangle$ is the vector $|\psi_\sigma\rangle$ corresponding to the identity $\sigma \in S(A,n)$.

Equations (2.9) imply that the unidimensional subspace generated by $|\psi_o\rangle$ is invariant under the action of operators C_{ij} and $C_{mm'}$, $1 \leq i,j < A < m,m' \leq n$, hence under the group $U(A) \times U(n-A)$.

Further, an equivalence relation is introduced on the orbit $\mathcal{O} = \rho(U(n))|\psi_o\rangle \subset K_A$: two vectors from $\mathcal{O}$ are equivalent if they differ by a phase. Let $\tilde{\mathcal{O}}$ be the quotient space of the orbit $\mathcal{O}$ with respect to the above equivalence relation. The representation ρ induces a transitive action $\tilde{\rho}$ of the group $U(n)$ onto the quotient space $\tilde{\mathcal{O}}$

$$\tilde{\rho}(U)|\tilde{\psi}\rangle = \widetilde{\rho(U)|\psi\rangle} \; , \quad U \in U(n) \; , \quad |\tilde{\psi}\rangle \in \tilde{\mathcal{O}} \quad .$$

Noting that the stationary group of the state $|\tilde{\psi}_o\rangle$ under the action of $\tilde{\rho}$ is isomorphic to the product $U(A) \times U(n-A)$, a one-to-one correspondence between the elements (states) of $\tilde{\mathcal{O}}$ and the points of the homogeneous space $U(n)/U(A) \times U(n-A) \simeq G_A(\mathbb{C}^n)$ is obtained.

A differentiable manifold structures isomorphic to the manifold $G_A(\mathbb{C}^n)$ is induced onto $\tilde{\mathcal{O}}$ by the projective mapping

$$\omega : \; U(n)/(U(A) \times U(n-A)) \to \tilde{\mathcal{O}} \quad , \tag{2.10.1}$$
$$\omega \circ h_\sigma^{-1}(Z) = |\widetilde{Z,\sigma}\rangle \quad , \tag{2.10.2}$$
$$|Z,\sigma\rangle = \rho(\Delta^\sigma)\; |\underline{Z}\rangle \quad , \tag{2.10.3}$$
$$|\underline{Z}\rangle = U(Z)|\psi_o\rangle \quad , \tag{2.10.4}$$
$$U(Z) = \exp\Big(-\sum_{m=A+1}^{n}\sum_{i=1}^{A} Z_{mi}C_{mi}\Big) \quad . \tag{2.10.5}$$

The elements of matrix Z are local coordinates for $Z \in M(n-A,A)$ and the neighbourhood $\mu(V_\sigma) \subset G_A(\mathbb{C}^n)$, for each Schubert symbol $\sigma \in S(A,n)$ (see Appendix 1). $M(m,m')$ denotes the space of $m \times m'$ complex matrices, and $M(m) = M(m,m)$.

To justify the construction furnished by equations (2.10) it is enough to note that

$$\rho((U_0\Delta^{\sigma+})^+)|\psi_0\rangle = |Z,\sigma\rangle \quad , \tag{2.11}$$

with

$$U_0 = U_1(Z)U_2(Y)U_2^+(-Z) \quad ,$$

$$U_1(Z) = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \quad ,$$

$$U_2(Y) = \begin{pmatrix} \mathbb{1}_A & 0 \\ Y & \mathbb{1}_{n-A} \end{pmatrix} \quad , \tag{2.12}$$

$$Y = (\mathbb{1}_{n-A} + ZZ^+)^{-1}Z \quad ,$$

$$P = (\mathbb{1}_A + Z^+Z)^{-1/2} \quad ,$$

$$Q = (\mathbb{1}_{n-A} + ZZ^+)^{1/2} \quad .$$

Here $U_0 = \Delta^{\sigma}U \in h_{\sigma}^{-1}(Z)$, $\sigma \in S(A,n)$, and U_0 is the unitary matrix (A1.9)

$$U_0 = \begin{pmatrix} (\mathbb{1}_A + Z^+Z)^{-1/2} & -Z^+(\mathbb{1}_{n-A} + ZZ^+)^{-1/2} \\ Z(\mathbb{1}_A + Z^+Z)^{-1/2} & (\mathbb{1}_{n-A} + ZZ^+)^{-1/2} \end{pmatrix} \tag{2.13}$$

A useful relation is the overlap function (see Appendix 2)

$$\langle Z'|Z\rangle = \det(\mathbb{1}_{n-A} + Z'^+Z) \quad . \tag{2.14}$$

3. EQUATIONS OF MOTION ON COMPLEX GRASSMANN MANIFOLD FOR LINEAR HAMILTONIANS IN BIFERMION OPERATORS

We consider energy functions

$$f_H^{\sigma}(Z) = \langle Z|Z\rangle^{-1} \langle Z|\rho(\Delta^{\sigma})^+H\,\rho(\mathcal{B})|Z\rangle \qquad \sigma \in S(A,n) \tag{3.1}$$

associated (Berezin 1978) to Hartree-Fock Hamiltonians, i.e. self-adjoint polynomials in bifermion operators.

In this section we study the linear Hartree-Fock Hamiltonians

$$H = \sum_{\alpha,\beta=1}^{n} \varepsilon_{\alpha\beta}C_{\alpha\beta} \tag{3.2}$$

where the matrix ε is hermitian

$$\varepsilon_{\alpha\beta} = \overline{\varepsilon}_{\beta\alpha}, \qquad \alpha,\beta = 1,\dots,n \quad . \tag{3.3}$$

For simplicity of notation, we shall, first, do the calculation in a given chart, for example, the one corresponding to the identity $\sigma \in S(A,n)$.

The equations of motion (2.2) can be written as

$$i\mathcal{N}\dot{Z} = -\frac{\partial H}{\partial \overline{Z}} \quad , \tag{3.4}$$

where

$$\mathcal{N} = (\mathcal{N}_{mi,\,\overline{m}'\overline{i}'}) = \frac{\partial^2 \ln\langle Z|Z\rangle}{\partial Z_{mi}\partial \overline{Z}_{\overline{m}'\overline{i}'}} \quad , \qquad 1 \le i \le A < m,m' \le n. \tag{3.5}$$

It is easy to prove that

$$\mathcal{N}_{mi,\overline{m}'\overline{i}'} = (\mathbb{1}_{n-A} + ZZ^+)^{-1}_{\overline{m}'m} (\mathbb{1}_A + Z^+Z)^{-1}_{i\overline{i}'}, \tag{3.6}$$

Indeed, let the notation

$$X = \mathbb{1}_A + Z^+Z \quad . \tag{3.7}$$

Then, we have

$$\frac{\partial \ln \det X}{\partial \bar{Z}_{\bar{m}'\bar{i}'}} = (\det X)^{-1} \frac{\partial \det X}{\partial \bar{Z}_{\bar{m}'\bar{i}'}} = \mathrm{Tr}(X^{-1} \frac{\partial X}{\partial \bar{Z}_{\bar{m}'\bar{i}'}}) \qquad (3.8)$$

Equation (3.6) and (3.8) imply (summation convention assumed)

$$\frac{\partial \ln \det X}{\partial \bar{Z}_{\bar{m}'\bar{i}'}} = Z_{\bar{m}'\alpha} X^{-1}_{\alpha\bar{i}'} ,$$

$$N_{mi,\bar{m}'\bar{i}'} = \frac{\partial}{\partial Z_{mi}} (Z_{\bar{m}'\alpha} X^{-1}_{\alpha\bar{i}'}) \quad . \qquad (3.9)$$

From equations (3.7) - (3.9) we get

$$N_{mi,\bar{m}'\bar{i}'} = (\mathbb{1}_{n-A} - ZX^{-1}Z^{+})_{\bar{m}'m} X^{-1}_{i\bar{i}'} \qquad (3.10)$$

which can be put in the form (3.6).

Denoting the matrix $C = (C_{pq})$, $p,q = 1,\ldots,n$ we calculate the mean values

$$\langle Z|C|Z\rangle = \begin{pmatrix} A & C \\ B & D \end{pmatrix} , \qquad (3.11)$$

where $A \in M(A)$, $C \in M(A,n-A)$, $B \in M(n-A,A)$ and $D \in M(n-A)$. The values of the matrices appearing in equations (3.11) are (see Appendix 2)

$$\langle \underline{Z}|C_{\alpha\beta}|\underline{Z}\rangle = A_{\beta\alpha} = (\mathbb{1}_A + Z^{+}Z)^{-1}_{\beta\alpha} , \qquad (3.12.1)$$

$$\langle \underline{Z}|C_{hk}|\underline{Z}\rangle = D_{kh} = (Z(\mathbb{1}_A + Z^{+}Z)^{-1}Z^{+})_{kh} , \qquad (3.12.2)$$

$$\langle \underline{Z}|C_{\alpha h}|\underline{Z}\rangle = B_{h\alpha} = -(ZA)_{h\alpha} = (EZ)_{h\alpha} , \qquad (3.12.3)$$

$$\langle \underline{Z}|C_{h\alpha}|\underline{Z}\rangle = C_{\alpha h} = -(AZ^{+})_{\alpha h} = (Z^{+}E)_{\alpha h} , \qquad (3.12.4)$$

$$E = D - \mathbb{1}_{n-A} = -(\mathbb{1}_{n-A} + ZZ^{+})^{-1} = E^{+} , \qquad (3.12.5)$$

$$1 \le \alpha,\beta \le A < h,k \le n \quad .$$

With notations (3.12), equation (3.6) reads:

$$N^{-1}_{\bar{m}\bar{i},nj} = -A^{-1}_{\bar{i}j} E^{-1}_{n\bar{m}} \qquad (3.13)$$

The energy function (3.1) corresponding to the Hamiltonian (3.2) can be expressed in the form (summation convention assumed)

$$H = \varepsilon_{\alpha\beta}A_{\beta\alpha} + \varepsilon_{hk}D_{kh} + \varepsilon_{\alpha h}B_{h\alpha} + \varepsilon_{h\alpha}C_{\alpha h} \quad . \qquad (3.14)$$

$$1 \le \alpha,\beta \le A < h,k \le n \; .$$

In order to write the equations of motion (3.4) corresponding to the energy function (3.14), we use the derivatives of matrices appearing in equations (3.12) (see Berceanu and Gheorghe, 1989)

$$\frac{\partial A_{pq}}{\partial Z_{mi}} = C_{pm}A_{iq} , \quad 1 \le p,q \le A \qquad (3.15.1)$$

$$\frac{\partial B_{pq}}{\partial Z_{mi}} = E_{pm}A_{iq} , \quad 1 < q \le A < p \le n , \qquad (3.15.2)$$

$$\frac{\partial C_{pq}}{\partial Z_{mi}} = C_{pm}C_{iq} , \quad 1 \le p \le A < q \le n , \qquad (3.15.3)$$

$$\frac{\partial D_{pq}}{\partial Z_{mi}} = E_{pm}C_{iq} , \quad A+1 \le p,q \le n , \qquad (3.15.4)$$

$$\frac{\partial E_{pq}}{\partial Z_{mi}} = E_{pm}C_{iq} , \quad A+1 \le p,q \le n \quad . \qquad (3.15.5)$$

Now, taking into account the equations (3.4), (3.13)-(3.15), we get

$$\frac{\partial H}{\partial Z_{pj}} = \varepsilon_{\alpha\beta}(Z^{+}E)_{\beta p}A_{j\alpha} + \varepsilon_{hk}E_{kp}(Z^{+}E)_{jh} +$$

$$\varepsilon_{\alpha h}E_{hp}A_{j\alpha} + \varepsilon_{h\alpha}(Z^{+}E)_{\alpha p}(Z^{+}E)_{jh} ,$$

$$i\,\dot{\bar{Z}}_{\bar{m}\bar{i}} = A^{-1}_{\bar{i}j} E^{-1}_{p\bar{m}}[\varepsilon_{\alpha\beta}(Z^{+}E)_{\beta p}A_{j\alpha} + \varepsilon_{hk}E_{kp}(Z^{+}E)_{jh} +$$

$$\varepsilon_{\alpha h}E_{hp}A_{jh} + \varepsilon_{\alpha h}(Z^{+}E)_{\alpha p}(Z^{+}E)_{jh}] ,$$

$$\dot{Z}_{\bar{m}\bar{i}} = i(Z_{\bar{m}\beta}\varepsilon_{\beta\bar{i}} - \varepsilon_{\bar{m}h}Z_{h\bar{i}} + \varepsilon_{\bar{m}\bar{i}} - Z_{\bar{m}\alpha}\varepsilon_{\alpha h}Z_{h\bar{i}}) \quad - \qquad (3.16)$$

It is useful to decompose the matrix ε in the block form

$$\varepsilon = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 \\ \varepsilon_3 & \varepsilon_4 \end{pmatrix} \tag{3.17}$$

where ε_1, ε_2, ε_3 and ε_4 are A x A, A x (n-A), (n-A) x A and (n-A) x (n-A) matrices, respectively.

With notation (3.17), equations (3.16) can be written in the matrix form

$$\dot{Z} = i(Z\varepsilon_1 - \varepsilon_4 Z + \varepsilon_3 - Z\varepsilon_2 Z) \quad . \tag{3.18}$$

This is a first order system of differential equations of second degree, known as Matrix Ricatti equation (Reid 1972). These Matrix Ricatti equations naturally arise in the theory of optimal control (see Hermann 1973). The relationship between the Matrix Ricatti equations and the Grassmann manifolds is intensively studied (see, for example Hermann 1973, 1974, Shayman 1986).

In the next section we will give an explicit solution of equation (3.18). However, we consider instructive to illustrate equations (3.18) in some particular cases.

1. Taking A = 1, n = 2, we are in the case of $\mathbb{C}\mathbb{P}_1$. The operator (2.10.5) becomes

$$U(Z) = \exp(-Z\ C_{21}) \quad , \tag{3.19}$$

where $Z = Z_{21}$. The energy function in a given chart is

$$\langle Z,\sigma|H|Z,\sigma\rangle = \varepsilon_{11}\langle \underline{Z}|C_{11}|\underline{Z}\rangle + \varepsilon_{12}\langle \underline{Z}|C_{\sigma(1)\sigma(2)}|\underline{Z}\rangle + \varepsilon_{21}\langle \underline{Z}|C_{\sigma(2)\sigma(1)}|\underline{Z}\rangle + \varepsilon_{22}\langle \underline{Z}|C_{\sigma(2)\sigma(2)}|\underline{Z}\rangle \quad . \tag{3.20}$$

There are two charts. The indices corresponding to the operators C and the elements of matrix ε are given in the next table.

Table I

Indices of C	11	12	21	22
Indices of ε, first chart	11	12	21	22
Indices of ε, second chart	22	21	12	11

The hermiticity condition (3.3) implies $\varepsilon_{11}, \varepsilon_{22} \in \mathbb{R}$, $\varepsilon_{12} = \overline{\varepsilon}_{21}$.

The equation of motion (3.4) in the case at $\mathbb{C}\mathbb{P}_1$ is

$$N\dot{Z} = -i\,\frac{\partial H}{\partial \overline{Z}} \quad , \tag{3.21}$$

where

$$N = \frac{\partial^2}{\partial Z \partial \overline{Z}} \ln(1 + |Z|^2) = (1 + |Z|^2)^{-2} \quad . \tag{3.22}$$

The equation of motion (3.21) correspondi.. to the Hamiltonian (3.20) is

$$\dot{Z} = i(-\varepsilon_{12}Z^2 + (\varepsilon_{11} - \varepsilon_{22})Z + \varepsilon_{21}) \tag{3.23}$$

in the first chart and

$$\dot{Z} = i(-\varepsilon_{21}Z^2 + (\varepsilon_{22} - \varepsilon_{11})Z + \varepsilon_{12}) \tag{3.24}$$

in the second chart.

2. Taking A = 2, n = 3, we are in the case of the $\mathbb{C}\mathbb{P}_2$ manifold. The expression of the operator (2.10.5) is

$$U(Z) = \exp(-Z_{31}C_{31} - Z_{32}C_{32}) \quad . \tag{3.25}$$

We introduce the notation

$$Z_{31} = Z_1 \quad , \qquad Z_{32} = Z_2 \quad . \tag{3.26}$$

The Schubert symbols correspond to the situation

$\sigma: \{1,2,3\} \to \{1,2,3\}$ where $\sigma^{-1}\{1,2\}$ and $\sigma^{-1}\{3\}$ are increasing functions. The correspondence between the indices of C and ε is given in the next table.

Table II

Indices of C	11	12	13	21	22	23	31	32
Indices of ε, first chart	11	12	13	21	22	23	31	23
Indices of ε, second chart	11	13	12	31	33	32	21	23
Indices of ε, third chart	22	23	21	32	33	31	22	13

In the first chart, the energy function has the expression

$$H = \varepsilon_{11}A_{11} + \varepsilon_{12}A_{21} + \varepsilon_{13}B_{31} + \varepsilon_{21}A_{12} + \varepsilon_{22}A_{22} + \varepsilon_{23}B_{32} + \varepsilon_{31}C_{13} + \varepsilon_{32}C_{23} + \varepsilon_{33}D_{33} \quad ,$$

and the equations (3.18) become

$$\begin{aligned} \dot{Z}_1 &= i\,[(\varepsilon_{11} - \varepsilon_{33})Z_1 + \varepsilon_{21}Z_2 + \varepsilon_{31} - \varepsilon_{13}Z_1^2 - \varepsilon_{23}Z_1Z_2] \quad , \\ \dot{Z}_2 &= i\,[\varepsilon_{12}Z_1 + (\varepsilon_{22} - \varepsilon_{33})Z_2 + \varepsilon_{32} - \varepsilon_{13}Z_1Z_2 - \varepsilon_{23}Z_2^2] \quad . \end{aligned} \tag{3.27}$$

3. In the case A = N, n = N+1, the Grassmann manifold becomes $\mathbb{C}\mathbb{P}_N$. With the notation

$$(Z_1,\dots,Z_N) = (Z_{n1},\dots,Z_{nN}) \quad , \tag{3.28}$$

equations (3.18) read

$$\dot{Z}_k = i[\varepsilon_{nk} + \sum_{\beta=1}^{N}(\varepsilon_{\beta k} - \delta_{k\beta}\varepsilon_{nn})Z_\beta - \sum_{\beta=1}^{N}\varepsilon_{\beta n}Z_\beta Z_k] \quad , \quad k=1,\dots,N. \tag{3.29}$$

This system of coupled Ricatti equations appears in the paper of Barut 1983. A particular example is the symmetric Gauss-Lotke-Voltera system. In reference Barut 1983 it is argued

that every nonlinear group action on the homogeneous space corresponds a nonlinear integrable dynamical system.

4. INTEGRATION OF EQUATIONS OF MOTION ON COMPLEX GRASSMANN MANIFOLD FOR LINEAR HAMILTONIANS IN BIFERMION OPERATORS

In this section we present the solution of the first order system of differential equations (3.18)

$$\dot{Z} = i(Z\varepsilon_1 - \varepsilon_4 Z + \varepsilon_3 - Z\varepsilon_2 Z) \quad . \tag{4.1}$$

This is the expression in local coordinates of the system of differential equations on the Grassmann manifold. This differential equation appears usually in applications. However, to find the solution of this equation, it is necessary to consider the equation on the Grassmann manifold $G_A(\mathbb{C}^n)$, taking into account the embedding $g_\sigma^{-1}: \mathbb{C}^{(n-A)xA} \to G_A(\mathbb{C}^n)$, the Grassmann manifold being a "compactification" of $\mathbb{C}^{(n-A)xA}$ (see Appendix 1). So, the Matrix Ricatti equation with $Z \in \mathbb{C}^{(n-A)xA}$ has a natural interpretation as a differential equation on $G_A(\mathbb{C}^n)$ (Schneider 1973, Hermann and Martin 1982,1983, Shayman 1986).

The Grassmann manifold being compact, the Matrix Ricatti equation is complete on $G_A(\mathbb{C}^n)$.

The situation is summarized in the following Proposition.

Proposition. Let us suppose that the characteristic equation

$$\mathrm{Det}(\varepsilon' - \lambda \mathbb{1}_n) = 0 \tag{4.2}$$

has distinct eigenvalues $\lambda_1,\dots,\lambda_n$, where

$$\varepsilon' = \begin{pmatrix} -\varepsilon_1 & \varepsilon_2 \\ \varepsilon_3 & -\varepsilon_4 \end{pmatrix} \quad , \tag{4.3}$$

and let

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} . \tag{4.4}$$

and $\Lambda_1(\Lambda_2)$ is a diagonal matrix with entries λ_i, $i = 1,\dots,A$ (respectively, $i=A+1,\dots,n$). Let also U denote the unitary solution of the equation

$$U\varepsilon' = \Lambda U \tag{4.5}$$

and let us consider a partition of U in the block form

$$U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} , \tag{4.6}$$

where U_1, U_2, U_3 and U_4 are A x A, A x (n-A), (n-A) x A and (n-A) x (n-A) matrices, respectively.

Then the solution of the Matrix Ricatti equation with initial condition $Z(0) = Z_o$, in the given chart, is

$$Z(t,Z_o) = (U_4 - Z'U_2)^{-1}(Z'U_1 - U_3) , \tag{4.7}$$

$$Z' = Z'(t,Z_o') = e^{it\Lambda_2} Z_o' e^{-it\Lambda_1} , \tag{4.8}$$

$$Z_o' = (U_3 + U_4 Z_o)(U_1 + U_2 Z_o)^{-1} , \tag{4.9}$$

whenever the inverse matrices exist.

The Matrix Ricatti equation is a flow on $G_A(\mathbb{C}^n)$.

Proof. To the Matrix Ricatti equation (4.1) it is associated the linear system of first order differential equations (Levin 1959)

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = i \begin{pmatrix} -\varepsilon_1 & \varepsilon_2 \\ \varepsilon_3 & -\varepsilon_4 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} , \tag{4.10}$$

where $X \in M(A)$ and $Y \in M(n-A,A)$. A solution to (4.10) projects to a solution of (4.1) via the map $\phi(X,Y) = YX^{-1}$, whenever $\det X \neq 0$. Note that the map ϕ is of the type indicated in equation (A1.6).

The autonomous linear system (4.10) has the standard solution (Hartman 1964)

$$\begin{pmatrix} X \\ Y \end{pmatrix} = e^{it\varepsilon'} \begin{pmatrix} X_o \\ Y_o \end{pmatrix} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \begin{pmatrix} X_o \\ Y_o \end{pmatrix} . \tag{4.11}$$

To evaluate the exponential in (4.11), we diagonalize the matrix ε'. The matrix ε being hermitian (equation (3.3)), then the matrix ε' (equation (4.3)) is also hermitian, so ε' has real eigenvalues, which for simplicity of writing are supposed to be distinct.

Let U be the unitary matrix of equation (4.5) which diagonalize the matrix ε'. Now we make a change of variables

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} . \tag{4.12}$$

In the new variables, the system (4.10) reads

$$\dot{X}' = i\Lambda_1 X' , \quad \dot{Y}' = i\Lambda_2 Y' , \tag{4.13}$$

with the solution

$$X' = e^{it\Lambda_1} X_o' , \quad Y' = e^{it\Lambda_2} Y_o' , \tag{4.14}$$

and equations (4.7) - (4.10) of the Proposition are proved, provided that $Z_o = Y_o X_o^{-1}$.

Now we prove the last assertion of the Proposition.

We consider the map (see, Appendix 1) $g_\sigma^{-1}: \mathbb{C}^{(n-A)xA} \to G_A(\mathbb{C}^n)$, $g_\sigma^{-1}(Z) = Sp\begin{pmatrix} 1\!\!1_A \\ Z \end{pmatrix}$. We denote by $G_A^\sigma(\mathbb{C}^n)$ those A-dimensional linear subspaces which are complementary to the (n-A)-dimensional subspace $Sp\begin{pmatrix} 0 \\ 1\!\!1_{n-A} \end{pmatrix}$. It is clear that $g_\sigma(Z)$ is an open, dense subset of $G_A(\mathbb{C}^n)$ · $(G_A^\sigma(\mathbb{C}^n), g_\sigma)$ is one of the C_A^n standard charts of the manifold $G_A(\mathbb{C}^n)$ (Shayman 1986) and $G_A(\mathbb{C}^n)$ can be considered as a compactification of $\mathbb{C}^{(n-A)xA}$ (Hermann and

Martin 1982, 1983, Schneider 1973, Shayman 1986).

The flow $F(t,F_o) = e^{it\varepsilon'}F_o$ generated on $G_A(\mathbb{C}^n)$ by equation (4.11) is considered. It follows that

$$g_\sigma^{-1}(Z(t,Z_o)) = F(t,g_\sigma^{-1}(Z_o)) \quad , \tag{4.15}$$

whenever $Z(t,Z_o)$ exists.

Indeed, we have

$$F(t,g_\sigma^{-1}(Z_o)) = e^{it\varepsilon'}g_\sigma^{-1}(Z_o) = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} Sp\begin{pmatrix} \mathbb{1}_A \\ Z_o \end{pmatrix} =$$

$$= Sp\begin{pmatrix} B_1 + B_2Z_o \\ B_3 + B_4Z_o \end{pmatrix} = Sp\begin{pmatrix} \mathbb{1}_A \\ (B_3+B_4Z_o)(B_1+B_2Z_o)^{-1} \end{pmatrix}. \tag{4.16}$$

Equation (4.16) follows from the solution (4.11) of the system (4.10), projecting the solution (X,Y) of the system of linear differential equations to the solution of the Matrix Ricatti equation (4.1)

$$Z(t,Z_o) = (B_3 + B_4Z_o)(B_1 + B_2Z_o)^{-1} \quad . \tag{4.17}$$

In fact, equation (4.15) asserts that the Matrix Ricatti equation is the local expression with respect to the chart $(G_A^\sigma(\mathbb{C}^n),g_\sigma)$ for the differential equation on $G_A(\mathbb{C}^n)$ which corresponds to the flow $F(t,Z_o)$.

Note that $Z(t,Z_o)$ ceases to exist just when $F(t,g_\sigma^{-1}(Z_o))$ leaves the subset $G_A^\sigma(\mathbb{C}^n)$.

Observations. 1. In the variables Z', the solution $Z'(t,Z_o)$ describes a curve on the torus $T^n = S^1 x \dots S^1$.

2. If all the eigenvalues λ_i of the matrix ε' are rationally commensurable, then the solution $Z(t,Z_o)$ is periodic with period equal to the least common multiple of $2\pi|\lambda_m - \lambda_i|^{-1}$, $i = 1,\dots,A$, $m = A+1,\dots,n$. Indeed, equation (4.8) is the solution of the differential equation

$$\frac{dZ'}{dt} = i(\Lambda_2 Z' - Z'\Lambda_1) \quad , \tag{4.18}$$

or, equivalently, for components

$$\dot{Z}'_{mi} = i(\lambda_m - \lambda_i)Z'_{mi}, \; i=1,\dots,A, \quad m= A+1,\dots,n \quad . \tag{4.19}$$

3. If all the eigenvalue λ_i of matrix ε' are not rationally commensurable, the solution is a dense set on the torus. Otherwise, the motion is almost periodic.

4. If the matrix ε' has not distinct eigenvalues, the solution to the Matrix Ricatti equation is obtained also by projecting the solution (4.11) of the linear system (4.10) onto the Grassmann manifold. Then, to calculate the exponential in equation (4.11), in the nondelmutation case the matrix ε' must be put into Jordan form, using the standard method (Hartman 1964). In the expression of the matrix elements of B_i, $i = 1,\dots,4$, in equation (4.11), the well-known quasi-polynomials of degrees less than the degrees of the Jordan cells appear.

5. In particular, if we explicitate the solution or Proposition in the case of equation (3.24) on $\mathbb{CP}_1$, we find the result

$$Z(t,Z_o) = \frac{Z_o[e^{i\alpha t}(\varepsilon+\alpha) + \alpha - \varepsilon] + 2\bar{\eta}(e^{i\alpha t} - 1)}{2\eta \cdot Z_o(e^{i\alpha t} - 1) + (\alpha-\varepsilon)e^{i\alpha t} + \alpha+\varepsilon} \quad , \tag{4.20}$$

where $\varepsilon = \varepsilon_{11} - \varepsilon_{22}$, $\varepsilon_{12} = \eta$, $\alpha = (\varepsilon^2 + 4|\eta|^2)^{1/2}$. Of course, this solution can be find by straightforward integration of equation (3.24). In the second chart, the solution of equation (3.24) can be obtained from the solution (4.20) by the substitution $Z \to 1/Z$, $Z_o \to 1/Z_o$. So, the solutions from different charts "glue" each other. In the case of the Grassmann manifold, the same conclusion can be seen from equation (4.19).

5. EQUATIONS OF MOTION ON COMPLEX GRASSMANN MANIFOLDS FOR QUADRATIC HAMILTONIANS IN BIFERMION OPERATORS

In this section we give some information on equations of motion (3.4) for the case of Hamiltoanians which are quadratic in bifermion operators

$$H_2 = \sum_{p,q,r,s=1}^{n} \eta_{pqrs} C_{pq} C_{rs} \quad . \tag{5.1}$$

The hermiticity condition of the Hamiltonian (5.1) imply the relations

$$\eta_{pqrs} = \overline{\eta}_{srqp} \;, \qquad p,q,r,s = 1,\ldots,n. \tag{5.2}$$

If we take into account the 4-block decomposition of the matrix of mean values of bifermions operators C from equation (3.11), then we have 16 types of terms in the Hamiltonian (5.2). For illustration we restrict ourselves to Hamiltonians which lead to the energy function

$$H_2^{11} = \sum_{i,j,k,l=1}^{A} \eta_{ijkl} \langle \underline{Z} | C_{ij} C_{kl} | \underline{Z} \rangle \quad . \tag{5.3}$$

However, the technique of calculation presented here works in all other situations, too. In fact, the calculation is based on a formula which enables us to express the mean value of a product of two bifermion operators in terms of mean values of bifermion operators. The formula reads

$$\langle \underline{Z} | C_{ij} C_{kl} | \underline{Z} \rangle = \langle \underline{Z} | C_{ij} | \underline{Z} \rangle \langle \underline{Z} | C_{kl} | \underline{Z} \rangle - $$
$$- \langle \underline{Z} | C_{il} | \underline{Z} \rangle \langle \underline{Z} | C_{kj} | \underline{Z} \rangle + \langle \underline{Z} | C_{il} | \underline{Z} \rangle \delta_{jk} \quad . \tag{5.4}$$

A proof of the expression (5.4) is given in Appendix 3 for the case of indices $1 \le i,j,k,l \le A$. Using the same technique the formula can be proved in all situations $1 \le i,j,k,l \le n$.

With formula (5.4) the mean values appearing in equation (5.3) can be calculated

$$H_2^{11} = \sum_{\alpha,\beta,p,q}^{A} \eta_{\alpha\beta pq} (A_{\beta\alpha} A_{qp} - A_{q\alpha} A_{\beta p} + A_{q\alpha} \delta_{\beta p}) \tag{5.5}$$

where the elements of the matrix A are given by equation (3.12.1). With the usual summation convention, equations (3.15) imply

$$\frac{\partial H_2^{11}}{\partial Z_{rj}} = \eta_{\alpha\beta pq} (C_{\beta r} A_{j\alpha} A_{qp} + A_{\beta\alpha} C_{qr} A_{jp} - C_{qr} A_{j\alpha} A_{\beta p} -$$
$$- A_{q\alpha} C_{\beta r} A_{jp} + C_{qr} A_{j\alpha} \delta_{\beta p}) \tag{5.6}$$

Introducing equation (5.6) into equation (3.4), using equations (3.12.4) and (3.14), we have

$$i\dot{Z}_{\overline{m}\overline{i}} = \eta_{\alpha\beta pq} A^{-1}_{\overline{i}j} E^{-1}_{r\overline{m}} [(Z^+E)_{\beta r} A_{j\alpha} A_{qp} + A_{\beta\alpha} (Z^+E)_{qr} A_{jp}$$
$$- (Z^+E)_{qr} A_{j\alpha} A_{\beta p} - A_{q\alpha} (Z^+E)_{\beta r} A_{jp} + (Z^+E)_{qr} A_{j\alpha} \delta_{\beta p}] \;,$$
$$i\dot{Z}_{\overline{m}\overline{i}} = \eta_{\alpha\beta pq} \{ \delta_{\overline{i}\alpha} (\overline{Z}_{\overline{m}\beta} A_{qp} - \overline{Z}_{\overline{m}q} A_{\beta p}) +$$
$$+ \delta_{\overline{i}p} (\overline{Z}_{\overline{m}q} A_{\beta\alpha} - \overline{Z}_{\overline{m}\beta} A_{q\alpha}) + \overline{Z}_{\overline{m}q} \delta_{\overline{i}\alpha} \delta_{\beta p} \} \quad . \tag{5.7}$$

In the case of the $\mathbb{C}\mathbb{P}_N$ manifold ($n=N+1, A=N$), the matrix A can be effectively computed

$$A_{pq} = \begin{cases} (1+|Z|^2)^{-1}(1+|Z|^2 - |Z_p|^2) \;, & \text{if } \; p = q, \\ -(1+|Z|^2)^{-1} \overline{Z}_p Z_q \;, & \text{if } \; p \neq q \;, \end{cases} \tag{5.8}$$

where

$$|Z|^2 = |Z_1|^2 + \ldots + |Z_N|^2 \;, \tag{5.9}$$

with the abbreviating notations (3.28).

For example, in the case N=2, we have the system of equations

$$\begin{aligned} \dot{Z}_1 &= i(\alpha Z_1 + \beta Z_2) \;, \\ \dot{Z}_2 &= i(\gamma Z_1 + \delta Z_2) \;, \end{aligned} \tag{5.10}$$

where

$$\alpha = \eta_{1111} + \eta_{2211} + \eta_{1122} - \eta_{2112} = \bar{\alpha} ,$$

$$\beta = \eta_{2221} + \eta_{2111} , \qquad (5.11)$$

$$\gamma = \eta_{1112} + \eta_{1222} = \bar{\beta} ,$$

$$\delta = \eta_{2222} + \eta_{1122} + \eta_{2211} - \eta_{1221} = \bar{\delta} .$$

APPENDIX 1

In this Appendix we present a convenient parametrization of manifold $G_A(\mathbb{C}^n)$ of complex A-dimensional linear subspaces of the vectorial space $\mathbb{C}^n$. In fact, we give a proof of the following proposition.

Proposition. The space $G_A(\mathbb{C}^n)$ admits a C^∞ manifold structure of complex A(n-A) dimension, with respect to which is diffeomorphic with the homogeneous space $U(n)/U(A) \times U(n-A)$.

For selfconsistency, we present a sketch of the proof adapted to our parametrization of the Grassmann manifold (Berceanu 1984, Berceanu and Gheorghe 1989).

We use the notation

$$Q_{An} = U(n)/U(A) \times U(n-A). \qquad (A1.1)$$

Here

$$U(A) \times U(n-A) = \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \middle| U_1 \in U(A), \quad U_2 \in U(n-A) \right\} .$$

A transitive continuous action $U(n) \times Q_{An} \to Q_{An}$ can be defined as $(U',\hat{U}) \to U'\hat{U} = \widehat{U'U}$, where $U' \in U(n)$, while $\hat{U}$ is the coset class of matrices $U \in U(n)$

$$\hat{U} = \{UU'' | U'' \in U(A) \times U(n-A)\} . \qquad (A1.2)$$

Let the notation

$$\Delta^\sigma = \{(\delta_{\sigma(i)j})_{1\le i,j\le n} \in U(n), \sigma \in S(A,n)\} . \qquad (A1.3)$$

An open covering $\{V_\sigma\}_{\sigma\in S(A,n)}$ of the space Q_{An} is also also introduced

$$V_\sigma = \{\hat{U} | \hat{U} \in Q_{An} , U \in U(n), \det(\Delta^\sigma U)_{1\le\alpha,\beta\le A} \ne 0\} . \qquad (A1.4)$$

The homeomorphisms $h_\sigma : V_\sigma \to M(n-A,A)$, $\sigma \in S(A,n)$ are defined

$$h_\sigma(\hat{U}) = Z,\ \hat{U} \in V_\sigma\ ,\quad Z \in M(n-A,A),\quad \sigma \in S(A,n)\quad , \tag{A1.5}$$

where

$$\Delta^\sigma U = \begin{pmatrix} T & W \\ W' & T' \end{pmatrix} \in U(n)\quad ,\qquad Z = W'T^{-1}\quad . \tag{A1.6}$$

Here $T \in T(A)$, $T' \in T(n-A)$, $W \in M(A,n-A)$, $W' \in M(n-A,A)$. $T(m)$ is the space of complex lower triangular matrices whose diagonal elements are strictly positive. The correctness of the definition follows from the following theorems of matrix decomposition (see Grantmacher 1964, Theorem 1 p.50, Theorem 2 p.53, Corollary 2 p.52, Theorem 19 p.300):

i) If $H \in M(m)$ is a Hermitian, positive definite matrix, than there exists a unique matrix $T \in T(m)$ such that $H = TT^+$ and the elements of matrix T are differentiable functions of the elements of matrix H.

ii) If $A \in GL(m, \mathbb{C})$, then there exists the unique matrices $U \in U(m)$ and $T \in T(m)$ such that $A = TU$, and the elements of matrices A and T are differentiable functions of the elements of matrix A.

It is easy to prove that the mapping h_σ is correctly defined and injective. Indeed, let $U' \in \hat{U} \subset V_\sigma$, $\sigma \in S(A,n)$. From property ii) it follows that there is a unique matrix $U'' \in U(A) \times U(n-A)$ such that $U' = UU''$, where U is given by (A1.6) The unitarity of matrix (A1.6) implies the relations

$$T^+T + W'^+W' = \mathbb{1}_A\quad , \tag{A1.7.1}$$

$$W^+W + T'^+T' = \mathbb{1}_{n-A}\quad , \tag{A1.7.2}$$

$$T^+W + W'^+T' = 0\quad , \tag{A1.7.3}$$

where $\mathbb{1}_n$ denotes the unit matrix of $GL(n, \mathbb{C})$. Using property i), we get from relations (A1.7) the matrices T, T' and W' which are differentiable functions of the matrix W and do not depend on the representative $U' \in \hat{U}$.

Further, it can be proved that the mapping (A1.6) is surjective. In fact, let the decomposition $T = T_1T_2$, where T_1 is hermitian and T_2 unitary (see Grantmacher 1964, p.249). From the relation

$$W' = ZT = ZT_1T_2\quad , \tag{A1.8}$$

using equation (A1.7.1), we get

$$T_1 = (\mathbb{1}_A + Z^+Z)^{-1/2}\quad .$$

So, given the matrix Z, we can find the unique matrices T_1 and T_2, such that equation (A1.8) is verified, and

$$W' = Z(\mathbb{1}_A + Z^+Z)^{-1/2}\, T_2\quad .$$

Denoting

$$V = \begin{pmatrix} (\mathbb{1}_A + Z^+Z)^{-1/2} & M \\ Z(\mathbb{1}_A + Z^+Z)^{-1/2} & M' \end{pmatrix} = \begin{pmatrix} T & W \\ W' & T' \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B' \end{pmatrix}\quad ,$$

the condition $V^+V = \mathbb{1}_n$ and M' hermitian, imply

$$M' = M'^+ = (\mathbb{1}_{n-A} + ZZ^+)^{-1/2}\quad ,\quad M = -Z^+(\mathbb{1}_A + Z^+Z)^{-1/2}\quad .$$

Now the matrix

$$\Delta^\sigma U = \begin{pmatrix} (\mathbb{1}_A + Z^+Z)^{-1/2} & -Z^+(\mathbb{1}_{n-A} + ZZ^+)^{-1/2} \\ Z(\mathbb{1}_A + Z^+Z)^{-1/2} & (\mathbb{1}_{n-A} + ZZ^+)^{-1/2} \end{pmatrix}\quad , \tag{A1.9}$$

$$\Delta^\sigma U = \begin{pmatrix} T & W \\ W' & T' \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B' \end{pmatrix}$$

has the property that $h_\sigma(U) = Z = W'T^{-1}$, where $B \in U(A)$, $B' \in U(n-A)$.

To end the proof of the Proposition, it will be enough to establish a bijection μ of the homogeneous space Q_{An} onto the

Grassmann manifold $G_A(\mathbb{C}^n)$. The construction can be done explicitely

$$\mu:\ Q_{An} \to G_A(\mathbb{C}^n)\ ,\qquad \mu(\hat{U}) = Sp(u_1,\dots,u_A)\qquad , \tag{A1.10}$$

where the transpose of u_i is $(u_{1i},\dots,u_{ni}) \in \mathbb{C}^{n^2}$, $i = 1,\dots,A$, and $Sp(u_1,\dots,u_A)$ means the linear subspace generated by the vectors $u_1,\dots,u_A$. It can be verified that μ is a homeomorphism of the homogeneous space Q_{An} onto the Grassmann manifold $G_A(\mathbb{C}^n)$.

It should be noted that the map $g_\sigma = h_\sigma \circ \mu^{-1}$, g_σ: $G_A(\mathbb{C}^n) \to M(n-A,A)$, $\sigma \in S(A,n)$ is a bijection of the complex Grassmann manifold $G_A(\mathbb{C}^n)$ onto $M(n-A,A) \approx \mathbb{C}^{(n-A)xA}$. Also, it follows that g_σ^{-1}: $M(n-A,A) \to G_A(\mathbb{C}^n)$ is given by $g_\sigma^{-1}(Z) = Sp\begin{pmatrix}\mathbb{1}_A\\ Z\end{pmatrix}$, where $Sp\begin{pmatrix}\mathbb{1}_A\\ Z\end{pmatrix}$, means the linear A-dimensional subspace in $\mathbb{C}^n$ generated by the column vectors of the matrix $\begin{pmatrix}\mathbb{1}_A\\ Z\end{pmatrix}$. If $G_A^\sigma(\mathbb{C}^n)$, $\sigma \in S(A,n)$ which consists of those A-dimensional linear subspaces in $G_A(\mathbb{C}^n)$ which are complementary to the (n-A)-dimensional subspace $Sp\begin{pmatrix}0\\ \mathbb{1}_{n-A}\end{pmatrix}$ then g_σ^{-1} embeds the euclidean space $\mathbb{C}^{(n-A)xA}$ into $G_A(\mathbb{C}^n)$ as an open and dense subset $G_A^\sigma(\mathbb{C}^n)$. $(G_A^\sigma(\mathbb{C}^n), g_\sigma)$, $\sigma \in S(A,n)$ is one of the $\mathbb{C}_A^n$ charts of the manifold $G_A(\mathbb{C}^n)$ and $G_A(\mathbb{C}^n)$ can be viewed as a compacfitication of $\mathbb{C}^{(n-A)xA}$ (see Hermann and Martin 1982, 1983 and Shayman 1986).

APPENDIX 2

In this Appendix we present a proof of equations (3.12). First, we need some auxiliary formulae.

Taking into account equations (2.9) and (2.7), the following relations hold

$$\rho\begin{pmatrix}A_1 & 0\\ 0 & A_2\end{pmatrix}|\psi_o\rangle = \mathrm{Det}(A_1)\ |\psi_o\rangle \qquad , \tag{A2.1}$$

$$\rho(U_2(Y))|\psi_o\rangle = \rho\begin{pmatrix}\mathbb{1}_A & 0\\ Y & \mathbb{1}_{n-A}\end{pmatrix}|\psi_o\rangle = e^{\sum_{m=A+1}^{n}\sum_{i=1}^{A} y_{mi}C_{mi}}|\psi_o\rangle\ , \tag{A2.2}$$

$$\rho\begin{pmatrix}\mathbb{1}_A & A_3\\ 0 & \mathbb{1}_{n-A}\end{pmatrix}|\psi_o\rangle = |\psi_o\rangle \qquad , \tag{A2.3}$$

where $A_1 \in M(A)$, $A_2 \in M(n-A)$, $Y \in M(n-A,A)$ and $A_3 \in M(A,n-A)$.

From equations (2.10.4), (2.10.5) and (A2.2) it follows a useful representation

$$|Z\rangle = \rho(U_2(-Z))|\psi_o\rangle = \rho\begin{pmatrix}\mathbb{1}_A & 0\\ -Z & \mathbb{1}_{n-A}\end{pmatrix}|\psi_o\rangle \quad . \tag{A2.4}$$

With the representation (A2.4) and equations (A2.1), (A2.3), equation (2.14) can be obtained observing that

$$\langle Z'|Z\rangle = \langle\psi_o|\rho(U_2^+(-Z')U_2(-Z))|\psi_o\rangle \quad .$$

In the last formula the product of matrices inside the interior brackets is

$$U_2^+(-Z')U_2(-Z) = \begin{pmatrix}\mathbb{1}_A + Z'^+Z & -Z'^+\\ -Z & \mathbb{1}_{n-A}\end{pmatrix} =$$

$$\begin{pmatrix}\mathbb{1}_A & 0\\ T & \mathbb{1}_{n-A}\end{pmatrix}\begin{pmatrix}\mathbb{1}_A + Z'^+Z & 0\\ 0 & Y\end{pmatrix}\begin{pmatrix}\mathbb{1}_A & T'\\ 0 & \mathbb{1}_{n-A}\end{pmatrix} \quad .$$

The values of the matrices T, T' and Y are not relevant for the value of the overlap function.

To prove equations (3.13), we introduce the following generating functions

$$\varphi_1(X,Y) = \langle Z|\exp\Big(\sum_{i,j=1}^{A} x_{ij}C_{ij} + \sum_{h,k=A+1}^{n} y_{hk}C_{hk}\Big)|Z\rangle \quad , \tag{A2.5}$$

$$\varphi_2(U) = \langle Z|\ \exp\Big(\sum_{\alpha=1}^{A}\sum_{h=A+1}^{n} u_{\alpha h}C_{\alpha h}\Big)\ |Z\rangle \quad , \tag{A2.6}$$

$$\varphi_3(V) = \langle Z|\exp\Big(\sum_{\alpha=1}^{A}\sum_{h=A+1}^{n} v_{h\alpha}\ C_{h\alpha}\Big)\ |Z\rangle \quad , \tag{A2.7}$$

where $X \in M(A)$, $Y \in M(n-A)$, $U \in M(A, n-A)$ and $V \in M(n-A,A)$.

One can see that

$$<Z|C_{\alpha\beta}|Z> = \frac{\partial\varphi_1(X,Y)}{\partial x_{\alpha\beta}}\Big|_{X=0,Y=0} , \qquad 1 \le \alpha,\beta \le A , \qquad (A2.8.1)$$

$$<Z|C_{hk}|Z> = \frac{\partial\varphi_1(X,Y)}{\partial y_{hk}}\Big|_{X=0,Y=0} , \qquad A < h,k \le n \qquad (A2.8.2)$$

$$<Z|C_{\alpha h}|Z> = \frac{\partial\varphi_2(U)}{\partial u_{\alpha h}}\Big|_{U=0} , \qquad 1 \le \alpha \le A < h \le n , \qquad (A2.8.3)$$

$$<Z|C_{h\alpha}|Z> = \frac{\partial\varphi_3(V)}{\partial v_{h\alpha}}\Big|_{V=0} , \qquad 1 \le \alpha \le A < h \le n . \qquad (A2.8.4)$$

Now we proceed to calculate the generating functions (A2.5)-(A2.7). Using equations (A2.1)-(A2.3), we have succesively

$$\varphi_1(X,Y) = <Z|\rho\{[\exp(\Sigma x_{ij}E_{ij} + y_{hk}E_{hk})] \begin{pmatrix} \mathbb{1}_A & 0 \\ -Z & \mathbb{1}_{n-A} \end{pmatrix}\}|\psi_o> =$$

$$= <Z|\rho\begin{pmatrix} X' & 0 \\ -Y'Z & Y' \end{pmatrix}|\psi_o> = <Z|\rho[\begin{pmatrix} \mathbb{1}_A & 0 \\ -Y'ZX'^{-1} & \mathbb{1}_{n-A} \end{pmatrix} \begin{pmatrix} X' & 0 \\ 0 & Y' \end{pmatrix}]|\psi_o>$$

$$= \det X' <Z|Y'ZX'^{-1}> = \det(X' + Z^+Y'Z) ,$$

where $X' = e^X$, $Y' = e^Y$.

With formula (2.14) we obtain

$$\varphi_1(X,Y) = \det(e^X + Z^+e^YZ) . \qquad (A2.9)$$

Similarly we have

$$\varphi(U) = <\psi_o|\rho[\begin{pmatrix} \mathbb{1}_A & -Z^+ \\ 0 & \mathbb{1}_{n-A} \end{pmatrix} \begin{pmatrix} \mathbb{1}_A & U \\ 0 & \mathbb{1}_{n-A} \end{pmatrix} \begin{pmatrix} \mathbb{1}_A & 0 \\ -Z & \mathbb{1}_{n-A} \end{pmatrix}]|\psi_o> =$$

$$= <\psi_o|\rho\begin{pmatrix} \mathbb{1}_A - UZ+ZZ^+ & U-Z^+ \\ -Z & \mathbb{1}_{n-A} \end{pmatrix}|\psi_o> =$$

$$= <\psi_o|\rho\begin{pmatrix} \mathbb{1}_A & 0 \\ P & \mathbb{1}_{n-A} \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} \mathbb{1}_A & R \\ 0 & \mathbb{1}_{n-A} \end{pmatrix}|\psi_o> = \mathrm{Det}X ,$$

where $X = \mathbb{1}_A - UZ + Z^+Z$, and the values of the matrices P,Q and

R are not relevant here bacause of the formulae (A2.1)-(A2.3). So, we get

$$\varphi_2(U) = \det(\mathbb{1}_A - UZ + Z^+Z) . \qquad (A2.10)$$

Also, we have

$$\varphi_3(V) = \det(\mathbb{1}_A - Z^+V + Z^+Z) . \qquad (A2.11)$$

APPENDIX 3

In this Appendix we prove the relation (5.4). The starting point is the representation (A2.9) of the generating function (A2.5)

$$\varphi_1(X,Y) = <Z|\exp(\sum_{\alpha,\beta=1}^{A} x_{\alpha\beta}C_{\alpha\beta} + \sum_{h,k=A+1}^{n} y_{hk}C_{hk})|Z> =$$

$$= \det(e^X + Z^+e^YZ) , \qquad (A3.1)$$

where $X \in M(A)$ and $Y \in M(n-A)$.

It can be observed that

$$<Z|C_{\alpha\beta}C_{\alpha'\beta'}|Z> = \frac{\partial^2\varphi_1(X,Y)}{\partial x_{\alpha'\beta'}\partial x_{\alpha\beta}}\Big|_{X=0,Y=0} +$$

$$\frac{1}{2}\delta_{\beta\alpha'} <Z|C_{\alpha\beta'}|Z> - \frac{1}{2}\delta_{\alpha\beta'} <Z|C_{\alpha'\beta}|Z> , 1\le\alpha,\beta,\alpha',\beta'\le A. \quad (A3.2)$$

Indeed, it can be proved that

$$\frac{\partial^2\varphi_1(X,Y)}{\partial x_{\alpha'\beta'}\partial x_{\alpha\beta}}\Big|_{X=0,Y=0} = \frac{1}{2} <Z|C_{\alpha\beta}C_{\alpha'\beta'} + C_{\alpha'\beta'}C_{\alpha\beta}|Z> . \qquad (A3.3)$$

For example, equation (A3.3) can be derived by the derivation formula of the exponential of a matrix G depending on a parameter t

$$\frac{d}{dt} e^{G(t)} = \int_0^1 e^{\tau G(t)} G'(t)e^{-\tau G(t)} \, d\tau \, e^{G(t)} . \qquad (A3.4)$$

Equation (A3.2) follows from equation (A3.3) and the commutation relations (2.6).

The main steps in calculation of derivatives appearing in equation (A3.3) are briefly presented here.

First

$$\frac{\partial\varphi_1(X,Y)}{\partial x_{\alpha\beta}} = \det U\,\mathrm{Tr}\,[U^{-1}\frac{\partial}{\partial x_{\alpha\beta}}(U)] =$$

$$= a \sum_{p,q=1}^{A} M_{pq}N_{pq}\ , \qquad \alpha,\beta = 1,\ldots,A\ , \tag{A3.5}$$

where

$$U = e^X + Z^+e^YZ\ , \tag{A3.6}$$

$$a = \det U\ , \tag{A3.7}$$

$$M = U^{-1} \tag{A3.8}$$

$$N = e^P \int_0^1 e^{-\tau P} E_{\alpha\beta} e^{\tau P}\, d\tau\ , \tag{A3.9}$$

$$P = \sum_{p,q=1}^{A} x_{pq}E_{pq} \tag{A3.10}$$

It can be derived

$$\left.\frac{da}{dx_{\gamma\gamma'}}\right|_{X=0,Y=0} = a(\mathbb{1}_A + Z^+Z)^{-1}_{\gamma'\gamma}\ , \tag{A3.11}$$

$$\left.\frac{dM_{pq}}{dx_{\gamma\gamma'}}\right|_{X=0,Y=0} = -(\mathbb{1}_A + Z^+Z)^{-1}_{p\gamma}(\mathbb{1}_A + Z^+Z)^{-1}_{\gamma'q}\ , \tag{A3.12}$$

$$\left.\frac{dN}{dx_{\gamma\gamma'}}\right|_{X=0,Y=0} = \frac{1}{2}(E_{\alpha\beta}E_{\gamma\gamma'} + E_{\gamma\gamma'}E_{\alpha\beta})\ . \tag{A3.13}$$

Taking into account equations (A3.11) and (A3.13), it follows that

$$(\det a)^{-1}\left.\frac{\partial^2\varphi_1(X,Y)}{\partial x_{\gamma\gamma'}\partial x_{\alpha\beta}}\right|_{X=0,Y=0} = A_{\gamma'\gamma}A_{\beta\alpha} - A_{\beta\gamma}A_{\gamma'\alpha} + \frac{1}{2}(A_{\gamma'\alpha}\delta_{\beta\gamma} + A_{\beta\gamma}\delta_{\gamma'\alpha})\ , \tag{A3.14}$$

where the matrix A has the expression (3.12.1) and $1 \le \alpha,\beta,\gamma,\gamma' \le A$. Formula (5.4) follows if equation (A3.14) is introduced into equation (A3.2)

REFERENCES

Abraham R and Marsden J E 1978 Foundations of Mechanics (Reading: Benjamin/Cummings)

Barut O 1983 "Conference on Differential Geometric Methods in Theoretical Physics" ed. G Denardo and H D Doebner (Singapore: World Scientific) pp 79-86

Berceanu S 1984 "Lectures on Morse Theory" Institute of Physics and Nuclear Engineering preprint FT-23, Bucharest

Berceanu S and Gheorghe A 1987 J.Math.Phys. 28 2889-907

____ 1989 Rev.Roum.Phys. to appear

Berezin F A 1978 Commun.Math.Phys. 63 131-53

Gantmacher F P 1966 Teoria matritz (Moskva: Nauka)

Guillemin V and Sternberg S 1984 "Symplectic Techniques in Physics" (London: Cambridge University Press)

Hartman P 1964 "Ordinary differential equations" (New York: John Wiley)

Hermann R 1973 "Algebraic Topics in Systems Theory", Interdisciplinary Mathematics III (New Jersey, New Brunswick: Privately printed)

____ 1974 "Linear Systems Theory and Introductory Algebraic Geometry", Interdisciplinary Mathematics VIII (New Jersey, New Brunswick: Privately printed)

Hermann R and Martin C 1982 Math.Systems Theory 15 277-84

____ 1983 Math.Systems Theory 16 297-306

Kramer P and Saraceno M 1981 "Geometry of time dependent variational principle in quantum mechanics" Lecture Notes in Physics 140 (Berlin: Springer)

Levin J J 1959 Proc.Amer.Math.Soc. 10, 519-24

Morse M 1934 "The calculus of variations in the large" Amer.Math.Soc. 18 (Providence RI: Princeton Univ.Press)

Perelomov A A 1972 Commun.Math.Phys. 26 222-36

____ 1981 Physica D4 1-25

——— 1983 "Generalized coherent states. General case". Institute of Theoretical and Experimental Physics, preprint ITEP 144 Moscow

Pressley A and Segal G 1986 "Loop groups and their representations (Oxford: University Press)

Reid T W 1972 "Ricatti differential equations" Mathematics in Science and Engineering, V 86 (New York: Academic Press)

Rosensteel G and Ihring E 1981 J.Math.Phys. 21 2297-301

Sato M 1981 RIMS-Kokyuroku, 439 30-46

Schneider C R 1973 Math.Systems Theory 7 281-6

Segal G and Wilson G 1985 Pub.IHES 61 1-65

Shayman M A 1986 SIAM J.Control and optimisation 24 1-65

Smale S 1967 Bull.Amer.Math.Soc. 747-817

Wang H C 1954 Amer.J.Math. 76 1-31

Witten E 1988 Commun.Math.Phys. 113 528-600

Yamamura M and Kuriyama A 1987 Progress Theor.Phys.Suppl. 93 1-175.