VORTICITY FLOWS IN THE NUCLEAR ROTATIONAL-VIBRATIONAL COLLECTIVE MODEL

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1 INTRODUCTION

In this review one exhaustively discuss the interplay between the two main current flows for a nucleus consisting of interacting neutrons and protons, i.e. the irrotational and the rigid rotor flows.

In the first chapter the Villars' canonical transformation is presented both in the classical and the generalized form. The irrotational value for the inertia moment is deduced from the Villars' treatment. Next the Rowe model is presented and one explain the reason which underly the generation of rotational flow and the corresponding rigid-rotor value for the moment of inertia. The last section of the second chapter deals with the generalized Villars' transformation which gives the most general real transformation of the coordinates and velocities of a many-body system of interacting particles. The physical quantity called *vorticity* is introduced in connection with special classes of velocity fields. The kinetic energy in the new coordinates is written and the rotational component is separated from the rest of possible kinetic terms.

Chapter three is dedicated to the algebraic approach of nuclear collective motion. To every rotational model considered in the second chapter there is associated a specific dynamical group. Special attention is payed to the construction of irreducible unitary representations(unirreps) of the $SL(3, \mathbf{R})$ group by the method of induced representations. There is also discussed the group associated to the rigid-rotor $ROT(3)$ and the "Mass-Quadrupole Model"-$CM(3)$.

The last chapter(4) concerns the problem of quantum mechanical systems in rotating frames. The specifical shapes of currents in the cranked anisotropic harmonic oscillator are deduced. The phenomena of vorticity lines in the irrotational velocity field is also discussed.

2 THE VILLARS' CANONICAL TRANSFORMATION

2.1 The classical Villars' transformation

The "canonical form" of the kinetic energy of any system of identical interacting particles appears as a result of a canonical transformation in which the original particle coordinates and momenta are replaced by the following new variables:

1. The center-of-mass $\mathbf{X}$, and total momentum $\vec{P}$.
2. The Euler angles θ_s $(s = 1, 3)$, describing the orientation of the body-fixed frame in space, and their conjugate momenta Π_s, which are linear functions of the total angular momentum components $L_{\hat{A}}$.
3. $3N - 6$ intrinsic variables ξ_σ and their conjugate momenta π_σ.

The first two conditions are justified by the fact that for any system of interacting particles there are two obvious collective constants of motion: the total linear and the total angular momentum. As a result of the canonical transformation one try to write the Hamiltonian(kinetic energy) of the system in such a way as to display its dependence on these two constants of motion. The Euler angles of the body-fixed(intrinsic) frame are the collective variables since a common rotation of particles can be considered as a rotation of the body-fixed system. The three Euler angles $\theta_s = (\psi, \theta, \phi)$ [Goldstein 1959] define an orthogonal transformation $R_{A\alpha}(\theta_s)$ from space-fixed components x'_{iA}, where the indices refer to the Cartesian components:

$$x_{i\alpha} = X_\alpha + R_{A\alpha}(\theta_s) x'_{iA}(\xi_\sigma) \tag{1}$$

$$x'_{iA} = R_{A\alpha}(\theta_s)(x_{i\alpha} - X_\alpha). \tag{2}$$

Here α stands for the letters α, β, γ, the space-fixed Cartesian axes, and A for A, B, C, the body-fixed Cartesian axes(see Fig.1). The dummy indices convention is used throughout [1]. In equation (1) the center-of-mass X_α is also isolated. This is equivalent to the statement that the body-fixed system is free of the center-of-mass motion.

Using the definition of the center-of-mass in space-fixed coordinates

$$X_\alpha = \frac{1}{N} \sum_i^N x_{i\alpha} \tag{3}$$

where N is the number of particles in the system, one derive a first consequence due to the particular choice of the body-fixed frame by summing eq.(2) over i

$$\sum_i x'_{iA(BC)} = R_{A(BC)\alpha}\left(\sum_i x_{i\alpha} - X_\alpha N\right) = 0 \tag{4}$$

[1] repeated indices are to be summed even if the summation symbol is not indicated

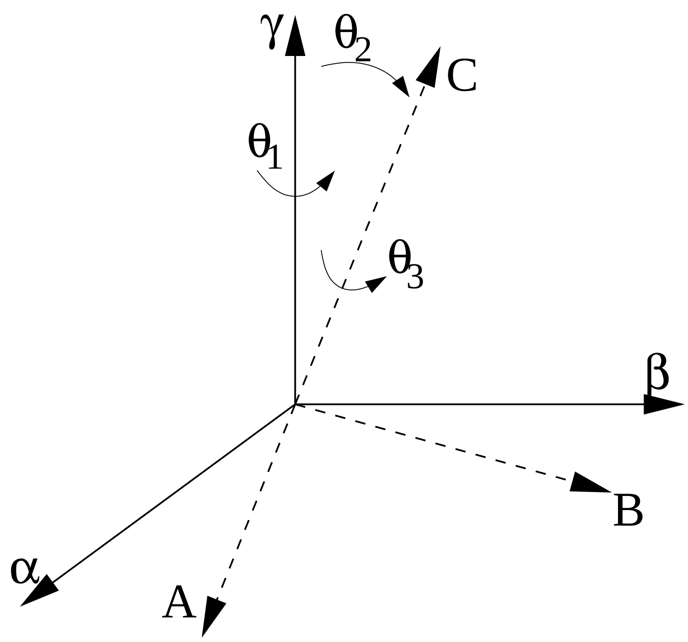


Figure 1: Coordinates systems: α, β, γ denote the laboratory coordinates A, B and C are the body-fixed coordinates and θ_j are the Euler angles.

Another constraint is obtained by imposing the vanishing of inertia moments in the body-fixed frame

$$\sum_i x'_{iA}x'_{iB} = \sum_i x'_{iB}x'_{iC} = \sum_i x'_{iC}x'_{iA} = 0 \tag{5}$$

The six equations, (4) and (5), determine how the center-of-mass coordinates and the Euler angles depend on the space-fixed coordinates $x_{i\alpha}$. The coordinates x'_{iA} are functions of $3A-6$ independent internal (intrinsic) coordinates ξ_σ, satisfying the above conditions identically.

Equation (1) is the coordinate transformation part of a contact(canonical) transformation generated by [Goldstein 1959]

$$F(\mathcal{P}_\alpha, x_{i\alpha}) = \sum_{\beta=1}^{3} X_\beta(x_{i\alpha})P_\beta + \sum_{s=1}^{3} \Pi_s\theta_s(x_{i\alpha}) + \sum_{\sigma=1}^{3N-6} \pi_\sigma\xi_\sigma(x_{i\alpha}) \tag{6}$$

where $\mathcal{P}_\alpha$ stands for the set of canonical momenta $P_\alpha, \Pi_s, \pi_{i\alpha}$ which are conjugate to X_α, θ_s and ξ_σ. The old momenta can be obtained from the generating function F by the usual expression

$$p_{i\alpha} = \frac{\partial F}{\partial x_{i\alpha}} = \frac{\partial X_\beta}{\partial x_{i\alpha}}P_\beta + \frac{\partial \theta_s}{\partial x_{i\alpha}}\Pi_s + \frac{\partial \xi_\sigma}{\partial x_{i\alpha}}\pi_\sigma \tag{7}$$

One also obtains from eq.(3)

$$\frac{\partial X_\beta}{\partial x_{i\alpha}}P_\beta = \frac{1}{N}\delta_{\alpha\beta}P_\beta \tag{8}$$

In order to solve the other two terms in eq.(7), $\Pi_s(x,p)$ and $\pi_\sigma(x,p)$ one shall use the Villars' approach [Villars 1957a]. The first step consists in transforming the third term of eq.(7), in order to eliminate the explicite occurence of $\partial\xi_\sigma/\partial x_{i\alpha}$, a quantity that one need not to evaluate, in favour of the more interesting residual momentum, to be defined bellow. The transformation begins by taking the derivative of eq.(1) with respect to $x_{k\beta}$, considering X_α, θ_s and ξ_σ to be functions of $x_{i\alpha}$'s. Using eqs.(2) and (3) this gives

$$\frac{\partial x_{i\alpha}}{\partial x_{k\beta}} = \delta_{ik}\delta_{\alpha\beta} = \frac{1}{N}\delta_{\alpha\beta} + \frac{\partial R_{A\alpha}}{\partial\theta_s}\cdot\frac{\partial\theta_s}{\partial x_{k\beta}}x'_{iA} + \frac{\partial x_{i\alpha}}{\partial\xi_\sigma}\cdot\frac{\partial\xi_\sigma}{\partial x_{k\beta}} \tag{9}$$

Rearraging the terms in this last equation one gets

$$(\delta_{ik} - \frac{1}{N})\delta_{\alpha\beta} = \frac{\partial R_{A\alpha}}{\partial\theta_s}\cdot\frac{\partial\theta_s}{\partial x_{k\beta}}\cdot x'_{ia} + \frac{\partial x_{i\alpha}}{\partial\xi_\sigma}\cdot\frac{\partial\xi_\sigma}{\partial x_{k\beta}} \tag{10}$$

In deriving (9) use have been made of

$$\frac{\partial R_{A\alpha}}{\partial\xi_\sigma} \equiv \frac{\partial X_\alpha}{\partial\xi_\sigma} = 0 \tag{11}$$

Next the eq.(10) is multiplied by $\partial x_{i\alpha}/\partial\xi_\tau$ and we make the summation over $i\alpha$

$$\frac{\partial x_{k\beta}}{\partial\xi_\tau} = \sum_i \frac{\partial x_{i\beta}}{\partial\xi_\tau}\frac{\partial R_{A\alpha}}{\partial\theta_s}\frac{\partial\theta_s}{\partial x_{k\beta}}x'_{iA} + C_{\tau\sigma}\frac{\partial\xi_\sigma}{\partial x_{k\beta}} \tag{12}$$

where we have introduced the new definition

$$C_{\sigma\tau} = \sum_{i\alpha}\frac{\partial x_{i\alpha}}{\partial\xi_\sigma}\frac{\partial x_{k\alpha}}{\partial\xi_\sigma} \tag{13}$$

Obviously, the transformation is meaningfull if $C_{\sigma\tau}$ has inverses so that (12) can be solved for $\partial\xi_\sigma/\partial x_{k\beta}$,

$$\frac{\partial\xi_\sigma}{\partial x_{k\beta}} = C^{-1}_{\sigma\tau}\frac{\partial x_{k\beta}}{\partial\xi_\tau} - C^{-1}_{\sigma\tau}\frac{\partial x_{i\beta}}{\partial\xi_\tau}\frac{\partial R_{A\alpha}}{\partial\theta_s}\frac{\partial\theta_s}{\partial x_{k\beta}}x'{}_{iA} \tag{14}$$

Then, multiplying by π_σ and summing over σ, one obtains the expression

$$\pi_\sigma\frac{\partial\xi_\sigma}{\partial x_{k\beta}} = \frac{\partial x_{k\beta}}{\partial\xi_\tau}C^{-1}_{\sigma\tau}\pi_\sigma - \frac{\partial\theta_s}{\partial x_{k\beta}}\frac{\partial R_{A\alpha}}{\partial\theta_s}R_{B\alpha}\frac{\partial x'_{i\beta}}{\partial\xi_\tau}C^{-1}_{\sigma\tau}\pi_\sigma x'_{iA} \tag{15}$$

where eq.(1) has been used in the r.h.s.. Interchanging $(k\beta) \to (i\alpha)$ and $(i\alpha) \to (j\gamma)$ and reintroducing the summation symbols for convenience, one obtains

$$\begin{aligned} \sum_\sigma \pi_\sigma\frac{\partial\xi_\sigma}{\partial x_{i\alpha}} &= \sum_{\sigma\tau}\frac{\partial x_{i\alpha}}{\partial\xi_\tau}C^{-1}_{\sigma\tau}\pi_\sigma \\ &- \sum_s\frac{\partial\theta_s}{\partial x_{i\alpha}}\sum_{AB}\frac{\partial R_{A\gamma}}{\partial\theta_s}R_{B\gamma}\sum_j x'_{jA}\sum_{\sigma\tau}\frac{\partial x'_{j\beta}}{\partial\xi_\tau}C^{-1}_{\sigma\tau}\pi_\sigma \end{aligned} \tag{16}$$

Since $C_{\sigma\tau} = C_{\tau\sigma}$, one can permute σ and τ. Splitting also the sum over A and B into two sums by permuting these two indices one have

$$
\begin{aligned}
\sum_\sigma \pi_\sigma \frac{\partial \xi_\sigma}{\partial x_{i\alpha}} &= \sum_{\sigma\tau} \frac{\partial x_{i\alpha}}{\partial \xi_\sigma} C^{-1}_{\sigma\tau} \pi_\tau \\
&- \sum_s \frac{\partial \theta_s}{\partial x_{i\alpha}} \sum_{AB} \frac{1}{2} \left(\frac{\partial R_{A\gamma}}{\partial \theta_s} R_{B\gamma} \sum_j x'_{jA} p'_{jB} + \frac{\partial R_{B\gamma}}{\partial \theta_s} R_{A\gamma} \sum_j x'_{jB} p'_{jA} \right)
\end{aligned}
$$

where we have introduced a new quantity p'_{jA} which has the dimensions of a momentum

$$
p'_{jA} = \sum_{\sigma\tau} \frac{\partial x'_{jA}}{\partial \xi_\tau} C^{-1}_{\sigma\tau} \pi_\sigma = \sum_{\sigma\tau} \frac{\partial x'_{iA}}{\partial \xi_\sigma} C^{-1}_{\sigma\tau} \pi_\tau \tag{17}
$$

Here we define the antisymetric matrix with elements two-rank tensor

$$
A_{s,AB} = -A_{s,BA} = \frac{\partial R_{A\gamma}}{\partial \theta_s} R_{B\gamma} \tag{18}
$$

and the important physical quantity called *intrinsic angular momentum* [2], since it is expressible in terms of ξ_σ and π_σ (App.B)

$$
L'_{AB} = \sum_i (x'_{iA} p'_{iB} - x'_{iB} p'_{iA}) \tag{19}
$$

One are thus lead to the final relation

$$
\sum_\sigma \pi_\sigma \frac{\partial \xi_\sigma}{\partial x_{i\alpha}} = R_{A\alpha} p'_{iA} - \sum_s \frac{\partial \theta_s}{\partial x_{i\alpha}} \sum_{A<B} A_{s,AB} L'_{AB} \tag{20}
$$

At this stage it is worthwhile to notice that neither x'_{iA} nor the p'_{iA} are independent variables and therefore are not conjugate variables. Nevertheless, Villars introduced a suggestive notation for an approximation one is tempted to try at this place, if the number of particles is sufficiently large: to consider at least some of the x'_{iA} and p'_{iA} in fact as independent variables, in which case they become conjugate pairs, of course.

The second step consists in expressing the momenta Π_s in terms of the total angular momentum with respect to the center-of-mass

$$
L_{\alpha\beta} = \sum_i (x_{i\alpha} p_{i\beta} - x_{i\beta} p_{i\alpha}) - (X_\alpha P_\beta - X_\beta P_\alpha) \tag{21}
$$

Multiplying eq.(7) by $x_{i\beta}$ and substracting from the same quantity, but with α and β interchanged one gets

$$
\begin{aligned}
x_{i\alpha} p_{i\beta} - x_{i\beta} p_{i\alpha} &= \frac{1}{N} (x_{i\alpha} P_\beta - x_{i\beta} P_\alpha) + \sum_{\sigma\tau} \left(x_{i\alpha} \frac{\partial x_{i\beta}}{\partial \xi_\sigma} - x_{i\beta} \frac{\partial x_{i\alpha}}{\partial \xi_\sigma} \right) C^{-1}_{\sigma\tau} \pi_\tau \\
&- \sum_s \left(x_{i\alpha} \frac{\partial \theta_s}{\partial x_{i\beta}} - x_{i\beta} \frac{\partial \theta_s}{\partial x_{i\alpha}} \right) \left(\sum_{C<D} A_{s,CD} L'_{CD} - \Pi_s \right)
\end{aligned}
$$

[2]Later we shall see that this quantity is related to the *vorticity*

Next, summing over i, using eq.(3) one obtains $L_{\alpha\beta}$

$$\begin{aligned} L_{\alpha\beta} &= \sum_i \left\{ \sum_{\sigma\tau} \left(x_{i\alpha} \frac{\partial x_{i\beta}}{\partial \xi_\sigma} - x_{i\beta} \frac{\partial x_{i\alpha}}{\partial \xi_\sigma} \right) C^{-1}_{\sigma\tau} \pi_\tau \right. \\ &+ \left. \sum_s \left(x_{i\alpha} \frac{\partial \theta_s}{\partial x_{i\beta}} - x_{i\beta} \frac{\partial \theta_s}{\partial x_{i\alpha}} \right) (\Pi_s - \sum_{C<D} A_{s,CD} L'_{CD}) \right\} \end{aligned} \tag{22}$$

One explicite the first term in the r.h.s. of the above equation, using eqs.(1), (3), the functional dependence of X,x' and the constraint of fixed center-of-mass (4), and definitions (16), (18)

$$\sum_i \sum_{\sigma\tau} \left(x_{i\alpha} \frac{\partial x_{i\beta}}{\partial \xi_\sigma} - x_{i\beta} \frac{\partial x_{i\alpha}}{\partial \xi_\sigma} \right) C^{-1}_{\sigma\tau} \pi_\tau = R_{A\alpha} R_{B\beta} L'_{AB} \tag{23}$$

In expliciting the second term from the r.h.s of eq.(21) one use the orthonormation property of rotation matrices, i.e. $R_{A\alpha} R_{B\beta} = \delta_{AB} \delta_{\alpha\beta}$ and again (1)

$$\begin{aligned} &\sum_{is} \left(x_{i\alpha} \frac{\partial \theta_s}{\partial x_{i\beta}} - x_{i\beta} \frac{\partial \theta_s}{\partial x_{i\alpha}} \right) (\Pi_s - \sum_{C<D} A_{s,CD} L'_{CD}) \\ &= R_{A\alpha} R_{B\beta} \sum_s \Theta_{s,AB} (\Pi_s - \sum A_{s,CD} L'_{CD}) \end{aligned}$$

where $\Theta_{s,AB}$ is the matrix element

$$\Theta_{s,AB} = -\Theta_{s,BA} = \left(x'_{iA} \frac{\partial \theta_s}{\partial x_{i\beta}} R_{B\beta} - x'_{iB} \frac{\partial \theta_s}{\partial x_{i\alpha}} R_{A\alpha} \right) \tag{24}$$

Consequently the total angular momentum relative to the center-of-mass may be written as

$$L_{\alpha\beta} = R_{A\alpha} R_{B\beta} \left[L'_{AB} + \sum_s \Theta_{s,AB} (\Pi_s - \sum_{C<D} A_{s,CD} L'_{CD}) \right] \tag{25}$$

Since the total angular momentum relative to the center-of-mass projected on the body-fixed frame coordinates is related to the space-fixed one by

$$L_{AB} = R_{A\alpha} R_{B\beta} L_{\alpha\beta} \tag{26}$$

one obtains

$$L_{AB} = L'_{AB} + \sum_s \Theta_{s,AB} (\Pi_s - \sum_{C<D} A_{s,CD} L'_{CD}) \tag{27}$$

In order to simplify this last equation one needs an orthogonality relation for the matrices $A_{s,CD}$ and $\Theta_{s,AB}$. For that one multiply eqs(17) and (23) and sums over s

$$\sum_s A_{s,CD} \Theta_{s,AB} = \delta_{AC} \delta_{BD} - \delta_{BC} \delta_{AD} \tag{28}$$

In deriving the above equation the orthogonality relation for the rotation matrices, and eqs.(1) have been used. Using this last equation, then eq.(25) may be expressed as follows

$$L_{AB} = \sum_s \Theta_{s,AB}\Pi_s \tag{29}$$

Using the orthogonality condition (27), the eq.(28) may be inversed

$$\Pi_s = \sum_{A<B} A_{s,AB} L_{AB} \tag{30}$$

One thus arrive to the main task of the calculations: the expression for the momentum $p_{i\alpha}$ in terms of the new variables. Introducing (19) and (29) in (7) one obtains

$$p_{i\alpha} = \frac{1}{N}P_\alpha + R_{A\alpha}p'_{iA} + \sum_{A<B} \frac{\partial\theta_s}{\partial_{i\alpha}} A_{s,AB}(L_{AB} - L'_{AB}) \tag{31}$$

In the third term of the r.h.s. of the eq.(30) one may replace $(\partial\theta_s/\partial_{i\alpha})A_{s,AB}$ by another matrix element

$$M_{i\alpha,AB} = \sum_s \frac{\partial\theta_s}{\partial x_{i\alpha}} A_{s,AB} = \frac{\partial R_{A\gamma}}{\partial x_{i\alpha}} R_{B\gamma} \tag{32}$$

and thus eq.(30) may be reexpressed as

$$p_{i\alpha} = \frac{1}{N}P_\alpha + \sum_A R_{A\alpha}p'_{iA} + \sum_{A<B} M_{i\alpha,\hat{C}}(L_{\hat{C}} - L'_{\hat{C}}) \tag{33}$$

where $L_{\hat{C}} \equiv L_{AB}$. One are also mentioning that the term $R_{A\alpha}p'_{iA}$ can be found in the literature under the name of residual momentum [Weaver et al. 1976], as we have mentioned earlier.

Further Villars [Villars 1957a,b] stated that the three terms occuring in eq.(30) are *orthogonal* to each other. As we shall see later this is not quite true. For the moment the kinetic energy is derived following closely the Villars' treatment

It is obvious that the product between the center-of-mass canonical momenta, the residual momenta $R_{A\alpha}p'_{iA}$ and the third term vanishes. The product between the last two terms needs some manipulations

$$\sum_{i\alpha}\sum_A R_{A\alpha}p'_{iA} \sum_{C<D} M_{i\alpha,CD}(L_{CD} - L'_{CD}) = 0$$

since $\partial R_{A\gamma}/\partial\xi_\sigma$.

Thus Villars' obtained that all the cross terms in the quadratic expression of the kinetic energy vanish. Introducing tensor of inertial nature

$$Q_{AB} = \frac{1}{m}\sum_{i\alpha} M_{i\alpha,\hat{A}} M_{i\alpha,\hat{B}} \tag{34}$$

and working out the rest of diagonal terms one obtains the canonical form of the kinetic energy

$$T = \frac{1}{2m}\sum_{i\alpha} p_{i\alpha}^2 = \frac{1}{2mN}\sum_\alpha P_\alpha^2 + \frac{1}{2m}\sum_{iA} p'^2_{iA} + \frac{1}{2}\sum_{AB} Q^2_{AB}(L_{\hat{A}} - L'_{\hat{A}})(L_{\hat{B}} - L'_{\hat{B}}) \tag{35}$$

In the case of a system with N interacting non-identical particles eq.(34) and (35) are rewritten as follows

$$Q_{AB} = \sum_{i\alpha} \frac{1}{m_i} M_{i\alpha,\hat{A}} M_{i\alpha,\hat{B}} \tag{36}$$

$$T = \sum_{i\alpha} \frac{p_{i\alpha}^2}{2m_i} = \frac{1}{2M}\vec{P}^2 + \sum_{iA} \frac{p'^2_{iA}}{2m_i} + \frac{1}{2}\sum_{AB} Q^2_{AB}(L_{\hat{A}} - L'_{\hat{A}})(L_{\hat{B}} - L'_{\hat{B}}) \tag{37}$$

where $M = \sum_i m_i$

2.2 The inertia tensor for irrotational fluid motion

In order to find the expression of Q_{AB} - *inverse inertia tensor* - one must look for the matrix elements $M_{i\alpha,\hat{A},(\hat{B},\hat{C})}$ which appear in eq.(33). Invoking the constraint given by eq.(4), which specify that the body-fixed system is free of deviations moments $I_{AB} = \sigma x'_{iA} x'_{iB}$ one obtains by derivation with respect to $x_{i\alpha}$

$$\frac{\partial}{\partial x_{i\alpha}} \sum_{j=1}^{N} x'_{jA} x'_{jB} \equiv 0 = \qquad \sum_{\gamma} \frac{\partial R_{A\gamma}}{\partial x_{i\alpha}} R_{B\gamma}(I_B - I_A) + x'_{iB} R_{A\alpha} + x'_{iA} R_{B\alpha}$$

One thus obtain for the matrix elements

$$M_{i\alpha,AB} \equiv \frac{\partial R_{A\gamma}}{\partial x_{i\alpha}} R_{B\gamma} = \frac{x'_{iA} R_{B\alpha} + x'_{iB} R_{A\alpha}}{I_A - I_B} \tag{38}$$

where $I_{A,(B,C)}$ are the diagonal inertia moments. Introducing now eq.(37) into eq.(33) one get a diagonal form for the tensor Q_{AB}

$$Q_{AA'} = \begin{cases} \frac{1}{m}\frac{I_B + I_C}{(I_B - I_C)^2}, & A = A' \\ 0 & , A \neq A' \end{cases}$$

Introducing the inertia tensor as the inverse of the tensor Q_{AB} one obtain a value equivalent to Bohr's result derived for the rotation and vibration of an irrotational liquid drop $\mathcal{I}_{IF}$ [Bohr 1952]

$$\mathcal{I}_A = m\frac{(I_B - I_C)^2}{I_B + I_C} \tag{39}$$

$$\mathcal{I}_B = m\frac{(I_C - I_A)^2}{I_C + I_A} \tag{40}$$

$$\mathcal{I}_C = m\frac{(I_A - I_B)^2}{I_A + I_B} \tag{41}$$

The Villar's derivation of the irrotational flow moment $\mathcal{I}_{IF}$ presented above is rather microscopic. A detailed calculation of $\mathcal{I}_{IF}$ in the frame of the phenomenological rotation-vibration model(RVM) may be found in textbooks in nuclear structure [Eisenberg and Greiner 1970a]. The advantage of the Villar's derivation is that is exact. However the deduced moment of inertia $\mathcal{I}_{IF}$ differs very much from experimental values. The reason seems transparent from the above considerations. In the irrotational flow transformation, the kinetic energy not only separates into a rotational and an intrinsic part but, as Villars shows, leads to a rather large Coriolis coupling term $L_{\hat{A}}L'_{\hat{B}}$(see the third term in eq.(34)). This coupling must therefore be responsible for renormalizing $\mathcal{I}_{IF}$ to generate the effective moment of inertia given by the experiment. Thus, a strong rotational-intrinsic coupling is fundamental to irrotational flow. Crudely speaking, one may visualize irrotational flow as a deformation wave propagating over the surface of the nucleus, maintained by a corresponding motion of the intrinsic structure so that it carries very small angular momentum [Rowe 1970a,b]

2.3 The inertia tensor for rotational flow

To evaluate the expression of Q_{AB}, it is necessary to specify the constraints on the intrinsic coordinates which define the nuclear orientation angles θ_s

According to Villar's hypothesis

$$I_{AB} = \sum x'_{iA} x'_{iB} \neq 0, \quad (A \neq B) \tag{42}$$

i.e. the quadrupole mass-tensor is diagonal in the body-fixed frame. Consequently one has obtained the irrotational flow inertia tensor $\mathcal{I}_{RR}$ in contradiction with the adiabatic rotor-model(ARM).

The ARM states the existence of bands of nuclear rotational states whose wave functions satisfies the properties listed below:

1. $\psi_{JM} = D^J_{KM}\chi(\xi)$, states of the same rotational band are characterised by a common intrinsic wave function

2. $\mathbf{J}\chi(\xi) = 0$, i.e. the intrisic wave function carryes zero total angular momentum

3. $H = H_0(\xi) + \frac{1}{2}\sum_{AB} Q_{AB}(\xi) J_A J_B$, is the adiabatic hamiltonian, where H_0 and Q_{AB} operate on intrisic coordinates

4. $Q(\xi) \to \langle\, \chi(\xi) \mid \hat{Q}(\xi) \mid \chi(\xi)\, \rangle$, the inverse inertia tensor can be replaced by its expectation over the common intrinsic wave function for a given rotational band. This is consistent with (2), i.e. $\chi(\xi)$ is independent of J.

These properties are valid in the "adiabatic approximation", when the centrifugal stretching are suppressed in zero order. This approximation is not very good for odd rotational nuclei, because the odd particles are insuficiently bound to the even core to give the combined structure of rigidity necessary to suppress the centrifugal or other intrisic- rotational couplings(Coriolis terms in the Hamiltonian). For doubly even nuclei, for which Coriolis perturbations are not normally entertained, the ARM works quite well.

A question arise naturally. how does nuclei rotate in the ARM? Rowe attempted to answer this question in the following manner: Transforming the wave function given by property (1) by a rotation $\theta \to \theta'$ one obtains:

$$\psi_{JM} \to \psi'_{JM}(\vec{r}_i') = D^J_M(\theta')\chi(\xi) = \psi_{JM}(\vec{r}_i') \tag{43}$$

This transformation is generated by $\vec{J}$ since it operates only on D^J_M and consequently the wave function transforms by simply rotating the spatial coordinates

$$x_{i\alpha} = \sum R_{A\alpha} x'_{iA} \tag{44}$$

Thus in the absence of any coupling between the intrinsic and the rotational degrees of freedom, the distatnces between particles are left invariant by the rotational motion

$$\sum_\alpha (x_{i\alpha} - x_{j\alpha})^2 = \sum_{\alpha AB} R_{A\alpha}(\theta_s) R_{B\alpha}(\theta_s)(x'_{iA} - x'_{jA})(x'_{iB} - x'_{jB}) = \sum_\alpha (x'_{iA} - x'_{jA})^2 \tag{45}$$

It then follows that the rotational motion of the zero-order ARM is rotational flow or $rigid - body \quad flow$. The only distinction between ordinary mechanical rigid rotation and nuclear rotational flow is that in the second case the particles are not frozen in position but are free to execute any independent motion in the intrinsic coordinates.

Property (2) of the ARM express the necessity that the system has zero- angular momentum relative to the body-fixed axes. Classicaly this is expressed by

$$\vec{l}' = \sum_j \vec{r}_j' \times \vec{p}_j' = 0 \tag{46}$$

or expressing in differential constraints

$$\sum_j (x'_{jA} \delta x'_{jB} - x'_{jB} \delta x'_{jA}) = 0 \tag{47}$$

It is important to notice that eq.(46) is a non-holonomic constraint which cannot be integrated to give the θ_s in terms of the original particle coordinates. At this important remark one shall come back latter, when talking the imposibility of rigid-flow in quantum mechanics.

Since the differentials δx enter in the definition of rigid frames by means of the formula

$$x'_{iA} = x^0_{iA} + \delta x'_{iA} \tag{48}$$

eq.(46) may be expressed again in the form

$$\sum_j (x'_{jA} x'_{jB} - x'_{jB} x'_{jA}) = 0 \tag{49}$$

Using eq.(1) and (4) one get after a short calculation that

$$\sum_j (x'_{jA} x'_{jB} - x'_{jB} x'_{jA}) = \sum_{j\gamma} (x'_{jA} R_{B\gamma} x_{j\gamma} - x'_{jB} R_{A\gamma} x_{j\gamma}) = 0 \tag{50}$$

Differentiating this last expression with respect to $\theta_{s'}$, multiplying by $\partial\theta_s/\partial x_{i\alpha}$ and summing over s one obtains after a long but straightforward calculation that

$$\begin{aligned} &\sum_{j\gamma C}(x'_{jB}x'_{jC}\frac{\partial R_{A\gamma}}{\partial x_{i\alpha}}R_{C\gamma} - x'_{jA}x'_{jC}\frac{\partial R_{B\gamma}}{\partial x_{i\alpha}}R_{C\gamma}) \\ &= \sum_{j\gamma s}(x'_{jA}R_{B\gamma} - x'_{jB}R_{A\gamma})\frac{\partial x_{j\gamma}}{\partial\theta_s}\frac{\partial\theta_s}{\partial x_{i\alpha}} \end{aligned}$$

Defining the inertia moments relative to the body-fixed axes(see eq.(37)) and using the definition (33), the above equation will be rewritten in the following manner

$$\sum_C(I_{BC}M_{i\alpha,AC} - I_{AC}M_{i\alpha,BC}) = \sum_{j\gamma s}(x'_{jA}R_{B\gamma} - x'_{jB}R_{A\gamma})\frac{\partial x_{j\gamma}}{\partial\theta_s}\frac{\partial\theta_s}{\partial x_{i\alpha}} \qquad (51)$$

This last equation is a mathematical identity and carries no information concerning the character of the rotational and intrinsic coordinates. For rotational flow, this information is contained in eq.(46). Since $\delta x'_{jA}(\xi_\sigma) = (\partial x'_{jA}/\partial\xi_\sigma)\delta\xi_\sigma$, eq.(46) takes the form

$$\sum_j(x'_{jA}\frac{\partial x'_{jB}}{\partial\xi_\sigma} - x'_{jB}\frac{\partial x'_{jA}}{\partial\xi_\sigma}) = 0 \qquad (52)$$

Substitution of eq.(2) in eq.(50) gives

$$\sum_j(x'_{jA}R_{B\gamma} - x'_{jB}R_{A\gamma})\frac{\partial x_{j\gamma}}{\partial\xi_\sigma} = 0 \qquad (53)$$

Since $\sum_j x'_{jA} = 0$ and $\partial x_{j\gamma}/\partial X_\alpha = \delta_{\alpha\gamma}$ there simply results that

$$\sum_{j\gamma}x'_{jA}R_{B\gamma}\frac{\partial x_{j\gamma}}{\partial X_\alpha} = 0$$

which make possible the construction of the trivial equation

$$\sum_{j\gamma}(x'_{jA}R_{B\gamma} - x'_{jB}R_{A\gamma})\frac{\partial x_{j\gamma}}{\partial X_\alpha} = 0 \qquad (54)$$

Next, eq.(51) is multiplyied by $\partial\xi_\sigma/\partial x_{i\alpha}$ and summed over σ, whereas eq.(52) is simply multiplyed by $\partial X_\alpha/\partial x_{i\alpha}$. Then both equations are added to the r.h.s. of eq.(50)

$$\sum_C(I_{BC}M_{i\alpha,AC} - I_{AC}M_{i\alpha,BC}) = (x'_{jA}R_{B\alpha} - x'_{iB}R_{A\alpha}) \qquad (55)$$

In expanded form, by making cyclic permutation and using the fact that $M_{i\alpha,AA} = M_{i\alpha,BB} = M_{i\alpha,CC} = 0$, this equation reads

$$\begin{aligned} -(I_{AA}+I_{BB})M_{i\alpha,AB} + I_{AC}M_{i\alpha,BC} + I_{BC}M_{i\alpha,CA} &= x'_{iB}R_{A\alpha} - x'_{iA}R_{B\alpha} \quad (56) \\ I_{CA}M_{i\alpha,AB} + (I_{BB}-I_{CC})M_{i\alpha,BC} + I_{BA}M_{i\alpha,CA} &= -x'_{iC}R_{B\alpha} - x'_{iB}R_{C\alpha} \quad (57) \\ -I_{CB}M_{i\alpha,AB} - I_{AB}M_{i\alpha,BC} + (I_{CC}+I_{AA})M_{i\alpha,CA} &= x'_{iC}R_{A\alpha} - x'_{iA}R_{C\alpha} \quad (58) \end{aligned}$$

Defining the rigid-body inertia tensor $\mathcal{I}_{RR}$

$$\mathcal{I}_{RR} = \begin{pmatrix} I_{BB}+I_{CC} & -I_{AB} & -I_{AC} \\ -I_{BA} & I_{CC}+I_{AA} & -I_{BC} \\ -I_{CA} & -I_{CB} & I_{AA}+I_{BB} \end{pmatrix} \tag{59}$$

eq.(55-57) may be written in the condensed form

$$\hat{\mathcal{I}}_{RR}\hat{M}_{i\alpha} = \hat{r'}_{iA}\hat{R}_{\alpha} \tag{60}$$

Next, one multiply each side of eq.(58) by its transpose, and sum over $i\alpha$

$$\sum_{i\alpha}\mathcal{I}_{RR}(M_{i\alpha})(M_{i\alpha})^T\mathcal{I}_{RR}^T = \sum_{i\alpha} r'_{iA}(R_{\alpha})(R_{\alpha})^T r'^T_{iA} \tag{61}$$

Since in the matrix form the inverse inertia tensor is

$$Q = \sum_{i\alpha}(M_{i\alpha})(M_{i\alpha})^T \tag{62}$$

and the rotation is orthogonal, i.e. $R^TR = I$, eq.(59) becomes

$$\mathcal{I}_{RR}Q\mathcal{I}_{RR}^T = \mathcal{I}_{RR} \tag{63}$$

Multiplying (61) on the left and on the right with $\mathcal{I}^{-1}$, one obtains what was to be expected, i.e. Q is the inverse of I.

The Coriolis coupling terms $-\hat{L}'_A\hat{L}_B, -\hat{L}_A\hat{L}'_B$ vanish for rotational flow coordinates, thereby confirming the argument that the absence of intrinsic angular momentum and Coriolis coupling implies rotational flow. Invoking eq.(13),(18),(51) one verify the above mentioned statement

$$\hat{L}'_C = \sum_i(x'_{iA}p'_{iB} - x'_{iB}p'_{iA}) = \sum_{\sigma\tau}\sum_i\left(x'_{iA}\frac{\partial x'_{iB}}{\partial\xi_\sigma} - x'_{iB}\frac{\partial x'_{iA}}{\partial\xi_\sigma}\right)(C^{-1})_{\sigma\tau}\pi_\tau = 0 \tag{64}$$

Therefore one can see eq.(51) as a constraint which ensure that the body-fixed system carries no angular momentum. The kinetic energy then reads

$$T = \sum_{i\alpha}\frac{p^2_{i\alpha}}{2m_i} = \frac{1}{2M}\vec{P}^2 + \sum_{iA}\frac{p'^2_{iA}}{2m_i} + \frac{1}{2}\sum_{AB}(\mathcal{I}^{-1}_{RR})_{AB}L_{\hat{A}}L_{\hat{B}} \tag{65}$$

At first sight one should suppose that this canonical transformation demonstrates that all many-body rotations must be rotational flow. In fact, there is still a centrifugal rotational-intrinsic coupling coming from the dependence of the inertia tensor $\mathcal{I}_{RR}$ on the dynamic intrinsic coordinates. The above calculations show that it is possible to choose an intrinsic coordinate system such that the Coriolis coupling component in the kinetic energy vanishes.

For the motion to be rigid rotation it must be required that the deformed nucleus have sufficient rigidity to suppress the effects of the centrifugal coupling and to replace the inertia tensor by its expectation over the intrinsic wave function (property 4 of the ARM which express the adiabaticity).

2.4 The generalized Villar's transformation

The most general real linear transformation on the Cartesian coordinates of the n-th particle of a collection of N identical particles is given by [Weaver and Biedenharn 1972]

$$x_{i\alpha} = \sum_{a=1}^{3} g^{-1}_{\alpha a} x''_{iA}, \quad (a, \alpha = 1, 3) \tag{66}$$

where g^{-1} is the inverse of the real (3×3) matrix g. The set of matrices g with nonzero determinant form the *General Linear Group* in the real three-dimensional space $GL(3, \mathbf{R})$, which will be presented in the next chapter. Its generators are the six shear operators

$$t_{\alpha\beta} = \sum_{i=1}^{N} (x_{i\alpha} p_{i\beta} + x_{i\beta} p_{i\alpha}) \tag{67}$$

and the three independent components of the total angular momentum tensor

$$L_{\alpha\beta} = \sum_{i=1}^{N} (x_{i\alpha} p_{i\beta} - x_{i\beta} p_{i\alpha}) \tag{68}$$

where $p_{i\alpha}$ is the momentum.

If $x''_{i\alpha}$ in (64) is held fixed and the nine parameters $g^{-1}_{i\alpha}$ are considered to be dynamic, the $3N$ degrees of freedom of the system reduce to these nine. The parameters g may be uniquely decomposed according to the Cartan decomposition [Gantmacher 1953] $g = SR$, where S and R are respectively symmetric and orthogonal (3×3) real matrices. Furthermore, S can be diagonalized by means of an orthogonal matrix so that one can write $g = R'' S_0 R'$ where R' and R'' are two different orthogonal matrices and S_0 is diagonal. The decomposition can be made unique by requiring

$$\det R' = \det R'' = 1, \quad S_{02} \geq S_{01} \geq S_{03} \geq 0 \tag{69}$$

Recall the definition of the center-of-mass coordinates (eq.(3)) and allow the $x''_{i\alpha}$ to depend on $(3N - 12)$ independent intrinsic coordinates ξ_σ in order to regain the original $3N$ degrees of freedom. The complete transformation becomes

$$x_{i\alpha} = X_\alpha + \sum_{A,a=1}^{3} R'_{A\alpha}(\theta_s) S^{-1}_{0A} R''_{Aa}(\phi_t) x''_{ia}(\xi_\sigma) \tag{70}$$

where θ_s and ϕ_t are sets of Euler angles.

An appropriate choice of the generalized coordinates X_i, θ_s, S_{0A}, and ϕ_t is made by imposing 12 constraints on $x''_{n\alpha}$. For the center-of-mass one have the usual three constraints

$$\sum_{i=1}^{N} x_{iA} = \sum_{i=1}^{N} x''_{ia} \tag{71}$$

where

$$x_{iA} = \sum_{i=1}^{N} R'_{A\alpha}(x_{i\alpha} - X_\alpha) \tag{72}$$

and analogously

$$p_{iA} = \sum_{i=1}^{N} R'_{A\alpha}(p_{i\alpha} - P_\alpha) + p'_{iA} \tag{73}$$

The coordinates θ_s are chosen in the same manner as Villars (section 2.1) such that the first rotation, R', diagonalizes the mass tensor

$$\sum_{i=1}^{N} m x_{iA} x_{iB} = \delta_{AB} m \sum_{i=1}^{N} x_{iA}^2 \equiv \delta_{AB} I_A \tag{74}$$

where m is the mass of a nucleon. Thus the three axes (ABC) coincide with the principal axes. Next, the S_{0A} are chosen by requiring a spherical quadrupole distribution for the intrinsic system [Cusson 1968], i.e.

$$\sum_{i=1}^{N} \sum_{a,b=1}^{3} R''_{Aa} R''_{Bb} x''_{ia} x''_{ib} \equiv \delta_{AB} \tag{75}$$

or equivalently

$$\sum_{i=1}^{3} x''_{ia} x''_{ib} \equiv \delta_{AB} \tag{76}$$

Combining (72) and (73), one have

$$S_{0A} = \sqrt{\frac{m}{I_A}} \tag{77}$$

One next generalize the constraint for rotational flow as used by Rowe [Rowe 1970b]

$$\sum_{i=1}^{N} (x''_{ja} \delta x''_{jb} - x''_{jb} \delta x''_{ja}) = 0$$

and since $\delta x''_{ja} = (\partial x''_{ja}/\partial \xi_\sigma)\delta\xi_\sigma$ then (see eq.(51))

$$\sum_{j} \left(x''_{ja} \frac{\partial x''_{jb}}{\partial \xi_\sigma} - x''_{jb} \frac{\partial x''_{ja}}{\partial \xi_\sigma} \right) = 0 \tag{78}$$

which are likewise nonintegrable except for the the trivial case of a frozen intrinsic structure. In using eq.(76), one must therefore always be careful to remember that the intrinsic axes (a, b, c) and hence the collective coordinates, ϕ_t, cannot, in general, be defined in integrated form as functions of the particle coordinates. Only the differentials $\delta\phi_t$ are strictly definable. This does not present any immediate problem since the nonintegrable coordinates do not explicitely appear in the expansion of the kinetic energy. Thus, the coordinates ϕ_t, are cyclic. The use of the nonintegrable but cyclic coordinates is a well known device in classical mechanics [Goldstein 1959]. The nonintegrability question is discussed further.

The transformed kinetic energy can be obtained ffrom either of the two expressions derived in Appendix A.

The transformed momentum for the i-th particle is given by

$$p_{i\alpha} \equiv -i\hbar \frac{\partial}{\partial x_{i\alpha}}$$

$$= -i\hbar \left(\frac{\partial X_\beta}{\partial x_{i\alpha}} \frac{\partial}{\partial X_\beta} + \frac{\partial \theta_s}{\partial x_{i\alpha}} \frac{\partial}{\partial \theta_s} + \frac{\partial S_{0A}}{\partial x_{i\alpha}} \frac{\partial}{\partial S_{0A}} + \frac{\partial \phi_t}{\partial x_{i\alpha}} \frac{\partial}{\partial \phi_t} + \frac{\partial \xi_\sigma}{\partial x_{i\alpha}} \frac{\partial}{\partial \xi_\sigma} \right) \tag{79}$$

Now using the definition (3) one have

$$-i\hbar \frac{\partial X_\beta}{\partial x_{i\alpha}} \frac{\partial}{\partial X_\beta} = -\frac{i\hbar}{N} \delta_{\alpha\beta} \frac{\partial}{\partial X_\beta} \equiv \frac{1}{N} P_\alpha \tag{80}$$

Noticing that

$$\frac{\partial}{\partial \theta_s} \equiv \frac{\partial x_{i\alpha}}{\partial \theta_s} \frac{\partial}{\partial x_{i\alpha}}; \quad \frac{\partial}{\partial \phi_s} \equiv \frac{\partial x_{i\alpha}}{\partial \phi_t} \frac{\partial}{\partial x_{i\alpha}}$$

and using eqs.(68-71), then

$$-i\hbar \frac{\partial}{\partial \theta_s} = \sum_{i\alpha} \frac{\partial x_{i\alpha}}{\partial \theta_s} p_{i\alpha} = \sum_{i\alpha AB} \frac{\partial R'_{A\alpha}}{\partial \theta_s} R'_{B\alpha} x_{iA} p_{iB} \tag{81}$$

and thus the second term in (77) reads

$$-i\hbar \frac{\partial x_{i\alpha}}{\partial \theta_s} \frac{\partial}{\partial \theta_s} = \frac{1}{2} M_{i\alpha,AB} L_{AB} \tag{82}$$

where L_{AB} and $M_{i\alpha,AB}$ were defined earlier (see eqs.(18) and (33)). From (37) one have that

$$M_{i\alpha,AB} = \frac{x_{iA} R'_{B\alpha} + x_{iB} R'_{A\alpha}}{I_A - I_B} = R'_{B\alpha} \frac{\partial R'_{A\alpha}}{\partial x_{i\alpha}} \tag{83}$$

After some algebraic manipulations one obtains for the third term, following the same reasoning

$$\frac{\partial \phi_t}{\partial x_{i\alpha}} \frac{\partial}{\partial \phi_t} = -\frac{1}{2} N_{i\alpha,AB} \mathcal{L}_{AB} \tag{84}$$

where

$$N_{i\alpha,AB} = R''_{B\alpha} \frac{\partial R''_{A\alpha}}{\partial x_{i\alpha}} = m \frac{\sqrt{\frac{I_A}{I_B}} x_{iB} R'_{A\alpha} + \sqrt{\frac{I_B}{I_A}} x_{iA} R'_{B\alpha}}{I_A - I_B} \tag{85}$$

and $\mathcal{L}_{AB}$ are the *rotation − distortion* operators. They are defined by

$$\mathcal{L}_{AB} = \sqrt{\frac{I_B}{I_A}} x_{iA} p_{iB} - \sqrt{\frac{I_A}{I_B}} x_{iB} p_{iA} \tag{86}$$

and have vector components

$$\mathcal{L}_A \equiv \frac{1}{2} \epsilon_{abc} \mathcal{L}_{BC} \tag{87}$$

Next using eq.(78), one get

$$-i\hbar \frac{\partial S_{0A}}{\partial x_{i\alpha}} \frac{\partial}{\partial S_{0A}} = -i\hbar \frac{\partial I_A}{\partial x_{i\alpha}} \frac{\partial}{\partial I_A} \tag{88}$$

But since $I_A = m \sum_i x_{iA}^2$

$$\frac{\partial I_A}{\partial x_{i\alpha}} = 2m x_{iA} R'_{A\alpha} \tag{89}$$

Combining (86) and (87) one arrive at the result

$$-i\hbar \frac{\partial S_{0A}}{\partial x_{i\alpha}} = -2i\hbar m R'_{A\alpha} x_{iA} \frac{\partial}{\partial I_A} \tag{90}$$

Calculation of the last term in eq.(78) demands a lot of algebra and therefore one simply list bellow the result

$$-i\hbar \frac{\partial \xi_\sigma}{\partial x_{i\alpha}} \frac{\partial}{\partial \xi_\sigma} = -i\hbar C_{\lambda\sigma^{-1}} \frac{\partial x_{i\alpha}}{\partial \xi_\lambda} \frac{\partial}{\partial \xi_\sigma} \tag{91}$$

where

$$C_{\lambda\sigma} = \frac{\partial x_{jA}}{\partial \xi_\lambda} \cdot \frac{\partial x_{jA}}{\partial \xi_\sigma} \tag{92}$$

Therefore, inserting (78), (80), (82), (88) and (90) into (77) yields the transformed momentum

$$p_{i\alpha} = \frac{1}{N} P_\alpha + R'_{A\alpha} p''_{A\alpha} + 2m R'_{A\alpha} x_{iA} (-i\hbar \frac{\partial}{\partial I_A}) + \frac{1}{2} M_{i\alpha,AB} L_{AB} - \frac{1}{2} N_{i\alpha,AB} \mathcal{L}_{\mathcal{AB}} \tag{93}$$

where

$$p''_{A\alpha} = C^{-1}_{\sigma\lambda} \frac{\partial x'_{iA}}{\partial \xi_\lambda} \left(-i\hbar \frac{\partial}{\partial \xi_\sigma} \right) \tag{94}$$

Note that the nonintegrable coordinate ϕ_t does not appear in (91) as can be seen from (90). Furthermore, the opreator $\mathcal{L}_{AB}$ which acts on ϕ_t, has a well defined action on the particle coordinates, which will be discussed later. Thus, all the components on the right-hand-side of (91) are well defined.

The transformed kineyic energy T, can now be obtained by squaring (91) [Gulshani and Rowe 1976]

$$T = \sum_{i\alpha} \frac{1}{2m} p_{i\alpha}^2 \equiv T_{C.M.} + T_{intr} + T_{coll} \tag{95}$$

where

$$T_{C.M.} \equiv \frac{1}{2mN} \sum_\alpha P_\alpha^2 \tag{96}$$

$$T_{intr} \equiv \frac{1}{2mN} \sum_{iA} {p''}_{iA}^2 \tag{97}$$

$$
\begin{aligned}
T_{coll} &\equiv T_{vib} + T_{rot} \\
&= = -2\hbar^2 \sum_A I_A \left[\frac{\partial^2}{\partial I_A^2} + (\frac{N-3}{2I_A} + \sum_{B\neq A} \frac{1}{I_A - I_B}) \frac{\partial}{\partial I_A} \right] \\
&+ \frac{1}{2} \sum_{A<B} \left[\frac{I_A + I_B}{(I_A - I_B)^2} (L_{AB}^2 + \mathcal{L}_{AB}^2) - \frac{4\sqrt{I_A I_B}}{(I_A - I_B)^2} L_{AB} \mathcal{L}_{AB} \right] \qquad (98)
\end{aligned}
$$

The expression for T_{rot} in (96) is similar to that derived by Cusson [Cusson 1968] where T_{vib} was absent. From a different point of view, Zickendraht [Zickendraht 1971] derived an expression similar to T_{coll} but where the $\mathcal{L}_{AB}$ were intrinsic quantities such that T_{coll} was strongly coupled to the intrinsic system. It is important to stress out that in the above expansion, (93), the only coupling between T_{coll} and T_{intr} arises through the dependence of $C_{\sigma\lambda}$ on the collective coordinates. There is no explicit dynamical coupling as was the case in the Zickendraht's and Villars's transformations. Similar expression has been given in [Dzyublik et al. 1972], [Weaver et al. 1976] and [Rowe and Rosensteel 1978].

It is interesting to establish a relation between the Villar's kinetic energy derived in section 2.1 and the kinetic energy in eq.(93). In order to do this one must first give a classical interpretation of the opeartors occuring in Villars' transformation. Ignoring the center-of-mass motion, the Villars' tarnsformation can be written as

$$x_{i\alpha} = R_{A\alpha}(\theta_s) x_{iA}(\xi_\sigma) \qquad (99)$$

The Euler angles $\theta_s (s = 1, 2, 3)$ are chosen such that the axes (A, B, C) coincide with the principal axes

$$m R_{A\alpha} R_{B\beta} x_{i\alpha} x_{j\beta} \equiv \delta_{AB} m x_{iA}^2 \equiv \delta_{AB} I_A \qquad (100)$$

Differentiating (97) with respect to time one finds that the components of the space-fixed velocities along the principal axes are given by

$$\dot{x}_{iA} \equiv R_{A\alpha} \dot{x}_{i\alpha} = R_{A\alpha} (\dot{R}_{B\beta} x_{iB} + R_{B\alpha} \dot{x}_{iB}) = \omega_{BA} x_{iB} + \dot{x}'_{iB} \qquad (101)$$

where the three angular momentum velocities ω_{AB} of the principal axes (ABC) are given by

$$\omega_{AB} = -\omega_{BA} = \dot{R}_{A\alpha} R_{B\alpha} \quad (A \neq B) \qquad (102)$$

Using (99), the total angular momentum components are

$$L_{AB} = m(x_{iA} \dot{x}_{iB} - x_{iB} \dot{x}_{iA}) = L'_{AB} + \omega_{AB}(I_A + I_B) \qquad (103)$$

where L'_{AB} is a component of the angular momentum relative to the principal axes

$$L'_{AB} = m(x_{iA} \dot{x}'_{iB} - x_{iB} \dot{x}'_{iA}) \qquad (104)$$

To express ω_{AB} and L'_{AB} in terms of particle coordinates and momenta, and thereby extend their definitions to quantum mechanics, one differentiate (98) with respect to time and obtain

$$\frac{d}{dt}(m R_{A\alpha} R_{B\beta} x_{i\alpha} x_{j\beta}) = 0 \qquad (105)$$

Using the definition (67) for the shear operators one derive

$$\omega_{AB} = \frac{1}{I_A - I_B} t_{AB} \quad (A \neq B) \tag{106}$$

where the single-particle momentum is defined as

$$p_{iA} = m\dot{x}_{iA} \tag{107}$$

Combining (91) and (94), one obtains

$$L'_{AB} = L_{AB} - \omega_{AB}\frac{I_A + I_B}{I_A - I_B} t_{AB} = -\frac{2\sqrt{I_A I_B}}{I_A - I_B}\left(\sqrt{\frac{I_B}{I_A}} x_{iA} p_{iB} + \sqrt{\frac{I_A}{I_B}} x_{iB} p_{iA}\right) \tag{108}$$

The expression (66) in terms of momenta rather than velocities, are valid both in classical and quantum mechanics. However, whereas in classical mechanics $\vec{L}'$ has the significance of the relative angular momentum of the system as seen by an observer moving with the principal coordinate axis (body-fixed), it does not have the structure of an angular momentum operator in quantum mechanics; i.e. its components do not form a closed $SU(2)$ algebra. It is convenient therefore, to express $\vec{L}'$ in terms of the angular momentum $\mathcal{L}$ (vorticity) whose components are given in (84). Equation (106) can then be simply manipulated into the form

$$L'_{AB} = \frac{4 I_A I_B}{(I_A - I_B)^2}\left[\frac{I_A + I_B}{2\sqrt{I_A I_B}}\mathcal{L}_{AB} - L_{AB}\right], \quad (A \neq B) \tag{109}$$

Likewise the shear operators t_{AB} can be expressed

$$t_{AB} = \frac{I_A + I_B}{I_A - I_B}\left(L_{AB} - \frac{2\sqrt{I_A I_B}}{I_A I_B}\mathcal{L}_{AB}\right), \quad (A \neq B) \tag{110}$$

Note that eqs.(106-108) are all valid both in classical and quantum mechanics. Corresponding expressions for the classical angular velocity can be obtained by replacing t_{AB} by $(I_A - I_B)\omega_{AB}$ according to (104). From (96), one then obtain

$$\omega_{AB} = \frac{L_{AB} - L'_{AB}}{I_A - I_B}, \quad (A \neq B) \tag{111}$$

and from (107)

$$\omega_{AB} = \frac{I_A + I_B}{I_A - I_B}\left[L_{AB} - \frac{2\sqrt{I_A I_B}}{I_A + I_B}\mathcal{L}_{AB}\right] \tag{112}$$

Now one can establish the connection between the intrinsic angulat momentum $\vec{L}'$ occuring in the Villars' treatment and the vorticity. From the above considerations one have that

$$L_{AB}^{(Villars)} = L_{AB} = x_{iA} p_{iB} - x_{iB} p_{iA} = R_{A\alpha} R_{B\beta} L_\alpha L_\beta \tag{113}$$

and comparing the Villars' expression for the rotational kinetic energy from (34)

$$T_{rot}^{(Villars)} = \frac{1}{2} \sum_{A<B} \frac{I_A + I_B}{(I_A - I_B)^2} ({L_{AB}}^{(Villars)} - {L'_{AB}}^{(Villars)})^2 \tag{114}$$

with the rotational part of (96)

$$T_{rot} = \frac{1}{2} \sum_{A<B} \frac{I_A + I_B}{(I_A - I_B)^2} \left(L_{AB}^2 - \frac{4\sqrt{I_A I_B}}{I_A + I_B} L_{AB} \mathcal{L}_{AB} + \mathcal{L}_{AB}^2 \right)$$

and using (B.11) from App.B

$${L'_{AB}}^{Villars} = \frac{2\sqrt{I_A I_B}}{I_A + I_B} \mathcal{L}_{AB}$$

one finds that $T_{rot}^{(Villars)}$ contains terms in L_{AB}^2 and $L_{AB}\mathcal{L}_{AB}$ which are identical to those in T_{rot}, but that it differs by a term in $\mathcal{L}_{AB}$

$$T_{rot} = T_{rot}^{(Villars)} + \frac{1}{2} \sum_{A<B} \frac{(\sqrt{I_A} + \sqrt{I_B})^2}{(I_A - I_B)^2} \mathcal{L}_{AB}^2 \tag{115}$$

This is not surprising since, in Villars' classical case, L_{AB} operates on intrinsic coordinates and the extra term is contained in T_{intr}. The important point, however is that whereas in Villars' case the term in $L_{AB}\mathcal{L}_{AB}$ effects a dynamical coupling of the intrinsic and collective degrees of freedom, in the generalized case [Gulshani and Rowe 1976] it is a purely collective operator.

3 ALGEBRAIC APPROACH TO NUCLEAR COLLECTIVE ROTATIONAL-VIBRATIONAL MOTION

3.1 Dynamical symmetry of rotating systems

The solution to two physics problems from the theory of rotational states in atomic nuclei require Lie Group methods:

- What is the shape of a deformed nucleus and how does it rotate?
- What is the fundamental microscopic physics underlying macroscopic nuclear models?

The first question, an important unsolved basic science problem in nuclear structure physics, is pure kinematics. The precise quantitative formulation of this question is phrased in terms of *Casimir invariants* of certain subalgebra of the noncompact symplectic algebra $Sp(3,\mathbf{R})$, [Rosensteel 1992a,b].

The second question is pure theory. The fundamental theory of nuclei, i.e., of many-body systems of strongly interacting neutrons and protons, is the shell-model. However, useful macroscopic models ignore the shell structure of the nucleus and considere it to be a rotating fluid droplet. The relationship betwen these two seemingly incompatible theoretical approaches to nuclear structure is simply the connection between the reducible representations on the *Hilbert Space* of square integrable functions $L^2(\mathbf{R}^{3A})$ of various subalgebra chains of the symplectic algebra and their irreducible representations.

The real symplectic algebra, as represented on $L^2(\mathbf{R}^{3A})$ consists of all hermitian one-body operators that are quadratic in the space and/or momentum operators [Rowe 1985], [Rosensteel and Rowe 1985]. A basis for this 21-dimensional noncompact algebra is furnished by the shear tensor N_{ij}, and the symmetric inertia tensor Q_{ij} and kinetic T_{ij} tensors, where i,j index the cartesian axes x,y,z and

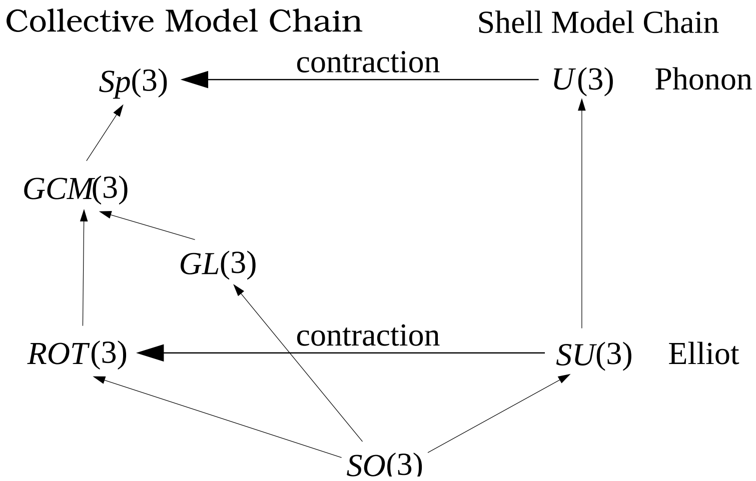
$$N_{ij} = \sum_{\alpha} x_{\alpha i} p_{\alpha j} \qquad (116)$$

$$Q_{ij} = \sum_{\alpha} x_{\alpha i} x_{\alpha j} \qquad (117)$$

$$T_{ij} = \sum_{\alpha} p_{\alpha i} p_{\alpha j} \qquad (118)$$

The sums are performed over the particle index $\alpha = 1,2,...,A$. The kinetic and inertia tensors each span abelian subalgebras isomorphic to $\mathbf{R}^6$. The N_{ij} span the Lie algebra of the linear motion group, $GL(3,\mathbf{R})$. One show bellow the two principal $Sp(3,\mathbf{R})$ subalgebra chains that terminate with the orbital angular momentum subalgebra $SO(3)$.

The collective model chain passes through the general collective motion algebra $GCM(3)$ and the rotational algebra $ROT(3)$. The shell model chain traverses



the symmetry algebra of the isotropic harmonic oscillator $SU(3)$. The defining generators for each of these subalgebras are as follows:

$$\begin{aligned} SO(3) &:= \mathrm{span}_{\boldsymbol{R}} i(N - N^{+})_{ij} \\ GL(3) &:= SO(3) \oplus \mathrm{span}_{\boldsymbol{R}} i(N + N^{+})_{ij} \\ GCM(3) &:= GL(3) \oplus \mathrm{span}_{\boldsymbol{R}} Q_{ij} \\ SU(3) &:= SO(3) \oplus \mathrm{span}_{\boldsymbol{R}} (Q^{(2)} + T^{(2)})_{ij} \end{aligned}$$

where $\oplus$ denotes vector space direct sum and rank 2 tensors are formed from the traceless part of the corresponding operators, for example, the mass quadrupole operator is

$$Q_{ij}^{(2)} = Q_{ij} - \delta_{ij} tr(Q)/3 = \sum_{\alpha} (x_{\alpha i} x_{\alpha j} - \delta_{ij} r_{\alpha}^2/3)$$

The subalgebras in the geometrical model chain are essentially kinematical in origin. $SO(3)$ and $GL(3)$ are Lie algebras of motion (dynamic) group acting on three-dimensional Euclidean space. $ROT(3)$ and $GCM(3)$ add the inertia tensor to the corresponding Lie algebras of the motion groups $SO(3)$ and $GL(3)$. But, the inertia tensor measures the spatial extension and deformation of a body, since its eigenvalues are the principal moments of the inertia ellipsoid. Thus, $ROT(3)$ and $GCM(3)$ are kinematical algebras too.

The shell-model chain provides a dynamical component to the theory via the harmonic oscillator Hamiltonian $H_0 = (tr(T) + tr(Q))/2 = \sum_{\alpha} (p_{\alpha}^2 + x_{\alpha}^2)/2$ and its symmetry algebra SU(3). Transformations from the shell model chain to the geometrical model chain provide a kinematical interpretation to shell model configurations. As every body know Elliot's model $SU(3)$ gives quasi-rotational bands which approach those of the rotor model in the limit of large-dimensional representations(contraction to $ROT(3)$). There is the problem that $SU(3)$ states needs to be renormalized by coupling to higher shells to get $B(E2)$ transitions up to observed values. Generalizing the $SU(3)$ model to the symplectic model one get symplectic states that are shell model configurations and eigenstates of the

Harmonic Oscillator Hamiltonian H_0 carrying good Elliot SU(3) symmetry (λ, μ) [Talmi 1993].

Thus, the shell model chain provides the connection with microscopic fermion physics and it enables detailed shell model calculations to be performed for geometrical collective states. From the view point of this chain, the symplectic model is an extension of the conventional oscillator shell model that goes beyond single major shells to include the $np - nh$ coherent admixtures required to build quadrupole and monopole colectivity.

3.2 $ROT(3)$-the dynamical symmetry of the rigid rotator

The concept of a body with a non-spherical shape presupposes that this shape is in some sense measurable. This, in turn, requires - by the uncertainty relations for the angular momentum - that infinitely many angular momenta are necesarily in order to specify the shape. If the shape of the body is rigid, these angular momenta must not dynamically affect the shape being mesured, so that the shape can be considered as fixed over the entire range of energy. In order that the shape is measurable in quantum mechanical framework, there must exist a set of quantum mechanical operators associated with the shape of the body. The eigenvalues of these operators will correspond to the result of measurement of the shape.

From the above consideration, it is clear that these shape operators does not commute with the angular momentum operators nor even with the Hamiltonian of the system. Therefore, the shape operators cannot be generators of any symmetry groups of the system, but rather constitute those of the dynamical group. If any of the shape operators does not commute with each other, it is then impossible to diagonalize these operators simultaneously. This implies that the shape cannot accurately be determined without uncertainty in this case. Thus, it is natural to define the concept of the *rigid* shape as being the condition that the quantum mechanical operators associated with the shape can be simultaneously diagonalized. In other words, the shape may be called *quantum mechanically rigid*, if and only if the non-spherical shape is in principle measurable without any quantum mechanical uncertainties[Ui 1970].

As the quantum mechanical shape operators are taken the mass multipole operators which may be obtained from the density distribution of the body in the usual way. Then, for a quadratically deformed shape such as the symmetric and asymmetric tops, the five components of the (mass) quadrupole moment are sufficient to define the shape.

In order to have a rigid shape, every component of the operator must commute with each other. Since the quadrupole moment transforms as an irreducible tensor of rank 2 under rotation of ordinary space, the commutation relation are needful. Take the quadrupole moment as

$$Q_{\mu}^{(2)} = \sum_{\alpha} r_{\alpha}^{2} Y_{2\mu}(\theta_{\alpha}, \phi_{\alpha}) \tag{119}$$

Since for an arbitrary irreducible tensor of rank λ the following relations are valide [Rose 1957]

$$[L_{\pm 1}, Q_{\lambda\mu}] = \sqrt{(\lambda \mp \mu)(\lambda \pm \mu + 1)} Q_{\lambda\mu\pm 1} \tag{120}$$

$$[L_z, Q_{\lambda\mu}] = \mu Q_{\lambda\mu} \tag{121}$$

where $L_{1\nu}(\nu = 0, \pm 1)$ are the spherical components of the angular momentum, then replacing $Q_{\lambda\mu}$ by Q^2_μ and using the commutation relations for the $SO(3)$ algebra of the angular momentum one obtains the following set of commutation relations for the $ROT(3)$ algebra

$$[L_z, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = 2L_z, \tag{122}$$

$$[L_z, Q^{(2)}_\mu] = \mu Q^{(2)}_\mu, \quad [L_\pm, Q^{(2)}_\mu] = \sqrt{(2 \mp \mu)(3 \pm \mu)} Q^{(2)}_{\mu\pm 1} \tag{123}$$

and

$$[Q^{(2)}_\mu, Q^{(2)}_{\mu'}] = 0 \tag{124}$$

Therefore the rotational algebra $ROT(3)$ is spanned by the one-body quadrupole operator $Q^{(2)}_\mu$ plus the angular momentum algebra $SO(3)$. In its quantum realisation $ROT(3)$ is the adiabatic roational model [Ui 1970], [Weaver et al. 1973], while, in its classical realisation $ROT(3)$ is the Euler rigid body model [Corben 1968]. This Lie algebra defines an eight- parameter $\{L_0, L_{\pm 1}, {Q_{\pm 2}}^{(2)}, {Q_{\pm 1}}^{(2)}, Q_0\}$ noncompact Lie group with semidirect product structure. On the other hand, the well known Lie algebra of the $SU(3)$ group is expressed in Racah's spherical basis as [Eisenberg & Greiner 1970c]

$$\begin{aligned}
[L_z, L_\pm] &= \pm L_\pm, \quad [L_+, L_-] = 2L_z && (125)\\
[L_z, Q^{(c)}_\mu] &= \mu Q^{(c)}_\mu, \quad [L_\pm, Q^{(c)}_\mu] = \sqrt{(2 \mp \mu)(3 \pm \mu)} Q^{(c)}_{\mu\pm 1} && (126)\\
[Q^c_\mu, Q^{(c)}_{\mu'}] &= 3\sqrt{10}\, C^{2\ 2\ \ 1}_{\mu\ \mu'\ \mu+\mu'} L_{\mu\pm\mu'} && (127)
\end{aligned}$$

where

$$Q^{(c)}_\mu = Q^{(2)}_\mu(\hat{\boldsymbol{r}}) + Q^{(2)}_\mu(\hat{\boldsymbol{p}}) \tag{128}$$

Therefore eqs.(119-121) can be obtained from eqs.(122-124) by the procedure [Gilmore 1974], [Barut and Raczka 1977] of *contraction*: first, put $Q^{(c)}_\mu$ and next, take the limit $\epsilon \to 0$ keeping T_μ finite and thus

$$[T_\mu, T_{\mu'}] = \epsilon^2 [Q^{(c)}_\mu, Q^{(c)}_{\mu'}] = O(\epsilon^2) \approx 0 \tag{129}$$

The Casimir invariants of $ROT(3)$ can be obtained from the casimir operators [Wybourne 1974] of the $SU(3)$ group

$$C_2 = (\hat{L} \times \hat{L}) + \frac{1}{3}(\hat{Q}^{(2)} \cdot \hat{Q}^{(2)}) \tag{130}$$

$$C_3 = [\hat{Q}^{(2)} \times \hat{Q}^{(2)} \times \hat{Q}^{(2)}]^{(0)}_0 + 3\sqrt{\frac{3}{7}} [\hat{Q}^{(2)} \times \hat{L} \times \hat{L}]^{(0)}_0 \tag{131}$$

by the Wigner procedure of contraction above mentioned

$$C_2 = \sum_{\mu} C^{2\ 2\ \ 0}_{\mu -\mu'\ 0} Q^{(2)}_{\mu} Q^{(2)}_{-\mu} \tag{132}$$

and

$$C_3 = \sum_{\mu_1\mu_2\mu_3} C^{\ 2\ \ 2\ \ 2}_{\mu_1\ \mu_2\ \mu_2}\ C^{\ 2\ \ \ 2\ \ 0}_{\mu_3\ -\mu_3\ 0}\ Q^{(2)}_{\mu_1} Q^{(2)}_{\mu_2} Q^{(2)}_{-\mu_3} \tag{133}$$

3.3 The group of linear collective flows $SL(3,\boldsymbol{R})$

One finds a more interesting algebraic structure if one consider in addition the time derivatives of the quadrupole moment

$$\dot{Q}^{(2)}_{\mu} = \frac{i}{\hbar}[H, Q^{(2)}_{\mu}] \tag{134}$$

where H is the Hamiltonian. If the potential energy is velocity independent (or more precisely to be not of spin-orbit type)then

$$\dot{Q}^{(2)}_{ij} = \sum_{\alpha}(x_{\alpha i}p_{\alpha j} + x_{\alpha j})p_{\alpha i} - \frac{2}{3}\delta_{ij}\boldsymbol{r}_{\alpha}\boldsymbol{p}_{\alpha} = S^{(2)}_{ij} \tag{135}$$

is called the shear momentum.

The commutator of two ${S_{ij}}^{(2)}$ is an orbital angular momentum, e.g.

$$[S_{ij}, S_{kl}] = -i\hbar(\delta_{jl}L_{ik} + \delta_{il}L_{jk} + \delta_{jk}L_{li} + \delta_{ik}L_{jl}) \tag{136}$$

where

$$L_{ij} = \sum_{\alpha}(x_{\alpha i}p_{\alpha j} + x_{\alpha j}p_{\alpha i}) \tag{137}$$

is the angular momentum operator. The commutator of the angular momentum and the shear operator is

$$[S_{ij}, L_{kl}] = -i\hbar(\delta_{jl}S_{ik} + \delta_{il}S_{jk} + \delta_{jk}S_{li} + \delta_{ik}S_{jl}) \tag{138}$$

Since the L_{ij} spans the subalgebra $SU(3)$ one are then lead to the conclusion that the five S_{ij} and three L_i generate the $SL(3)$ algebra. The essential commutation rules basis are

$$[L_0, L_{\pm}] = \pm L_{\pm}, \quad [L_+, L_-] = 2L_0, \quad [L_0, S_{\mu}] = \mu S_{\mu} \tag{139}$$

$$[L_{\pm}, S_{\mu}] = \sqrt{(6 - \mu(\mu \pm 1))}S_{\mu\pm 1}, \quad [S_{\mu}, S_{-\mu}] = -4L_0 \tag{140}$$

where eqs.(117) and (118) have been used.

To explore the physical consequence of the above commutation relations one make the following assumptions:

- The time derivatives of the mass quadrupole operator of the nucleus generate the algebra of $SL(3,\boldsymbol{R})$

- The electric quadrupole moment is proportional to the mass quadrupole moment
- The states of the nucleus form a basis for one irreducible unitary representation of $SL(3)$.

  $SL(3,\boldsymbol{R})$ is a non-compact group. This means that the group volume is infinite. To see this consider the realisation of an element of $SL(3,\boldsymbol{R})$ by a 3×3 real matrix with unit determinant. The entries in this matrix are bounded-they may range between $-\infty$ and $+\infty$.

3.4 The rotational-vibrational collective motion group $CM(3)$

The $CM(3)$ model has its counterpart in the classical Riemann model of rotating fluids [Chandrasekhar 1968]. It is the extension of the motion group of $ROT(3)$ from $SO(3)$ to the group of linear transformations $SL(3)$ and incorporating the monopole operator $\sum_\alpha {r_\alpha}^2$. The resulting Lie algebra $CM(3)$ allows for the continous range of rotational dynamics from rigid rotation to irrotational flow [Cusson 1968], [Weaver, Cusson and Bidenharn 1976], [Rosensteel and Rowe 1976]. Thus the $CM(3)$ is obtained by adjoining the 6 quadrupole moments Q_{ij} to the generators L_{ij} and S_{ij} of $SL(3)$. The condition of trace zero may be relaxed temporary.

The first who pointed out the existence of this symmetry was Tomonaga [Tomonaga 1955]. For an irrotational displacement of a system of particles in an incompressible fluid the velocity field can be derived from a potential Φ which satisfies the Laplace equation

$$\Delta\Phi = 0 \tag{141}$$

such that

$$\vec{v} = \nabla\Phi \tag{142}$$

The infinitesimal displacement of a particle at the position (x, y) is then given by

$$\delta x_\alpha = \epsilon\frac{\partial\Phi(x_\alpha, y_\alpha)}{x_\alpha} \tag{143}$$

$$\delta y_\alpha = \epsilon\frac{\partial\Phi(x_\alpha, y_\alpha)}{y_\alpha} \tag{144}$$

Integrating the Laplace equation one obtain for example the quadrupole potential

$$\Phi_1 = \frac{1}{2}(x^2 + y^2) \tag{145}$$

which gives

$$x_\alpha = \frac{\partial\Phi(x_\alpha, y_\alpha)}{x_\alpha} \tag{146}$$

$$y_\alpha = -\frac{\partial\Phi(x_\alpha, y_\alpha)}{y_\alpha} \tag{147}$$

Substituting these last two equations into eqs.(141-142) one get

$$x'_\alpha = (1+\epsilon)x_\alpha \tag{148}$$
$$y'_\alpha = (1-\epsilon)y_\alpha \tag{149}$$

Associated to this transformation one have the infinitesimal generator [Wybourne 1974]

$$X_1 = x_\alpha \frac{\partial}{\partial x_\alpha} - y_\alpha \frac{\partial}{\partial y_\alpha} \tag{150}$$

Integrating again the Laplace equation one get

$$\Phi_2 = xy \tag{151}$$

which gives after the same algebra manipulations

$$X_2 = x_\alpha \frac{\partial}{\partial y_\alpha} + y_\alpha \frac{\partial}{\partial x_\alpha} \tag{152}$$

One thus obtain the two generators of the two-dimensional quadrupole irrotational flow

$$P_1 = \sum_\alpha x_\alpha P_{x\alpha} - y_\alpha P_{y\alpha} \tag{153}$$
$$P_2 = \sum_\alpha x_\alpha P_{x\alpha} + y_\alpha P_{y\alpha} \tag{154}$$

They generate together with the angular momentum the algebra $SL(2) = SO(2) \oplus span_{\mathbf{R}}\{P_1 + P_2\} = span_{\mathbf{R}}\{L_z, P_1, P_2\}$ and satisfies the commutation relations

$$[P_1, P_2] = 2i\hbar L_z, \; [P_2, L_z] = -2i\hbar P_1, \; [L_z, P_1] = -2i\hbar L_z \tag{155}$$

Therefore if the system is initially described by a representation of $SL(2,R)$ these motions will not carry the system outside the representation. According to Tomonaga procedure, considering the momenta P_1 and P_2, one look for the conjugate coordinates Q_1 and Q_2

$$Q_1 = \frac{1}{2}\sum_\alpha (x_\alpha^2 - y_\alpha^2) \tag{156}$$
$$Q_2 = \sum_\alpha x_\alpha y_\alpha \tag{157}$$

Commuting with the generators of $SL(2,R)$ algebra one gets

$$[P_1, Q_2] = 0, \quad [P_2, Q_1] = 0, \quad [P_1, Q_1] = -i\hbar R^2, \quad [P_2, Q_2] = -i\hbar R^2 \tag{158}$$

where

$$R^2 = \sum_\alpha (x_\alpha^2 + y_\alpha^2)$$

Thus (P_α, Q_α) does not form a conjugate couple. However Tomonaga stressed that if the system is large enough the fluctuations in R^2 are small, i.e. $R^2 \approx {R_0}^2$ is a c-number. If not one obtains the closed algebra $CM(2) = SL(2, \mathbf{R}) \oplus \{Q_1, Q_2, R^2\}$. In the limit case one obtains the contraction of $CM(2)$: Introducing the scaled momenta $\Pi_2 = P_\alpha / R_0^2$, the commutation relations becomes[Weaver 1980]:

$$[\Pi_\alpha, Q_\beta] = \frac{1}{{R_0}^2}[P_\alpha, Q_\beta] \approx -i\hbar\delta_{\alpha\beta}, \quad [\Pi_\alpha, \Pi_\beta] = [Q_\alpha, Q_\beta] = 0 \tag{159}$$

and one concludes that for a system with small fluctuations and with large R_0^2, truly canonical coordinates and momenta emerge to the generators of Heisenberg-Weyl $\mathcal{W}$ [Weyl 1931].

4 QUANTUM MECHANICS TREATMENT OF FLOWS IN ROTATING FRAMES

4.1 The Hamiltonian in a rotating system

Let H_0 be the Hamiltonian in the stationary frame (laboratory fixed) and

$$j_3 = \sum_i (x_i p_{yi} - y_i p_{xi}) \tag{160}$$

the angular momentum of the system about the axes of rotation, which one take to be the z-axis. The Hamiltonian in the rotating frame is given by

$$H = H_0 - \lambda j_3 \tag{161}$$

This hamiltonian can be derived in several ways, each of which gives different insights.

In the Inglis cranking model H_0 is an independent-particel Hamiltonian

$$H_0 = \frac{1}{2m} p^2 + V_0(\vec{r}) \tag{162}$$

where

$$\vec{r}(t) = \widehat{\mathbf{R}}(\omega t) \cdot \vec{r} \tag{163}$$

with $\widehat{\mathbf{R}}$ an orthogonal rotation matrix. By a transformation to rotating coordinates, one derives the Hamiltonian.

Alternatively, one may seek a time-dependent solution to the Schroedinger equation

$$H(t)\psi(t) = (e^{-i\omega t j_3/\hbar} H_0 e^{i\omega t j_3/\hbar})\psi(t) = -i\hbar \frac{\partial \psi(t)}{\partial t} \tag{164}$$

which is stationary in the rotating frame. This done by writting the wave function

$$\psi(t) = e^{-i\omega t j_3/\hbar} \phi(t) \tag{165}$$

so that ϕ_t is a solution of the wave function

$$(H_0 - \omega j_3)\phi(t) = i\hbar \frac{\partial \phi(t)}{\partial t} \tag{166}$$

The requirement that $\psi(t)$ should be stationary in the rotating frame implies that $\phi(t)$ must be an eigenstate of $H_0 - \omega j_3$.

Finally a more fundamental derivation reveals that $H = H_0 - \omega j_3$ is in fact the hamiltonian for the system in the rotating frame and that $-\omega j_3$ is just the term needed to include the effects of the centrifugal and the Coriolis force. To derive the hamiltonian one resort to the textbook [Goldstein 1959]. The initial step consists in relating the velocities of a particle relative to the space $\vec{v}_s$ and rotating $\vec{v}_r$ set of axes respectively

$$\vec{v}_s = \vec{v}_r + \vec{\omega} \times \vec{r} \tag{167}$$

Analogously the time rate of change of $\vec{v}_s$ is

$$(\frac{d\vec{v}_s}{dt})_s \equiv \vec{a}_s = (\frac{d\vec{v}_s}{dt})_r + \vec{\omega} \times \vec{v}_s = \vec{a}_r + 2(\vec{\omega} \times \vec{r}) + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{168}$$

where

$$\vec{a}_r = \frac{d\vec{v}_r}{dt} \tag{169}$$

Finally, the equation of motion, which in the inertial system is simply

$$\vec{F} = m\vec{a}_s \tag{170}$$

expands, when expressed in the rotating coordinates, into the equation

$$\vec{F} = m[\vec{a}_r + 2(\vec{\omega} \times \vec{v}_r) + \vec{\omega} \times (\vec{\omega} \times \vec{r})] \tag{171}$$

To an observer situated in the rotating system it therefore appears as if the particle is moving under the influence of an effective force $\vec{F}_{eff}$. If $\vec{F}$ derives from a potential V_0 then

$$\vec{F}_{eff} = -\nabla V_0 - 2m(\vec{\omega} \times \vec{v}_r) - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{172}$$

In order to derive the Hamiltonian from $\vec{F}_{eff}$ in a frame rotating with angular velocity $\vec{\omega}$, one must first be able to derive $\vec{F}_{eff}$ like $\vec{F} = -\nabla V_0$ from a potential. The centrifugal term in $\vec{F}$ can be obtained from the gradient of the centrifugal potential

$$U_{cf} = -\frac{1}{2}\vec{\omega} \cdot \mathcal{I}_{RR} \cdot \vec{\omega} \tag{173}$$

where the rigid-body tensor has components

$$(\mathcal{I}_{RR})_{ij} \equiv \ m \ (\delta_{ij} r^2 - r_i r_j) \tag{174}$$

The Coriolis force, however, cannot be derived from the gradient of a potential. It can, however, be derived from a generalized potential in exact analogy with the magnetic fiels force. A constant magnetic field $\vec{B}$ can be related to the magnetic vector potential $\vec{A}$ through the expression

$$\vec{A} = \frac{\vec{B} \times \vec{r}}{2} \tag{175}$$

The magnetic force

$$\vec{F}_{mag} = -\frac{q}{c}\vec{B} \times \vec{v} = \frac{q}{c}\nabla(\vec{v} \cdot \vec{A}) = -\nabla(\frac{q}{2c}\vec{v} \cdot (\vec{B} \times \vec{r})) \tag{176}$$

then admits a generalized potential [Goldstein 1959]

$$V_{mag} = -\frac{q}{2c}\vec{v} \cdot (\vec{B} \times \vec{r}) \tag{177}$$

giving the correct equations of motions when used in the lagrangian. One thus established an analogy between the Lorentz and Coriolis potentials. Similarily, then, we can define for the Coriolis term in $\vec{F}$ the generalized potential

$$V_{cor} = -m\vec{v} \cdot (\vec{\omega} \times \vec{r}) \tag{178}$$

The total potential for the system in the rotating system is then

$$U_{eff} = V_0 - \frac{1}{2}\vec{\omega} \cdot \mathcal{I}_{RR} \cdot \vec{\omega} - m\vec{v} \cdot (\vec{\omega} \times \vec{r}) \tag{179}$$

Introducing the Lagrangian

$$L = \frac{1}{2}mv^2 - U_{eff} \tag{180}$$

one obtains the canonical momentum

$$\vec{p} \equiv \vec{\nabla}_{\vec{v}} L = m(\vec{v} + \vec{\omega} \times \vec{r}) \tag{181}$$

and finally the Hamiltonian

$$H = \vec{v} \cdot \vec{p} - L = \frac{1}{2m}(\vec{p} - m\omega \times \vec{r})^2 + V_0 - \frac{1}{2}\vec{\omega} \cdot \mathcal{I}_{RR} \cdot \vec{\omega} = H_0 - \omega j_3 \tag{182}$$

One observes that the Coriolis potential does not appear in the Hamiltonian, which is as it should be since the Coriolis force does not work and cannot contribute directly to the energy of a particle. It does, however, affect the trajectory of the classical motion as it enters into the equations of motion through the canonical momentum.

4.2 Rigid flow and non-integrable phases in quantum mechanics

In the cranking model one computes the wave function $\mid \phi \rangle$ in the rotating frame to first order in ω in using perturbation theory. One then defines the mean inertial parameter I and the energy increment ΔE by

$$\omega I = \langle\ \psi(t) \mid j_3 \mid \psi(t)\ \rangle \tag{183}$$

$$\Delta E = \langle\ \psi(t) \mid H(t) \mid \psi(t)\ \rangle - \langle\ \phi_0 \mid H_0 \mid \phi_0\ \rangle \tag{184}$$

where $\mid \phi_0 \rangle \equiv \mid \phi(\omega = 0) \rangle$. for the anisotropic oscillator and under the conditions of self-consistency it is well known that

$$\langle\ \phi \mid j_3 \mid \phi\ \rangle = \omega\ I_{RR} \tag{185}$$

and

$$\Delta E = \frac{1}{2}\omega^2 I_{RR} \tag{186}$$

One might thereby conjecture that the current flows might also be rigid. Indeed, as will be see later, it is even possible to derive a cranking model wave function that gives precisely rigid-flow currents. However, one can do so only at the expenses of departing from the conventional requirement of quantum mechanics that the wave function be a well-defined single-valued function on the configuration space.

Since the Schroedinger equation

$$H|\ \phi\ \rangle = \left[\frac{1}{2m}(\vec{p} - m\omega \times \vec{r})^2 + V_0 - \frac{1}{2}\vec{\omega} \cdot \mathcal{I}_{RR} \cdot \vec{\omega}\right]|\ \phi\ \rangle = E|\ \phi\ \rangle \tag{187}$$

of the cranking model is gauge invariant, one can make the gauge transformation

$$\phi(\vec{r}) = \exp\left(\ \frac{i}{\hbar m}\ S(\vec{r})\ \right)\phi'(\vec{r}) \tag{188}$$

to obtain

$$[\frac{1}{2m}(\vec{p} - m\omega \times \vec{r} + m\nabla S)^2 + V_0 - \frac{1}{2}\omega\mathcal{I}_{RR}\omega]|\ \phi'\ \rangle = E|\ \phi'\ \rangle \tag{189}$$

and

$$\langle\ \phi\ |\ j_3\ |\ \phi\ \rangle = \omega\ \langle\ \phi'\ |\ \hat{\mathcal{I}}_{RR}\ |\ \phi'\ \rangle \equiv \omega\ \langle\ \phi_0\ |\ \hat{\mathcal{I}}_{RR}\ |\ \phi_0\ \rangle \tag{190}$$

where it was assumed that $\langle\ \phi'\ |\ j_3\ |\ \phi'\ \rangle = 0$. Similarly the energy increment becomes

$$\Delta E = \frac{1}{2}\omega^2\ \langle\ \phi_0\ |\ \hat{I}_{RR}\ |\ \phi_0\ \rangle + \langle\ \phi'\ |\ H_0\ |\ \phi'\ \rangle - \langle\ \phi_0\ |\ H_0\ |\ \phi_0\ \rangle \tag{191}$$

which is the rigid flow kinetic energy plus the energy increment induced by centrifugal stretching. The current density, defined at time $t = 0$ by

$$\vec{J}(\vec{r}) \equiv \langle\ \phi\ |\ \hat{J}(\vec{r})\ |\ \phi\ \rangle \tag{192}$$

where

$$\hat{J}(\vec{r}) \equiv \frac{1}{2m}[\delta(\vec{r} - \vec{r}')\ \vec{p} + \vec{p}\ \delta(\vec{r} - \vec{r}')] \tag{193}$$

becomes

$$\vec{J}(\vec{r}) = \langle\ \phi'\ |\ e^{-\frac{im}{\hbar}S}\frac{1}{2m}[\delta(\vec{r} - \vec{r}')\vec{p} + \vec{p}\delta(\vec{r} - \vec{r}')]e^{\frac{im}{\hbar}S}\ |\ \phi\ \rangle = |\phi'(\vec{r})|^2\nabla S(\vec{r}) \tag{194}$$

In order to take into account the entire effect of the Coriolis term one choose the phase function such that

$$\vec{J}(\vec{r}) = |\phi'(\vec{r})|^2(\vec{\omega} \times \vec{r}) \tag{195}$$

Thus, without any conditions of self-consistency, one obtain rigid- flow results for any potential V_0. However for a quantal fluid the above analysis is unacceptable because $\nabla S = \vec{\omega} \times \vec{r}$ and then

$$\nabla \times \nabla S = \nabla \times (\vec{\omega} \times \vec{r}) = 2\vec{\omega} \tag{196}$$

It means that the Schwartz condition is not fulfiled

$$\frac{\partial^2 S}{\partial x \partial y} = +\omega =\neq \frac{\partial^2 S}{\partial y \partial x} = -\omega \tag{197}$$

and thus $S(\vec{r})$ is not integrable. In other words, the integrale

$$S(\vec{r}) = \int (\vec{\omega} \times \vec{r}') d\vec{r}' \tag{198}$$

is path dependent, implying that the wave function $\phi(\vec{r})$ defined by the gauge transformation is not single valued. In this context, recall the discussion in section 2.3 and 2.4 about the integrability of rotational collective coordiantes [Rowe 1970], [Gulshani and Rowe 1976,1977a,b].

It is important to point out that the current is not of the rigid flow type when the moment of inertia is assumed to take the rigid-body value.

4.3 The vortex flow of a single-particle fluid

For an arbitrary single-particle wave function $\psi = u + iv$ one can define the current

$$\vec{J}(\vec{r}) = \frac{1}{2mi}[\psi^*(\vec{r})\vec{p}\psi(\vec{r}) - \vec{p}\psi^*(\vec{r})\psi(\vec{r})] \tag{199}$$

and the velocity field

$$\vec{v}(\vec{r}) \equiv \frac{1}{\rho(\vec{r})}\vec{J}(\vec{r}) \tag{200}$$

where the density $\rho(\vec{r}) \equiv |\psi|^2 = u^2 + v^2$. The field velocity $\vec{v}(\vec{r})$ is well defined at all poinrs where $\psi = 0$. Now it is easy to show that at such points

$$\nabla \times \vec{v} = (\nabla \frac{1}{\rho}) \times \vec{J} + \frac{1}{\rho} \nabla \times \vec{J} \tag{201}$$

This result indicates that the velocity-flow of any single-particle fluid is irrotational whenever the wave function does not vanish. This is a familiar result in the fluid dynamical description of a single-particle wave function [Landau and Lifschits 1967], [Kan and Griffin 1977,1978], [Gulshani and Rowe 1977a,b] where it is seen to be a simple consequence of the fact that $\psi(\vec{r})$ can be written in the polar form

$$\psi(\vec{r}) = \chi(\vec{r}) \exp[\frac{im}{\hbar} S(\vec{r})] \tag{202}$$

χ and S are defined as the smooth functions satisfying the equations

$$\rho^2(\vec{r}) = \chi^2 = u^2 + v^2, \quad tg \frac{m}{\hbar} S = \frac{v}{u} \tag{203}$$

at all points at which $\psi(\vec{r})$ does not vanish. Thus it follows that

$$\vec{J}(\vec{r}) = \rho^2(\vec{r}) \nabla S(\vec{r}) \tag{204}$$

The field velocity is now given by

$$\vec{v}(\vec{r}) = \nabla S = \frac{\hbar}{m}\frac{u\nabla v - v\nabla u}{u^2+v^2} \tag{205}$$

Since $\vec{J} = \rho = 0$ at points where $\psi(\vec{r}) = 0$, it follows that at these points $\vec{v}(\vec{r}) = \nabla S$ is not defined. However, at such points the current circulation defined by

$$\nabla \times \vec{J} = 2\chi\nabla\chi \times \nabla S = \frac{2\hbar}{m}\nabla u \times \nabla v \tag{206}$$

does not necesarily vanish. The non-vanishing current circulations at these points may then be interpreted as velocity *vortices* [Kan and Griffin 1977].

From eq.(202) one sees that $\vec{v}(\vec{r})$, which is an irrotational velocity field can become singular only when χ(and hence ψ, or u and v) go to zero. This happens, for example, when the nodal surfaces of u intersect the nodal surfaces v. Then ψ is zero on the lines of intersection. This leads to line singularities in $\vec{v}(\vec{r})$. S which is a multivalued function can be made single valued on the principal branch of arctan, i.e. $-\frac{\pi}{2} \leq \arg(\arctan) \leq \frac{\pi}{2}$. With this choice, terms involving the Heaviside function are introduced in order to satisfy the convention that χ in eq.(199) be positive. From eq.(200) then

$$S = \frac{\hbar}{m}(\arctan\frac{v}{u} + \pi\theta(u)) \tag{207}$$

where $\theta(x)$ describes a unit jump discontinuity at $x = 0$.

Consider the line integral of $\vec{v}(\vec{r})$ along a closed path Γ in space. Assume that on Γ, $\vec{v}(\vec{r})$ has no singularity. Since $\vec{v} = \nabla S$, such that a closed line integral is equal to the sum of discontinuities which S may posses along Γ. As noted above in eq.(204), discontinuities of S must have value magnitude $2\pi\hbar/m$. This implies that any closed line integral of $\vec{v}(\vec{r})$ is quantized and

$$\oint \vec{v}(\vec{r})d\vec{l} = \frac{2n\pi\hbar}{m} \tag{208}$$

where n is an integer. This quantization condition for circulation is well known[Kan and Griffin 1977]. When Γ encircles no singularities of $\vec{v}(\vec{r})$, one must have $n = 0$ because the left-handside is evidently zero according to Stokes theorem:

$$\oint \vec{v}(\vec{r})d\vec{l} = \int \nabla \times \vec{v}(\vec{r})d\vec{s} = 0 \tag{209}$$

When Γ encircles a line of singularities of $\vec{v}(\vec{r})$, the line integral (205) is generally nonzero. Then, if one let the dimension of Γ go to zero, one conclude that $\vec{v}(\vec{r})$ must have an unbounded curl(vorticity) on the line of singularity, which, as one alrady noted, is also the nodal line of ψ. Following the terminology of classical fluid dynamics one denote such a line of singularity of vorticity distribution as a *line of vortex* [Batchelor 1967].

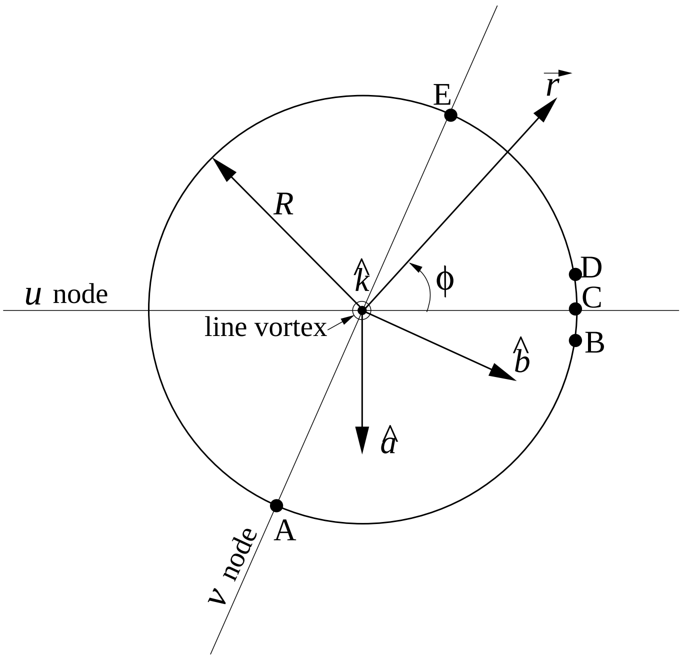


Figure 2: Relationship between a line vortex and the nodal surfaces of the real part u and the imaginary part v of the wave function. Unit vectors $\hat{a}$ and $\hat{b}$ are the normals of the u node and v mode. The line vortex is along the polar axis k, which is pointing perpendicular outwards from the page.

It is worthwhile to make an anlogy with the dislocation theory in solids. From macroscopic point of view, the deformation generated by the dislocations in a continuum, posses in the general case the following property: when describing a closed path Γ around a *line of dislocation* D, the vector $\vec{u}$ of elastic displacement performs a determined finite increasement $\vec{b}$ equal in magnitude and direction with one of the lattice periods. The constant vector $\vec{b}$ is called *Burgers vector* of the given dislocation. This property can be expressed [Landau and Lifschits 1990]

$$\oint_\Gamma du_i = \oint_\Gamma \frac{\partial u_i}{\partial x_k} dx_k = -b_i \tag{210}$$

The dislocation line represent the line of singularities of the deformed field. The two ends of these line must be at the surface of the crystal or must be joined on a closed loop.

Consider now a region in which a nodal surface of u intersects a nodal surface of v. Choose an arbitrary point P along such a nodal line of ψ and consider the irrotational velocity $\vec{v} = \nabla S$ near this point. let $\vec{r}$ be the position vector measured with respect to this point.

Consider first the simplest situation where both u and v vary linearly within a small neighborhood of P, i.e.

$$u(\vec{r}) \cong \nabla u|_{r=0} \cdot \vec{r} = \vec{a} \cdot \vec{r} \tag{211}$$

and

$$v(\vec{r}) \cong \nabla v|_{r=0} \cdot \vec{r} = \vec{b} \cdot \vec{r} \tag{212}$$

From eq.(205) one obtains

$$\vec{v}(\vec{r}) = \frac{\hbar}{m} \frac{u(\vec{r})\nabla v(\vec{r}) - v(\vec{r})\nabla u(\vec{r})}{u^2 + v^2} \cong \frac{\hbar}{m} \frac{\vec{r} \times (\vec{b} \times \vec{r})}{(\vec{a} \cdot \vec{r})^2 + (\vec{b} \cdot \vec{r})^2} \tag{213}$$

Let the z-direction $\vec{k}$ be the direction $\vec{a} \times \vec{b}$. The normals $\vec{a}$ and $\vec{b}$ then lie in the xy-plane. Let the azimuthal angles of $\vec{a}$ and $\vec{b}$ be φ_a and φ_b, respectively. Equation (210) then becomes

$$\vec{v}(\vec{r}) \cong \frac{\hbar}{m} g(\theta, \varphi) \frac{\vec{k} \times \vec{r}}{r} \tag{214}$$

From the last equation, one sees that the irrotational velocity field $\vec{v}(\vec{r})$ has the following two properties:

- $\vec{v}$ varies as r^{-1} for $r \to 0$. hence, the irrotational field is singular on the nodal lines of ψ.
- since $\vec{v} \parallel \vec{k} \times \vec{r}$, the stream lines(lines in the fluid whose tangent is everywhere in the direction of the velocity field of the fluid) of $\vec{v}$ are circles lying in the planes perpendicular to and centered upon the nodal lines. The sense of circulation of $\vec{v}$ is about the nodal lines throughout any part of a stream line.

A familiar example of the velocity field [Milne-Thompson 1960], [Batchelor 1972] created by a line vortex in a classical, incompressible, irrotational fluid is the velocity fluid

$$\vec{v} = \lambda \frac{\vec{k} \times \vec{r}}{r} \tag{215}$$

where λ is the *vorticity strength* and $\vec{k}$ is in the direction of the line vortex. Comparing this equation with eq.(211), one observe that the velocity field in the neighborhood of the line vortex in present Schroedinger fluid differs from the velocity field in eq.(212) by a factor $g(\theta, \varphi)$, which depends on the angles. The deviation of this factor from unity distinguishes a vortex in incompressible flow. In the following section one study the details of the current circulation rather than the singular points of the velocity field.

4.4 The current for a single particle in a rotating anisotropic oscillator potential

The Schroedinger equation for a single particle in a rotating anisotropic oscillator in a coordinate system rotating about the z-axis with angular velocity ω is given by

$$(H_0 - \omega j_3)\psi = E\psi \tag{216}$$

where

$$H_0 = \frac{1}{2m}p^2 + \frac{1}{2m}m\omega_1^2 x^2 + \frac{1}{2m}m\omega_2^2 y^2 + \frac{1}{2m}m\omega_3^2 z^2 \tag{217}$$

and

$$j_3 = xp_y - yp_x \tag{218}$$

In App.E is determined to first order in ω, the closed expression for

$$\psi_{n_1n_2}(\vec{r}) = (1 + i\omega X)\phi_{n_1}(x)\phi_{n_2}(y) \tag{219}$$

where ϕ_{n_1} and ϕ_{n2} are simple harmonic oscillator eigenfunctions (E.17)

$$X = \alpha(xy + \beta p_x p_y) \tag{220}$$

with

$$\alpha = \frac{m}{\hbar}\frac{\omega_2^2 + \omega_1^2}{\omega_2^2 - \omega_1^2}, \quad \beta = \frac{2}{m^2(\omega_2^2 + \omega_1^2)} \tag{221}$$

and where the motion in the z-direction was ignored since it is not affected by the rotation.

Note that, being of first order in ω, eq.(213) includes only the effects of the Coriolis force and ignores the quadratic centrifugal potential $m\omega^2/(x^2 + y^2)$.

In section 4.3 it was concluded that the single-particle velocity flow is irrotational everywhere except at the nodal points of the wave function where it has vortices. Inspection of eq.(216) shows that the wave function $\psi_{n_1n_2}$ vanishes if

$$\phi_{n_1}(x) = 0 \quad , \quad \frac{d\phi_{n_2}}{dy} = 0$$

and

$$\phi_{n_2}(x) = 0 \quad , \quad \frac{d\phi_{n_1}}{dy} = 0$$

Therefore $\psi_{n_1n_2}$ vanishes at the set of points $\vec{r}_{0p}$ and $\vec{r}_{p0}$ where the first suffix 0 or p denotes values of $\vec{r}$ at which ϕ_{n_1} vanishes or is maximum respectively, and the second suffix refers similarly to ϕ_{n_2}. The existence of vortices at these points implies non-vanishing current circulations. The latter is defined as

$$\vec{C}(\vec{r}) = \nabla \times \vec{J}(\vec{r}) \tag{222}$$

where the current $\vec{J}(\vec{r})$ in terms of $\psi_{n_1n_2}$ in (216) is given by

$$\vec{J}(\vec{r}) \equiv \frac{1}{m} Re[\psi^*_{n_1n_2}\vec{p}\psi_{n_1n_2}] \tag{223}$$

The J_z component of the current is zero and its x and y components are found to be

$$\begin{aligned} J_x &= \frac{1}{m} Re[\psi^*_{n_1 n_2} p_x \psi_{n_1 n_2}] \\ &= \frac{\omega\alpha\hbar}{m} \{\phi^2_{n_1}\phi^2_{n_2} y + \hbar^2 \beta \phi_{n_2}\phi'_{n_2}({\phi'_{n_1}}^2 - \phi_{n_1}\phi''_{n_1})\} \end{aligned} \quad (224)$$

$$\begin{aligned} J_y &= \frac{1}{m} Re[\psi^*_{n_1 n_2} p_y \psi_{n_1 n_2}] \\ &= \frac{\omega\alpha\hbar}{m} \{\phi^2_{n_1}\phi^2_{n_2} x + \hbar^2 \beta \phi_{n_1}\phi'_{n_1}({\phi'_{n_2}}^2 - \phi_{n_2}\phi''_{n_2})\} \end{aligned} \quad (225)$$

The first term in (219) is recognized, in terms of the velocity field, as a linear irrotational component

$$\vec{J}^{(IF)} = J_x^{(IF)}\vec{i} + J_y^{(IF)}\vec{j} = \rho(\vec{r})\frac{\omega\alpha\hbar}{m}\nabla(xy) \quad (226)$$

Next one must show that the second term generates sets of localized clockwise and anticlockwise current circulations at the points $\vec{r}_{op}$ and $\vec{r}_{po}$. The total current $\vec{J}(\vec{r})$ obviously vanishes at these points as is expected. One observes that $\vec{J}(\vec{r})$ also vanishes at the point $\{r_{oo}\}$. However, at $\{r_{oo}\}$ the velocity field also vanishes and, as a consequence, one shall find that there are no current circulations about the points $\{\vec{r}_{oo}\}$.

Next, at an arbitrary point $\vec{r}$, only the z-component of the current circulation is non-vanishing and this is given by

$$\begin{aligned} C(\vec{r}) &= [\nabla \times \vec{J}(\vec{r})]_z \\ &= \frac{2\omega\alpha\hbar}{m} \cdot \left\{ \phi_{n_1}\phi_{n_2}\left(\phi_{n_2}\frac{d\phi_{n_1}}{dx}x - \phi_{n_1}\frac{d\phi_{n_2}}{dy}y\right) \right. \\ &+ \left. \beta\hbar^2 \left[\phi_{n_1}\frac{d^2\phi_{n_1}}{dx^2}\left(\frac{d\phi_{n_2}}{dx^2}\right)^2 - \phi_{n_2}\frac{d^2\phi_{n_2}}{dx^2}\left(\frac{d\phi_{n_1}}{dx^2}\right)^2\right]\right\} \end{aligned} \quad (227)$$

Evaluating $C(\vec{r})$ at the points $\{\vec{r}_{op}\}$ and $\{\vec{r}_{po}\}$

$$C(\vec{r}_{op}) = -\frac{2\omega\alpha\beta\hbar^3}{m}\phi_{n_2}\frac{d^2\phi_{n_2}}{dx^2}\left(\frac{d\phi_{n_1}}{dx^2}\right)^2 \quad (228)$$

$$C(\vec{r}_{po}) = \frac{2\omega\alpha\beta\hbar^3}{m}\phi_{n_1}\frac{d^2\phi_{n_1}}{dx^2}\left(\frac{d\phi_{n_2}}{dx^2}\right)^2 \quad (229)$$

Since for $\vec{r}_{op}$, $d\phi_{n_2}/dy$ is maximum and $d^2\phi_{n_2}/dy^2$ is negative, therefore

$$C(\vec{r}_{op}) < 0, \ \alpha < 0 \ (\omega_1 > \omega_2) \ \Rightarrow \ \textit{clockwise rotation}$$
$$C(\vec{r}_{op}) > 0, \ \alpha > 0 \ (\omega_1 < \omega_2) \ \Rightarrow \ \textit{anticlockwise rotation}$$

for $\vec{r}_{po}$, $d\phi_{n_1}/dx$ is maximum and $d^2\phi_{n_1}/dx^2$ is negative, therefore

$$C(\vec{r}_{po}) < 0, \ \alpha < 0 \ (\omega_1 > \omega_2) \ \Rightarrow \ \textit{anticlockwise rotation}$$
$$C(\vec{r}_{po}) > 0, \ \alpha > 0 \ (\omega_1 < \omega_2) \ \Rightarrow \ \textit{clockwise rotation}$$

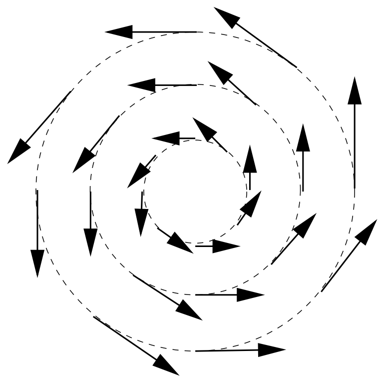


Figure 3: The current in the $x - y$ plane for one vortex

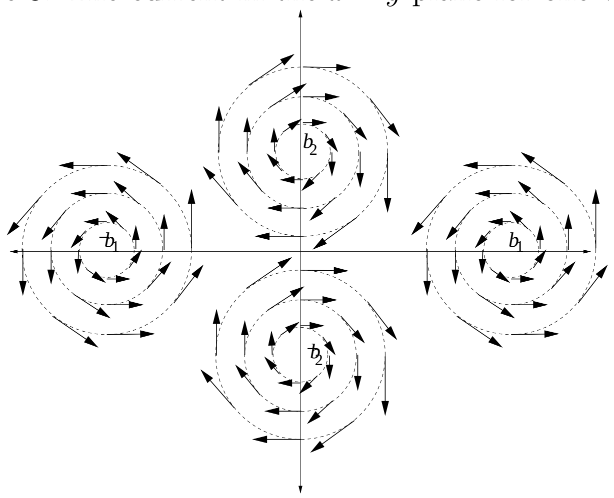


Figure 4: The current in the $x - y$ plane for four vortices

On therefore conclude that the first order single-particle current for the rotating anisotropic oscillator exquibits a set of clockwise and counterclockwise rotations with their centers forming a rectangular array. The locations of their centres are redily determined from the zeroes and peakes of the simple harmonic oscillator eigenfunctions ϕ_{n_1} and ϕ_{n_2}.

Consider some simple examples in what follows:

a) $(n_1, n_2, n_3) = (1, 0, 0)$, and ψ_{100} for $x = y = 0$, and one obtain just one vortex (fig.3a) around the origin.

b) $(n_1, n_2, n_3) = (1, 1, 0)$, and ψ_{110} for $x = 0$, and $y = \pm\sqrt{\hbar/m\omega_1} = b_1$, and $x = \pm\sqrt{\hbar/m\omega_2} = b_2$, $y = 0$. Therefore the $(1, 1, 0)$ state exquibits four vortices around the points $P_1(O, b_2)$, $P_2(O, -b_2)$, $P_3(b_1, O)$, $P_4(-b_1, O)$. The first two are clockwise and the other two are anticlockwise (fig.3b).

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