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EQUILIBRIUM OF CONTINUUA WITH LARGE DEFORMATIONS

(Echilibru Mediilor Continue cu Deformări Mari)

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The classical theory of a deformable continuum has not yet exhausted the entire set of fundamental problems related to the mechanical concept on these media (problems of statical stability and three-dimensional dynamics, problems of plastic deformation). In this sense there are still available growth factors needed for the extension of the calculations of the deformation phenomena which are traditionally studied in the framework of mathematical physics.

Stress in a body with large deformations

Consideration of large deformations impose a more ellaborate expression of the stress state variation. In order to construct a formulation which allows the unfolding of general properties , it is necessary to define the stress state with respect to a reference system in space which moves with the body. In order to determine the orientation in the selected space system, one should introduce the the law of variation of the scalar product in this system. Thus, let ijkvi the components

One can show that the deformed system of the supposed body corresponds to a space of metrics

$$(ds)^2 = g_{ik} dx_i dx_k \tag{1}$$

where g_{ik} are functions determined by the motion. In particular, it results that:

1. *In a continuua , in the neighborhood of a point, following a deformation of components*

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_i}u_j + \frac{\partial}{\partial x_j}u_i + \frac{\partial}{\partial x_i}u_j\frac{\partial}{\partial x_j}u_i\right) \tag{1.1}$$

in the very same point, the mixed scalar product has a mixed tensorial character

$$\alpha_{\nu_j}^{\bar{\nu}_i} = \alpha_k^{\nu_i} \cdot \alpha_{\nu_j}^{l} \cdot \alpha_l^{\bar{k}}; \qquad \alpha_l^{\bar{k}} = \frac{1}{\sqrt{1+2\varepsilon_{kk}}}\left(\delta_l^k + \frac{\partial u_l}{\partial x_k}\right); \qquad \alpha_j^i = \delta_j^i \tag{1.2}$$

This property allows the determination of the deformed system with respect to the three variations of the selected orientations (Appendix C). In order to substantiate the properties of the state of stress, it is neccesary to define the *reduced stress* ($F_{\nu_i}^{\nu_j}$)- a quantity that multiplied with the initial (undeformed) surface element $d\sigma^{\nu_j}$ equilibrate the real final stress applied on the deformed surface and having the direction ν_i

Deformation state in a body with large deformations

The metric (3) leads to a geometrical-analytical interpretation of the state of deformation, stressing the integrability of the displacemnt field in that system.

Let us introduce the first-kind Riemann-Christoffel curvature tensor in the considered space

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\lambda}\left(\frac{\partial\Gamma_{\beta\delta}^{\lambda}}{\partial x^{\gamma}} - \frac{\partial\Gamma_{\beta\gamma}^{\lambda}}{\partial x^{\delta}} + \Gamma_{\beta\delta}^{\kappa}\Gamma_{\kappa\gamma}^{\lambda} - \Gamma_{\beta\gamma}^{\kappa}\Gamma_{\kappa\delta}^{\lambda}\right) . \tag{5.1}$$

where

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2}g^{\lambda\kappa}\left(\frac{\partial g_{\kappa\mu}}{\partial x^{\nu}} + \frac{\partial g_{\kappa\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\kappa}}\right) \tag{5.2}$$

are second-kind Christoffel symbols.

According to the defontion (C.5) of the fundamental quadratic formulation

$$g_{ij} = \delta_{ij} + \varepsilon_{ij} \tag{5.3}$$

one can deduce the vanishing of the curvature tensor and according to a well known theorem :

5. *The system associated to the deformation of a continuum is euclidian. The displacement over a finite distance inside this system is integrable.*

The corresponding compatibility condition

$$\frac{\partial^2 \varepsilon_{\beta\gamma}}{\partial x_\alpha \partial x_\delta} + \frac{\partial^2 \varepsilon_{\alpha\delta}}{\partial x_\beta \partial x_\gamma} - \frac{\partial^2 \varepsilon_{\alpha\gamma}}{\partial x_\beta \partial x_\delta} - \frac{\partial^2 \varepsilon_{\beta\delta}}{\partial x_\alpha \partial x_\gamma} = \tag{5.4}$$

$$\sum_\mu [(\varepsilon_{\mu\beta,\delta} + \varepsilon_{\mu\delta,\beta} - \varepsilon_{\beta\delta,\mu}) \cdot (\varepsilon_{\mu\alpha,\gamma} + \varepsilon_{\mu\gamma,\alpha} - \varepsilon_{\alpha\gamma,\mu})$$

$$- (\varepsilon_{\mu\beta,\gamma} + \varepsilon_{\mu\gamma,\beta} - \varepsilon_{\beta\gamma,\mu}) \cdot (\varepsilon_{\mu\alpha,\delta} + \varepsilon_{\mu\delta,\alpha} - \varepsilon_{\alpha\delta,\mu})]$$

results from(5.1). Since the contracted Ricci-Einstein tensor $R_{\beta\gamma} = R^{\alpha}{}_{\beta\gamma\alpha}$ vanish, a second fundamental compatibility condition is obtained

$$\frac{\partial^2 \varepsilon_{\beta\beta}}{\partial x_\alpha^2} + \frac{\partial^2 \varepsilon_{\alpha\alpha}}{\partial x_\beta^2} - 2\frac{\partial^2 \varepsilon_{\alpha\beta}}{\partial x_\alpha \partial x_\beta} = -H_\gamma, \quad \alpha \neq \beta \neq \gamma \tag{5.5}$$

where the sum of displacement hessians is made visible

$$H_\gamma = \sum_\mu \begin{vmatrix} \frac{\partial^2}{\partial x_\alpha^2} & \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \\ \frac{\partial^2}{\partial x_\beta \partial x_\alpha} & \frac{\partial^2}{\partial x_\beta^2} \end{vmatrix} u_\mu \tag{5.6}$$

Equations (5.4) and (5.5) together with

$$\frac{\partial}{\partial x_\alpha}(\varepsilon_{\alpha\alpha} - \varepsilon_{\beta\beta} - \varepsilon_{\gamma\gamma}) + 2\left(\frac{\partial}{\partial x_\beta}\varepsilon_{\alpha\beta} - \frac{\partial}{\partial x_\gamma}\varepsilon_{\alpha\gamma}\right)$$

$$= \left(\delta_\alpha^\mu + \frac{\partial}{\partial x_\alpha} u^\mu\right) \Delta u_\mu, \quad \alpha \neq \beta \neq \gamma \tag{5.7}$$

and

$$\frac{\partial}{\partial x_\beta}\varepsilon_{\alpha\gamma} - \frac{\partial}{\partial x_\gamma}\varepsilon_{\alpha\beta} - \frac{\partial}{\partial x_\alpha}\omega_\alpha = \frac{1}{2}\left[\frac{\partial^2 u_\mu}{\partial x_\alpha \partial_\beta}\frac{\partial u_\mu}{\partial x_\gamma} - \frac{\partial^2 u_\mu}{\partial x_\alpha \partial_\gamma}\frac{\partial u_\mu}{\partial x_\beta}\right];$$

$$\omega_\alpha = \frac{1}{2}\left(\frac{\partial u_\gamma}{\partial x_\beta} - \frac{\partial u_\beta}{\partial x_\gamma}\right) \tag{5.8}$$

operate a corrections to the St.Venant conditions.

The particularity of St. Venant equations is apparent from the following observations:

6. *In the case of a deformation which satisfies the St. Venant equations, there exist three functions* u'_i, $(i = 1, 2, 3)$ - relative displacement - such that

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_i}u'_j + \frac{\partial}{\partial x_j}u'_i\right) \tag{6.1}$$

and vice versa, with these relations the compatibility conditions are staisfies, if the effective displacements are functional dependent or harmonic, i.e.

$$\Delta u_i \left\|\frac{\partial u_k}{\partial j_i}\right\| = 0$$

In general, the following notations are used

$$\varepsilon_{ii} = \frac{\partial u_i}{\partial x_i} \ ; \ \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_i}u'_j + \frac{\partial}{\partial x_i}u'_j\right) + \varepsilon^0_{ij}, \quad (i \neq j)$$

In the case of St.Venant equations, the following identities result from (5.8) and (5.9)

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\varepsilon^0_{ij} = 0 \ ; \ \frac{\partial}{\partial x_j}\varepsilon^0_{ij} + \frac{\partial}{\partial x_k}\varepsilon^0_{ik} = 0$$

Thus, there are three harmonic functions $f_i(x_j, x_k)$, such that

$$\varepsilon^0_{ij} = \frac{\partial}{\partial x_i}f_j(x_i, x_k) + \frac{\partial}{\partial x_k}f_i(x_j, x_k)$$

Taking

$$u'_i = \bar{u}_i + f_i(x_j, x_k)$$

and using (6.3), the above statements are fullfiled. It is also clear that:

7.*If deformations are finite and admit a system of relative displacements, then the effective displacements are functional dependent of harmonic, such that*

$$D\bar{U} \equiv \left|\frac{\partial u_j}{\partial x_i}\right| = 0 \tag{7.1}$$

or

$$\Delta\bar{U} \equiv \sum_i \frac{\partial^2 u'_i}{\partial x_i^2}$$

In what follows, we shall deduce conditions of a more general character regarding the equilibrium of continua with large deformations including also the instability phenomena.

17. *Continuity relations and contour conditions of a body subjected to volume and contour forces* $\boldsymbol{F} = \| F_i \|$ *;* $\boldsymbol{T}_\nu = \| T'_\nu \|$ *generating deformations and relative stresses*

$$\boldsymbol{\Theta} = \| \delta_{ij} + \partial_i u_j \| \quad ; \quad \boldsymbol{T} = \| T_{ij} \| \tag{17.1}$$

are

$$\partial_j F_{ij} = F_i \ ; \quad \alpha_{\nu j} F_{ij} = T_{i\nu} \tag{17.2}$$

or in matrix form

$$\boldsymbol{\partial\Theta T} = \boldsymbol{F} \ ; \quad \boldsymbol{\mathcal{A}\Theta T} = \boldsymbol{T}_\nu \tag{17.3}$$

$$\boldsymbol{\partial} = \| \partial_j \| \quad , \quad \boldsymbol{\mathcal{A}} = \| \alpha_{ij} \| \tag{17.4}$$

where F_{ij} are the reduced stresses [1]

$$F_{ij} = \alpha_{i\bar{k}} T_{jk} \tag{17.5}$$

To demonstrate the above elations, we consider that the deformation energy in the body's domain G reads

$$E_G = \int_G F(u_i, \partial_j u_i) dV = \int_G T_{ij} \varepsilon_{ij} dV + \int_G F_i u_i dV \ ; \quad i, j = 1, 2, 3. \tag{17.6}$$

Introducing the energy

$$E_\Gamma = -\int T_{i\nu} u_i dS \tag{17.7}$$

given by contour Γ displacement, the extremuum conditions of the total energy

$$E = E_G + E_\Gamma \tag{17.8}$$

following a variation of the displacements, are deduced from the variational equations of function F (Appendix A) by identifying F with E.

[1] $\alpha^i_{\bar{k}}$ represents the scalar unitary product in the deformed space.

$$\alpha^i_{\bar{k}} = \frac{\delta_{ik} + \partial_k u_i}{\sqrt{1 + 2\varepsilon_{kk}}}$$

By transposing these equations (Appendix A) with respect to the energy expressions in G and on Γ, the continuity equations in G and on Γ are restrained in the stated form. Equations (17.3) agree with the equations that can be obtained from the equilibrium relations in the interior and on the contour. Momentum equilibrium are additionaly constraining the symmetry of the matrix T.

18. *The continuity equations can be also expressed in the form*

$$\boldsymbol{F} = \boldsymbol{R}_1 + \boldsymbol{R}_2 + \boldsymbol{R}_3 \tag{18.1}$$

$$\boldsymbol{R}_1 = \boldsymbol{\partial T} \ , \quad \boldsymbol{R}_2 = (\boldsymbol{\Theta} - \boldsymbol{E})\boldsymbol{\partial T} \ , \quad \boldsymbol{R}_3 = (\boldsymbol{\partial\Theta})\boldsymbol{T}$$

Iteration of general solutions

20. *Performing the decomposition*

$$F = \sum_i F_i \ ; \quad T_\nu = \sum_i T_{i\nu} \ ; \quad \Theta = \sum_i \Theta_i \ ; \quad T = \sum_i T_i \tag{20.1}$$

such that

$$\partial T_i = F_i - \partial \sum_{k=i-j-1}^{i-1} (\Theta_k - E)T_j \ ; \quad i = 0, 1, 2 \ldots$$

$$\mathcal{A} T_i = F_i - \mathcal{A} \sum_{k=i-j-1}^{i-1} \sum_{j=0}^{i-1} (\Theta_k - E)T_j$$

one obtains a system of equations that once solved by iteration leads to the solutions of the continuity and boundary equations.

Indeed by adding up one obtains eqs.(17.3). Explicitely, the iterating relations are written

$$F_i = T_{\nu i}, \quad i = 0, \qquad \partial T_0 = F_0 \ , \quad \mathcal{A} T_0 = T_{\nu 0}$$

$$\partial T_1 = -\partial(\Theta_0 - E)T_0 \ ; \quad \mathcal{A} T_1 = -\mathcal{A}(\Theta_0 - E)T_0$$

$$\partial T_2 = -\partial(\Theta_1 - E)T_0 - \partial(\Theta_0 - E)T_1 - \partial(\Theta_1 - E)T_1$$

$$\mathcal{A} T_2 = -\mathcal{A}(\Theta_1 - E)T_0 - \mathcal{A}(\Theta_0 - E)T_1 - \mathcal{A}(\Theta_1 - E)T_1$$

The first two equations correspond to the classical problem of elasticity and provides the stable solution for small deformations; the following iterations correspond at each step to an elastic body in classical sense, loaded wigh mass and contour forces

$$\partial(\Theta_0 - E)T_0 \ ; \quad - \mathcal{A}(\Theta_0 - E)T_0$$

$$-\partial(\Theta_1 - E)T_0 - \partial(\Theta_0 - E)T_1 - \partial(\Theta_1 - E)T_1$$
$$-\mathcal{A}(\Theta_1 - E)T_0 - \mathcal{A}(\Theta_0 - E)T_1 - \mathcal{A}(\Theta_1 - E)T_1$$

Critical state. Variational criteria for the approximate calculus

In order to build some fundamental criteria, for the determination of the critical state, as well to exploit the methods of approximate calculus we consider the variation $\delta u_i = k_i u_i$ ($k_i =$ constant) of the displacement in body in equilibrium and acted upon by internal stresses.

We shall use the following notions determined for critical state (Appendix B):

-internal energy corresponding to the divergence (dilations)

$$E_1 = \frac{1}{2}\int_G \bar{u}_i^2 \partial_j \partial_k T_{jk}$$

One can show that in the case of linear deformations we have

$$E_1 = \frac{3\lambda + 2\mu}{2}\int_G \bar{u}_i^2 \Delta \mathrm{div}\mathbf{u} dV$$

where the quantity $\bar{u}_i \Delta \mathrm{div}\mathbf{u}$ arise as a force in the i-th direction.;

-internal energy due to curvature variation

$$E_2 = -\int_\Gamma \bar{u}\alpha_{\nu j}\partial_k \bar{u}_i T_{jk} \bar{u}_i dS$$

(the quantity $\partial_j \partial_k \bar{u}_i T_{jk} \approx R_{3i}$ is precisely the consequence of the relative stress given by the curvature variation $\partial_j \partial_k \bar{u}_i$);

-energy given by the variation of the contour displacement

$$E_1' = \frac{1}{2}\int_\Gamma \bar{u}_i \alpha_{\nu j} \partial_k T_{jk} \bar{u}_i dS$$

(the quantity $\alpha_{j\nu}\partial_k T_{jk} \approx \alpha_{j\nu} R_{1j}'$ appears as the normal component on the contour of the vector which results from the displacement variation over the variation $\bar{u}_i dS$ of the body's contour).

-energy of the loads deforming the contour

$$E_2' = -\int_\Gamma \bar{u}_i \alpha_{\nu j} \partial_k \bar{u}_i T_{jk} dS$$

(the quantity $\alpha_{j\nu}\partial_k T_{jk} \approx \alpha_{j\nu} R'_{2j}$ appears as the normal component on the contour of the vector which results from the deformation variation on the contour; this component refersto the variation $\bar{u}_i dS$ of the body's volume).

With above considerations it can be shown under rather general conditions that:

21. *A continuum is found in a limit state of stability with respect to a variation of the displacements if the internal energy state due to dilations or shears accounts for the total energy variation of the displacement and contour deformation.*

$$E_1 + E_2 = E'_1 + E'_2$$

The conditions that the second variation (21.5) of the total energy is positive, express the stability condition; its limit is vanishing. The condition (21.5) results by identifying the function E with J (Appendix B). It also result that

22. *With respect to a vanishing variation of the displacements on the contour, a continuum preserves its initial equilibrium position.*

Indeed, in this case

$$E_1^i = E_2^i = 0$$

According to a fundamental theorem of variational calculus, the integrand of the bellow expression is zero

$$\int_G \bar{u}_i \left[\frac{1}{2} \bar{u}_i \frac{\partial^2 T_{jk}}{\partial x_j \partial x_k} - \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_k} T_{jk} \right] dV = 0$$

Neglecting the continuity equations (17.3) for the case of stable equilibrium, it results that

$$\frac{\partial^2 T_{jk}}{\partial x_j \partial x_k} = 0$$

(in the case of linear elasticity $\Delta\theta = 0$) and therefore,

$$\frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_k} T_{jk} = 0$$

In view of the displacements variation, we have $T_{jk} = 0$. Analogously it results in more general terms that

23. *With respect to a variation of displacements which are zero on the contour, a continuum, with a stress state of zero divergence ($\partial_j \partial_k T_{jk} = 0$ or $\Delta\theta = 0$), preserves its initial equilibrium position*[1].

[1] Compare this result with the result at 15

These results can be state in a more general frame. Accordingly one can show first that

24. *In a stability limit state of a continuum, when the total displacement and deformation energy vanishes with respect to a variation $k^j\bar{u}^j$, the divergence of the stress is equivalent to the strain given by the curvature, i.e.*

$$\frac{1}{2}u_i\partial_j\partial_k T_{jk} = \partial_j\partial_k u_i T_{jk} \tag{24.1}$$

where the coordinate system is choosen such that the variation of the energy (in the conventional sense) given by the displacement of the body volume vanishes

$$\int_G \bar{u}_i dV = 0 \tag{24.2}$$

The deformation energy on the contour cancels, and we have

$$\int_G \bar{u}_i \left[\frac{1}{2}\bar{u}_i\partial_j\partial_k T_{jk} - \partial_j\partial_k\bar{u}_i T_{jk}\right] dV = 0 \tag{24.3}$$

According to a well known theorem from variational calculus [3] along with eq.(24.2) the integrand of the above integral vanish.

Based on proposition (24), the above results are recovered for a variation of the displacementwhich satisfies condition (24.2). Since in the expression of the curvature and stress the effects of coordinate change are not present in general conditions, eqs. (24.2) and (24.3) are satisfied for arbitrary variations, such that

25. *With respect to a displacement variation* [2]*, a continua found in a stress state of zero divergence ($\partial_j\partial_k T_{jk}$) or $\Delta\theta = 0$), preserves its initial equilibrium position (with small deformations) if the deformation and displacement energy variation of the contour is negligible*

Example

Centric stress (stable solution)

The solution is of the form

$$u \approx \varepsilon - \frac{3}{2}\varepsilon^2 \quad ; \quad \varepsilon = \frac{p}{E} \quad ; \quad T_{xx} = \frac{p}{1+\varepsilon-\frac{3}{2}\varepsilon^2} \quad ; \quad T_{xy} = T_{yy} = 0.$$

[2]1) Under the above conditions

Composed stress (with linear displacements)

For $u = ax + a'y,\ v = bx + b'y$ we have

$$T_{xx} = p_x\ ,\quad T_{yy} = p_y\ ,\quad T_{xy} = p_x a'\ ;\quad b' = \frac{p_x a'}{p_y}$$

If $a' = b' = 0$, the elastic stable solution is obtained. One should note also the occurence of another solution (unstable).

Pure bending

In first approximation one takes the elastic solution

$$u_0 = \frac{axy}{E}\ ;\ v_0 = -\frac{a}{2E}(x^2 - l^2 - \nu y^2)\ ;\ T^0_{xx} = ay\ ;\ a = \frac{M}{l}\ ,\ T^0_{xy} = T^0_{yy} = 0$$

In the second approximation the solution reads

$$T^1_{xx} = -\frac{a^2}{E}\left(y^2 + \frac{\nu+2}{2}\left(\frac{I}{\Omega} - y^2\right)\right)\ ;\ T^1_{yy} = \frac{a^2}{2E}(y^2 - h^2)\ ;T^1_{xy} = 0$$

PLANE PROBLEM OF TRANSVERSE STABILITY

$$\frac{d}{dx}T_{xx} + \frac{d}{dy}T_{xy} + \frac{d}{dz}T_{xz} = 0$$

$$\frac{d}{dx}T_{yx} + \frac{d}{dy}T_{yy} + \frac{d}{dz}T_{yz} = 0$$

$$\frac{d}{dx}T_{zx} + \frac{d}{dy}T_{zy} + \frac{d}{dz}T_{zz} = -\left(w_{xx}T_{xx} + w_{yy}T_{yy} + w_{zz}T_{zz} + 2w_{xy}T_{xy} + 2w_{yz}T_{yz} + 2w_{zx}T_{zx}\right)$$

Assuming bending we can take

$$M_{xx} = -k(w_{xx} + \nu w_{yy})\ ;\quad M_{xy} = -(1-\nu)k w_{xy}\ ;\quad M_{yy} = k(w_{yy} + \nu w_{xx})$$

$$T_{zx} = \frac{d}{dx}M_{xx} + \frac{d}{dy}M_{xy} = -\Delta k w_x\ ;\quad T_{zy} = -\Delta k w_y$$

$$k = \frac{E}{1-\nu^2}J\ ;\quad \Delta = \frac{d^2}{dx^2}$$

and together with

$$\frac{d}{dz}T_{zz} = P(x,y)$$

provide the continuity equations

$$\frac{d}{dx}T_{xx} + \frac{d}{dy}T_{xy} = 0 \ ; \quad \frac{d}{dx}T_{yx} + \frac{d}{dy}T_{yy} = 0$$

$$\Delta\Delta kw + P(x,y) = T_{xx}w_{xx} + 2T_{xy}w_{xy} + T_{yy}w_{yy}$$

Solving the plane problem one can write down the stability equation

$$\Delta\Delta kw + P(x,y) = F_{y^2}w_{xx} - 2F_{xy}w_{xy} + F_{x^2}w_{yy}$$

where F is the Airy function. The solutions of the above equation are of hyperbolic, parabolic or eliptic according to the sign of $H(F)$, which indicate the nature of solution stability.

PLANE STABILITY PROBLEM IN THE CASE OF LINEAR STRESS STATE

In the case of plane problem, the equilibrium equations assume the forms

$$\begin{aligned}
&(\lambda + 2\mu)\theta_x - \mu\omega_y + Lu = 0\\
&(\lambda + 2\mu)\theta_y + \mu\omega_x + Lv = 0\\
&L = T_{xx}\partial^2/\partial x^2 + 2T_{xy}\partial^2/\partial x\partial y + T_{yy}\partial^2/\partial y^2
\end{aligned}$$

or

$$\begin{aligned}
&(\lambda + 2\mu)\theta_x + D_2 u = 0\\
&(\lambda + 2\mu)\theta_y + \mu\omega_x + Lv = 0\\
&D_2 = \mu D + L
\end{aligned} \tag{2}$$

a form equivalent to

$$\begin{aligned}
&D_1 u + (\lambda + \mu)\omega_y = 0\\
&D_1 v - (\lambda + \mu)\omega_x = 0\\
&D_1 = (\lambda + 2\mu)\Delta + L \ ; \quad \omega = v_x - u_y
\end{aligned}$$

Before all we assume the plane elasticity plane problem to be solved. Thus, the displacements $\bar{u}, \bar{v}$ and the stresses $\bar{T}_{xx}, \bar{T}_{xy}, \bar{T}_{yy}$. Therefore in the frame of the stability problem the supplementary displacements $\bar{\bar{u}}, \bar{\bar{v}}$ and stresses $\bar{\bar{T}}_{xx}, \bar{\bar{T}}_{yx}, \bar{\bar{T}}_{yy}$, such that

$$u = \bar{u} + \bar{\bar{u}}, \ldots$$

$$T_{xx} = \bar{T}_{xx} + \bar{\bar{T}}_{xx}, \ldots$$

Since $Lu \approx L\bar{u}$, $Lv \approx L\bar{v}$, the following approximate relation is valide around the critical state

$$Lu \approx T_{xx}\frac{\partial^2}{\partial x^2}\bar{\bar{u}} + 2T_{xy}\frac{\partial^2}{\partial x \partial y}\bar{\bar{u}} + T_{yy}\frac{\partial^2}{\partial y^2}\bar{\bar{u}}$$

We also make the assumption that the state of stable stress has a uniform character, such that the stress fluctations around a given point can be neglected ; it will be possible to admit $\frac{d}{dx}L() = L()_x$, $\frac{d}{dy}L() = L()_y$. Taking this into account, it results that

$$\begin{aligned}
&D_1\bar{\bar{\theta}} = D_1\bar{\bar{\omega}} = D_2D_1\bar{\bar{u}} = D_1D_2\bar{\bar{v}} = 0 \\
&(\lambda+\mu)\bar{\bar{\theta}}_x + D_2\bar{\bar{u}} = 0 \\
&(\lambda+\mu)\bar{\bar{\theta}}_y + D_2\bar{\bar{v}} = 0 \\
&D_1\bar{\bar{u}} + (\lambda+\mu)\bar{\bar{\omega}}_y = 0 \\
&D_2\bar{\bar{v}} - (\lambda+\mu)\bar{\bar{\omega}}_x = 0
\end{aligned}$$

where $[D_1, D_2] = 0$.

The problem of plane stability can be tackled with analytical functions. Indeed, the function

$$F(z) = \int (P(x,y) + iQ(x,y))dz$$

where

$$P(x,y) = D_1\varphi \ ; \quad Q(x,y) = D_2\psi$$

are analytical, and

$$\bar{\bar{u}} = \varphi_x - \psi_y \ , \quad \bar{\bar{v}} = \varphi_y + \psi_x$$

The equilibrium equations are identical to the Cauchy-Riemann relations

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} \ , \quad \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}$$

which indicate the conjugate harmonicity of the functions P and Q.

It is recognized that in the case $L_u = L_v = 0$ one obtains the same representation like in elasticity, when $P(x,y) = (\lambda + 2\mu)\bar{\bar{\theta}}$; $Q(x,y) = \mu\bar{\bar{\omega}}$

The formulation of the displacements problem is reduced to the equations

$$\Delta(D_1\varphi) = \Delta(D_2\psi)$$

and the corresponding contour conditions are

$$u_N = \frac{d\varphi}{d\nu} - \frac{d\psi}{d\tau}; u_T = \frac{d\varphi}{d\tau} + \frac{d\psi}{d\nu}$$

where

$$\frac{d}{d\nu} = \frac{d}{dx} \cdot \frac{dx}{ds} + \frac{d}{dy} \cdot \frac{dy}{ds}\ ; \frac{d}{d\tau} = -\frac{d}{dx} \cdot \frac{dy}{ds} + \frac{d}{dy} \cdot \frac{dy}{ds}$$

represent the normal and tangential derivatives.

Appendix A

The minimum of a volume integral, defined over the domain G closed by the surface Γ, of the form

$$J = \int_G F(u^i, u^i_j)dV, \quad i,j = 1,2,3;\ u_{ij} = \frac{\partial u^i}{\partial x_j} \qquad (A.1)$$

is obtained by solving the equations [3]

$$[F_G]_i = 0 \quad \text{in } \ G \qquad (A.2)$$

$$[F_\nu]_i = 0 \quad \text{on } \ \Gamma \qquad (A.3)$$

where

$$[F_G]_i = F_{u_i} - \sum_j \left(\frac{\partial}{\partial x_j} + \sum_k u_{jk} \frac{\partial}{\partial x_k} \right) F_{u_{ij}} \qquad (A.4$$

$$[F_\nu]_i = \sum_j \alpha_{\nu i} F_{u_{ij}} \qquad (A.5)$$

Above it was assumed that F and its first two derivatives are continuous. The variational equations are obtained by varying the arguments $\delta u_i = \varepsilon_i \bar{u}_i$. The parameters ε_i are assumed to be arbitrarily small, whereas the functions and its first two derivatives are continuous as well, such that the functional

$$\Phi(\varepsilon_i) = \int_G F(u_i + \delta u_i, u_{ij} + \delta_{ij})dG \qquad (A.6)$$

satisfies the condition $\left(\frac{\partial\Phi}{\partial\varepsilon_i}\right) = 0$

The minimum of the integrals

$$J = \int_G F(u_i, u_{ij})dV - \int_\Gamma \sum_i u_i f^i_\nu dS \qquad (A.7)$$

where f^i_ν are continuous on the contour, is given by

$$[F_G]_i = -\left(f_{i\nu} + \sum_j \frac{\partial F_{u_{ij}}}{\partial x^j}\right) \qquad (A.8)$$

Appendix B

The determination of a minimum of the integral (A.1) with respect to the variations δu_i and consequently of function (A.2) is done by equating to zero the second variation of the integral J, i.e.

$$\delta^2 J = \sum_i \sum_j \varepsilon_i \varepsilon_j \left(\frac{\partial^2\Phi}{\partial\varepsilon_i\partial\varepsilon_j}\right) = 0 \qquad (B.1)$$

for $\varepsilon_i = 0$, $i = 1, 2, 3$.

Consequently,

$$\frac{\partial^2\Phi}{\partial\varepsilon_i\partial\varepsilon_j} = \int_G \left[\bar{u}_i\bar{u}_j F_{u_iu_j} + \sum_k \left(\frac{\partial\bar{u}_i}{\partial x_k}\bar{u}_j F_{u_{ik}u_j} + \bar{u}_i\frac{\partial\bar{u}_j}{\partial x_k}\bar{u}_j F_{u_iu_{jk}} + \frac{\partial\bar{u}_j}{\partial x_k}\bar{u}_j F_{u_{ik}u_{jl}}\right)\right] dV$$

We are interested in the case when

$$F_{u_iu_j} = F_{u_{il}u_j} = 0\ ;\ F_{u_{il}u_{jk}} = 0, \quad \text{for } i \neq j,$$

In this circumstance $\frac{\partial^2\Phi}{\partial\varepsilon_i\partial\varepsilon_j} = 0$, for $i \neq j$.

Instead of the relations (B.2) we have

$$\int_G \bar{u}_i^2 [F_G]_i dV \approx \int_\Gamma \bar{u}_i^2 [F_\nu]_i dS = 0$$

where

$$[F_G]_i = \bar{u}_i \left[F_{u_iu_i} - \sum_k \frac{\partial F_{u_iu_{ik}}}{\partial x_k} + \frac{1}{2}\sum_{l,k} \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_k}\frac{\partial}{\partial \bar{u}_i} F_{u_{il}u_{ik}}\right]$$
$$- \sum_{l,k} \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_k}\frac{\partial}{\partial \bar{u}_i} F_{u_{il}u_{ik}}$$

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