Large N expansion of tensor models

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arXiv:1301.1535[hep-th], *Annales Henri Poincaré* (in press)

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Plan

- Introduction
- Matrix models and their large N exapnsion (dominant graphs)
- 3-dimensional tensor models; the colored and the multi-orientable QFT simplifications
- Classification of Feynman graphs
- Some combinatorial and topological tools
- Large N expansion - dominant graphs
- Perspectives

Introduction

Quantum theory of gravity - Graal of modern theoretical physics

several approaches:

- string theory
- loop quantum gravity
- matrix models - $2-$dimensional quantum gravity
- causal dynamical triangulations
- *etc.*

Matrix models - 2−dimensional quantum gravity

Ph. Di Francesco *et. al.*, *Phys. Rept.* (1995), hep-th/9306153

M - $N \times N$ matrix

the partition function:

$$Z = e^F = \int dM e^{-\frac{1}{2}\mathrm{Tr} M^2 + \frac{g}{\sqrt{N}}\mathrm{Tr} M^3}.$$

diagrammatic expansion - Feynman ribbon graphs

generates random triangulations

discretized integral over geometries performed as sum over random triangulations

0−dimensional string theory (a pure theory of surfaces with no cuppling to matter on the string worldsheet)

Large N expansion of matrix models

the matrix amplitude can be combinatorially computed - in terms of number of vertices (p), edges and faces (F) of the graph
change of variables: $M \to M\sqrt{N}$ (easy to count powers of N)

$$\mathcal{A} = \lambda^V N^{-\frac{1}{2}V+F} = \lambda^{2p} N^{2-2g}$$

(since $E = \frac{3}{2}V$)
the partition function (and the free energy) supports a $1/N$ exapnsion:

$$Z = N^2 Z_0(g) + Z_1(g) + \ldots = \sum_g N^{2-2g} Z_g(g)$$

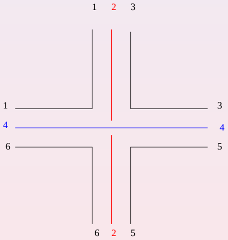
Z_g gives the contribution from surfaces of genus g

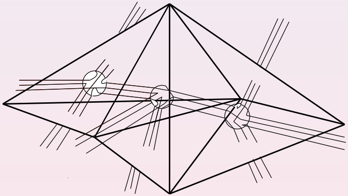
large N limit, only planar surfaces survive - dominant graphs (triangulations of the sphere $\mathcal{S}^2$)

V. A. Kazakov, *Phys. Lett.* **B** ('85), F. David, *Nucl. Phys.* **B** ('85), E. Brezin *et al.*, *Commun. Math. Phys.* ('78)

Tensor models

natural generalization of matrix models





QFT-inspired simplication of tensor models - the colored tensor models

highly non-trivial combinatorics
$\rightarrow$ a QFT simplification of these models - colored tensor models

(R. Gurău, Commun. Math. Phys. (2011), arXiv:0907.2582)

a quadruplet of complex fields $\left(\phi^0, \phi^1, \phi^2, \phi^3\right)$;

$$S[\{\phi^i\}] = S_f[\{\phi^i\}] + S_{int}[\{\phi^i\}]$$
$$S_f[\{\phi^i\}] = \frac{1}{2}\sum_{p=0}^{3}\sum_{ijk}\overline{\phi^p_{ijk}}\phi^p_{ijk} \tag{1}$$
$$S_{int}[\{\phi^i\}] = \frac{\lambda}{4}\sum_{i,j,k,i',j',k'}\phi^0_{ijk}\phi^1_{i'j'k}\phi^2_{i'jk'}\phi^3_{k'j'i} + \text{ c. c.},$$

the indices $0, \ldots, 3$ - color indices.
extra property: the faces of the Feynman graphs of this model have always exactly two (alternating) colors.

Various QFT develpoments for colored tensor models

- large N expansion

  R. Gurau, Annales Henri Poincare (2011), [arXiv:1011.2726 [gr-qc]]

- large N expansion in any dimension
  R. Gurau and V. Rivasseau, Europhys. Lett. (2011), arXiv:1101.4182[gr-qc],

  R. Gurău, Annales Henri Poincaré (2012) [arXiv:1102.5759 [gr-qc]].

- $\longrightarrow$ continuum phase transition and computation of critical exponents

  V. Bonzom *et. al.*, *Nucl. Phys.* **B** (2011) arXiv:1105.3122[hep-th]

- renormalizable tensor models
  J. Ben Geloun and V. Rivasseau, Commun. Math. Phys. (in press), arXiv:1111.4997 [hep-th].
  S. Carrozza *et. al.* arXiv:1207.6734 [hep-th].
  D. O. Samary and F. Vignes-Tourneret, arXiv:1211.2618 [hep-th].
  J. Ben Geloun and D. O. Samary, arXiv:1201.0176 [hep-th].

  J. B. Geloun and E. R. Livine, arXiv:1207.0416 [hep-th].

- Noether currents

  J. Ben Gelon, *J. Math. Phys.* (2012), [arXiv:1107.3122 [hep-th]]

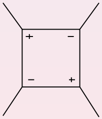
- *etc.*

A (Moyal) QFT-inspired simplication of tensor models

highly non-trivial combinatorics
$\rightarrow$ a QFT simplification of these models - multi-orientable models

A. Tanasă, J. Phys. **A** (2012)

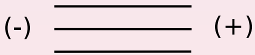
proposal made within the Group Field Theory framework

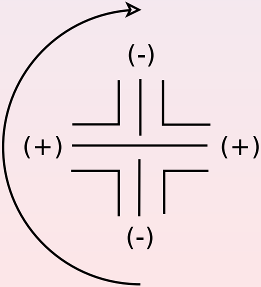


edge going from a $+$ to a $-$ corner

The action: the propagator and the vertex

$$S[\phi] = S_0[\phi] + S_{int}[\phi], \tag{2}$$
$$S_0[\phi] = \frac{1}{2}\sum_{i,j,k} \hat{\phi}_{kji}\phi_{ijk}, \quad S_{int}[\phi] = \frac{\lambda}{4}\sum_{i,j,k,i',j',k'} \phi_{kji}\hat{\phi}_{ij'k'}\phi_{k'ji'}\hat{\phi}_{i'j'k}.$$





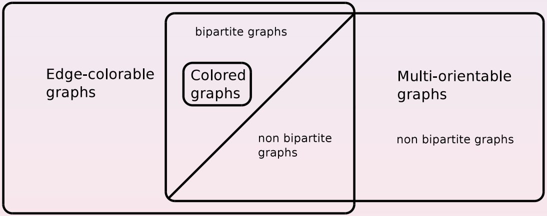
Multi-orientable tensor Feynman graphs

no twists on the propagators $\to$ one-to-one correspondence between multi-orientable tensor Feynman graphs and graphs

A *four-edge colorable* is a graph for which the edge chromatic number is equal to four.

The set of Feynman graphs generated by the colored action (1) is a strict subset of the set of Feynman graphs generated by the m.o. action (2).

A bipartite graph is four-edge colorable.

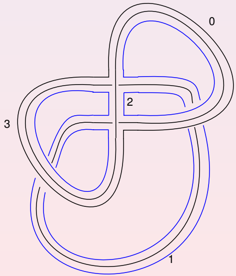


Example of graphs

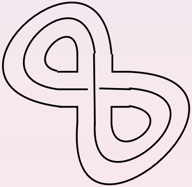
A **tadface** is a face "going" several times through the same edge.

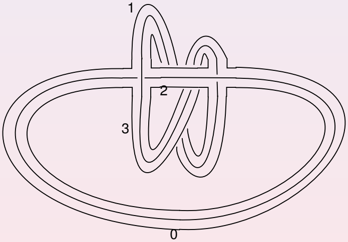
The condition of multi-orientability discards tadfaces (Theorem 3.1 of A. Tanasă, J. Phys. **A** (2012), arXiv:1109.0694).

example of a graph with a tadface which is edge-colorable

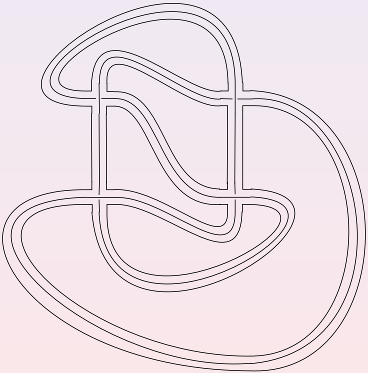


the planar double tadpole as an example of a m.o. graph which is not colorable. On the right, an example of a m.o. graph which is 4-edge colorable but does not occur in colorable tensor models.

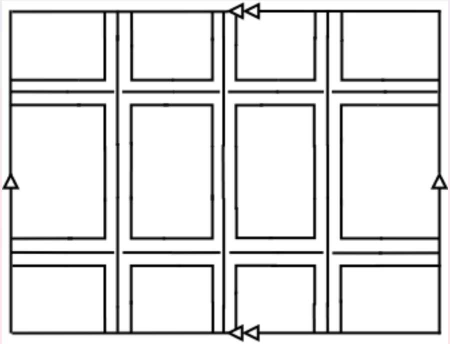




A 4-edge colorable m.o. graph which is not bipartite



A graph without tadfaces which is not m.o. Edges of the box are identified so that the graph is drawn on the torus

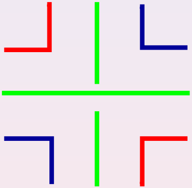


Topological tools - jacket ribbon subgraphs

In the colored case the $1/N$ expansion relies on the notion of **jacket ribbon subgraphs**, which are associated to the cycle of colors up to orientation.

Generalization of the notion of jackets for m.o. graphs

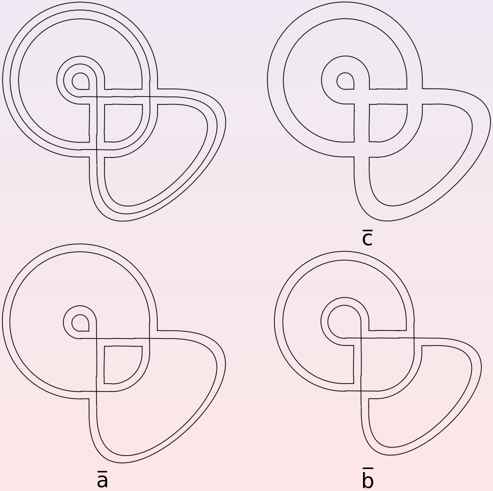
three pairs of opposite corner strands



A **jacket of an m.o. graph** is the graph made by excluding one type of strands throughout the graph. The *outer* jacket $\bar{c}$ is made of all outer strands, or equivalently excludes the inner strands; jacket $\bar{a}$ excludes all strands of type a and jacket $\bar{b}$ excludes all strands of type b.

Example of jacket subgraphs

A m.o. graph with its three jackets $\bar{a}$, $\bar{b}$, $\bar{c}$

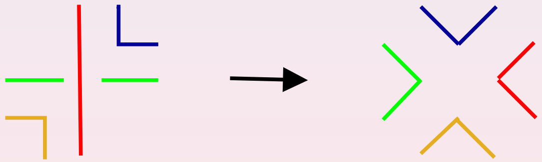


Is such a jacket subgraph a ribbon subgraph?

Is such a jacket subgraph a ribbon subgraph?

Any jacket of a m.o. graph is a ribbon graph (with uniform degree 4 at each vertex).

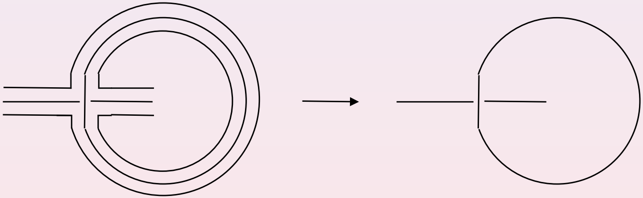
untwisting vertex procedure:



may introduce twists on the edges

this does not hold for any, non-m. o., tensor graph

Example: Deleting a pair of opposite corner strands in this tadpole (which has tadfaces), does not lead to a 2-stranded graph.



Euler characteristic & degree of tensor graphs

ribbon graphs can represent orientable or non-orientable surfaces.

Euler charcacteristic formula:

$$\chi(\mathcal{J}) = v - e + f = 2 - k,$$

k is the non-orientable genus,
v is the number of vertices,
e the number of edges and
f the number of faces.

If the surface is orientable, k is even and equal to twice the orientable genus g

Given a multi-orientable graph $\mathcal{G}$, its degree $\varpi(\mathcal{G})$ is defined by

$$\varpi(\mathcal{G}) = \sum_{\mathcal{J}} \frac{k_{\mathcal{J}}}{2},$$

the sum over J running over the three jackets of $\mathcal{G}$.

Large N expansion of the m.o. tensor model

Feynman amplitude calculation - each tensor graph face contributes with a factor N, N being the size of the tensor

$\Longrightarrow$ one needs to count the number of faces of the tensor graph

this can be acheived using the graph's jackets (ribbon subgraphs)

The Feynman amplitude of a general m.o. tensor graph G writes:

$$A(\mathcal{G}) = \lambda^{v_{\mathcal{G}}} N^{3-\varpi(\mathcal{G})}.$$

The free energy writes as a formal series in $1/N$:

$$F(\lambda, N) = \sum_{\varpi \in \mathbb{N}/2} C^{[\varpi]}(\lambda) N^{3-\varpi},$$

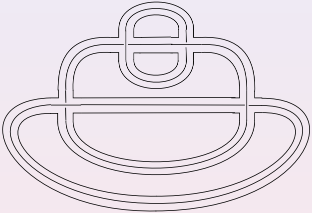
$$C^{[\varpi]}(\lambda) = \sum_{\mathcal{G}, \varpi(\mathcal{G})=\varpi} \frac{1}{s(\mathcal{G})} \lambda^{v_{\mathcal{G}}}.$$

Dominant graphs

dominant graphs:

$$\varpi = 0.$$

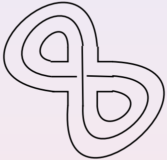
An example of a dominant tensor graph

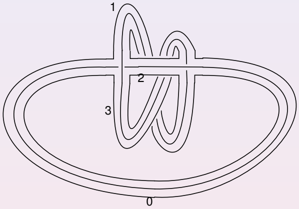


- outer jacket is orientable (always the case for the outer jacket), and it has genus $g_1 = 0$.
- the two remaining jackets also have vanishing genus $g_2 = g_3 = 0$ (can be directly computed using Euler's characteristic formula)

$\Longrightarrow$ vanishing degree ($\varpi = 0$) $\Leftrightarrow$ dominant graph

Two examples of non-dominant tensor graphs





double tadpole:
$\varpi = 0 + \frac{1}{2} + 0 = \frac{1}{2}$.

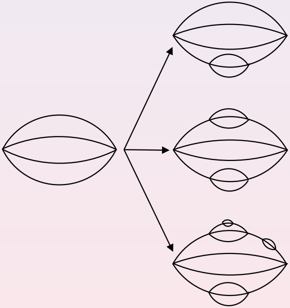
"twisted sunshine" (bipartite $4-$edge colorable graph):
Its outer jacket is orientable (always the case for the outer jacket), and it has genus $g_1 = 1$.
The two remaining jackets are isomorphic and have non-orientable genus $k_2 = k_3 = 1$.
$\Longrightarrow \varpi = 2$.

General identification of dominant graphs

Non-bipartite m.o. graphs have at least one non-orientable jacket and are thus non-dominant of degree

$$\varpi \geq \frac{1}{2}.$$

The only bipartite (and hence edge-colorable) m.o. tensor graphs of vanishing degree ($\varpi = 0$) are the graphs obtained from insertions of the "melon" graph.



series-parallel graphs

Main result:

The m.o. model admits a $1/N$ expansion whose dominant graphs are the "melonic" ones.

These graphs correspond to a particular class of triangulations of the sphre $\mathcal{S}^3$.

Perspectives

- combinatorial Hopf algebras for renormalizable tensor models the Ben Geloun-Rivasseau model *Commun. Math. Phys.* (in press), arXiv:1111.4997 [hep-th]

  (work in progress with M. Raasakka)
- sub-dominant tensor graphs
- generalization of the matrix integral techniques to tensor integral techniques - *hyper-map counting*
- Schaeffer bijection G. Schaeffer, *Electronic J. Comb.* (1997)
  $3D$ geodesic length?
- study the Noether currents of m.o. tensor models (generalization of J. Ben Geloun, *J. Math. Phys.* (2012), arXiv:1107.3122 [hep-th]).
- enlarge the m.o. framework studied in this paper to include still larger classes of tensor graphs and check whether they admit a $1/N$ expansion.

Vă mulțumesc pentru atenție!

"The amount of theoretical work one has to cover before being able to solve
problems of real practical value is rather large, but this circumstance is [...] likely to become more pronounced in the theoretical physics of the future."

P.A.M. Dirac, *"The principles of Quantum Mechanics"*, 1930